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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. A combined national-level engineering entrance examination has finally been proposed by the Ministry of Human Resource Development, Government of India. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulate. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

Ghanshyam Tewani
CHAPTER 1

Number System, Inequalities and Theory of Equations

- Constant and Variables
- What is Function
- Intervals
- Inequalities
- Generalized Method of Intervals for Solving Inequalities
- Absolute Value of x
- Some Definitions
- Geometrical Meaning of Roots (Zeros) of an Equation
- Key Points in Solving an Equation
- Graphs of Polynomial Functions
- Equations Reducible to Quadratic
- Remainder and Factors Theorem
- Quadratic Equation
- Common Root(S)
- Relation between Coefficient and Roots of n-Degree Equations
- Solving Cubic Equation
- Repeated Roots
- Quadratic Expression in Two Variables
- Finding the Range of a Function Involving Quadratic Expression
- Quadratic Function
- Location of Roots
- Solving Inequalities Using Location of Roots
CONSTANT AND VARIABLES

In mathematics, a variable is a value that may change within the scope of a given problem or set of operations.

In contrast, a constant is a value that remains unchanged, though often unknown or undetermined.

Dependent and Independent Variables

Variables are further distinguished as being either a dependent variable or an independent variable. Independent variables are regarded as inputs to a system and may take on different values freely.

Dependent variables are those values that change as a consequence to changes in other values in the system.

When one value is completely determined by another, or of several others, then it is called a function of the other value or values. In this case the value of the function is a dependent variable and the other values are independent variables. The notation \( f(x) \) is used for the value of the function \( f \) with \( x \) representing the independent variable.

For example, \( y = f(x) = 3x^2 \), here we can take \( x \) as any real value, hence \( x \) is independent variable. But value of \( y \) depends on value of \( x \), hence \( y \) is dependent variable.

WHAT IS FUNCTION

To provide the classical understanding of functions, think of a function as kind of machine. You feed the machine raw materials, and the machine changes the raw materials into a finished product based on a specific set of instructions. The kinds of functions we consider here, for the most part, take in a real number, change it in a formulaic way, and give out a real number (possibly the same as the one it took in). Think of this as an input-output machine; you give the function an input, and it gives you an output.

![Fig. 1.1](image)

For example, the squaring function takes the input 4 and gives the output value 16. The same squaring function takes the input 1 and gives the output value 1.

A function is always defined as "of a variable" which tells us what to replace in the formula for the function.

For example, \( f(x) = 3x + 2 \) tells us:

- The function \( f \) is a function of \( x \).
- To evaluate the function at a certain number, replace the \( x \) with that number.

- Replacing \( x \) with that number in the right side of the function will produce the function's output for that certain input.
- In English, the above definition of \( f \) is interpreted, "Given a number, \( f \) will return two more than the triple of that number."

Thus, if we want to know the value (or output) of the function at 3:

\[
f(3) = 3(3) + 2 = 11
\]

Thus, the value of \( f \) at 3 is 11.

Note that \( f(3) \) means the value of the dependent variable when \( x \) takes on the value of 3. So we see that the number 11 is the output of the function when we give the number 3 as the input. We refer to the input as the argument of the function (or the independent variable), and to the output as the value of the function at the given argument (or the dependent variable). A good way to think of it is the dependent variable \( f(x) \) depends on the value of the independent variable \( x \).

The formal definition of a function states that a function is actually a rule that associates elements of one set called the domain of the function with the elements of another set called the range of the function. For each value, we select from the domain of the function, there exists exactly one corresponding element in the range of the function. The definition of the function tells us which element in the range corresponds to the element we picked from the domain. Classically, the element picked from the domain is pictured as something that is fed into the function and the corresponding element in the range is pictured as the output. Since we "pick" the element in the domain whose corresponding element in the range we want to find, we have control over what element we pick and hence this element is also known as the "independent variable". The element mapped in the range is beyond our control and is "mapped to" by the function. This element is hence also known as the "dependent variable", for it depends on which independent variable we pick. Since the elementary idea of functions is better understood from the classical viewpoint, we shall use it hereafter. However, it is still important to remember the correct definition of functions at all times.

To make it simple, for the function \( f(x) \), all of the possible \( x \) values constitute the domain, and all of the values \( f(x) \) (\( y \) on the \( x-y \) plane) constitute the range.

**Example 1.1** A function is defined as \( f(x) = x^2 - 3x \).

(i) Find the value of \( f(2) \).

(ii) Find the value of \( x \) for which \( f(x) = 4 \).

**Sol.**

(i) \( f(2) = (2)^2 - 3(2) = -2 \)

(ii) \( f(x) = 4 \)

\[
\Rightarrow x^2 - 3x = 4 \Rightarrow x^2 - 3x - 4 = 0
\]

\[
\Rightarrow (x - 4)(x + 1) = 0 \Rightarrow x = 4 \text{ or } -1
\]

This means \( f(4) = 4 \) and \( f(-1) = 4 \).
Example 1.2: If \( f \) is linear function and \( f(2) = 4, f(-1) = 3 \), then find \( f(x) \).

Sol. Let a linear function is \( f(x) = ax + b \)

Given \( f(2) = 4 \Rightarrow 2a + b = 4 \)  \( (1) \)

Also \( f(-1) = 3 \Rightarrow -a + b = 3 \)  \( (2) \)

Solving \((1)\) and \((2)\) we get \( a = \frac{1}{3} \) and \( b = \frac{10}{3} \)

Hence, \( f(x) = \frac{x+10}{3} \)

Example 1.3: A function is defined as \( f(x) = \frac{x^2+1}{3x-2} \). Can \( f(x) \) take a value 1 for any real \( x \)?

Also find the value/values of \( x \) for which \( f(x) \) takes the value 2.

Sol. Here \( f(x) = \frac{x^2+1}{3x-2} = 1 \)

\( \Rightarrow x^2 + 1 = 3x - 2 \)
\( \Rightarrow x^2 - 3x + 3 = 0 \)

Now this equation has no real roots as \( D < 0 \).

Hence, value of \( f(x) \) cannot be 1 for any real \( x \).

For \( f(x) = 2 \) we have \( \frac{x^2+1}{3x-2} = 2 \)

or \( x^2 + 1 = 6x - 4 \)

or \( x^2 - 6x + 5 = 0 \)

or \( (x-1)(x-5) = 0 \)

or \( x = 1, 5 \)

Example 1.4: Find the values of \( x \) for which the following functions are defined. Also find all possible values which functions take.

(i) \( f(x) = \frac{1}{x+1} \)

(ii) \( f(x) = \frac{x-2}{x-3} \)

(iii) \( f(x) = \frac{2x}{x-1} \)

Sol.

(i) \( f(x) = \frac{1}{x+1} \) is defined for all real values of \( x \) except when \( x + 1 = 0 \)

Hence, \( f(x) \) is defined for \( x \in R \setminus \{-1\} \).

Let \( y = \frac{1}{x+1} \)

Here we cannot find any real \( x \) for which \( y = \frac{1}{x+1} = 0 \)

For \( y \) other than ‘0’, there exists a real number \( x \).

Hence, \( \frac{1}{x+1} \in R \setminus \{0\} \).

(ii) \( f(x) = \frac{x-2}{x-3} \) is defined for all real values of \( x \) except when \( x - 3 = 0 \).

Hence, \( f(x) \) is defined for \( x \in R \setminus \{3\} \).

Let \( y = \frac{x-2}{x-3} \)

Example 1.5: If \( f(x) = \begin{cases} x^2, & x < 0 \\ 3x-2, & 0 \leq x \leq 2 \\ x^2 + 1, & x > 2 \end{cases} \), then find the value of \( f(-1) + f(1) + f(2) \).

Also find the value/values of \( x \) for which \( f(x) = 2 \).

Sol. Here function is differently defined for three different intervals mentioned.

For \( x = -1 \), consider \( f(x) = x^2 \)
\( \Rightarrow f(-1) = 1 \)

For \( x = 1 \), consider \( f(x) = 3x - 2 \)
\( \Rightarrow f(1) = 1 \)

For \( x = 3 \), consider \( f(x) = x^2 + 1 \)
\( \Rightarrow f(3) = 10 \)

Also when \( f(x) = 2 \),
for \( x^2 = 2 \), \( x = \pm \sqrt{2} \), which is not possible as \( x < 0 \).
for \( 3x - 2 = 2 \), \( x = 4/3 \), which is possible as \( 0 \leq x \leq 2 \).
for \( x^2 + 1 = 2 \), \( x = \pm 1 \), which is not possible as \( x > 2 \).

Hence, for \( f(x) = 2 \), we have \( x = 4/3 \).

INTERVALS

The set of numbers between any two real numbers is called interval. The following are the types of interval.
1.4 Algebra

Closed Interval
\[ x \in [a, b] = \{x : a \leq x \leq b\} \]

Open Interval
\[ x \in (a, b) \text{ or } [a, b] = \{x : a < x < b\} \]

Semi-Open or Semi Closed Interval
\[ x \in [a, b] \text{ or } (a, b] = \{x : a \leq x < b\} \]

\[ x \in ]a, b] \text{ or } (a, b] = \{x : a < x \leq b\} \]

Note:
- A set of all real numbers can be expressed as \((-\infty, \infty)\)
- \(x \in (-\infty, a) \cup (b, \infty) \Rightarrow x \in R - [a, b]\)
- \(x \in (-\infty, a] \cup [b, \infty) \Rightarrow x \in R - (a, b)\)

INEQUALITIES

Some Important Facts about Inequalities

The following are some very useful points to remember:

(i) \(a \leq b\) either \(a < b\) or \(a = b\)

(ii) \(a < b\) and \(b < c\) \(\Rightarrow a < c\) (transitive property)

(iii) \(a < b\) \(\Rightarrow -a > -b\); i.e., inequality sign reverses if both sides are multiplied by a negative number.

(iv) \(a < b\) and \(c < d\) \(\Rightarrow a + c < b + d\) and \(a - d < b - c\).

(v) If both sides of inequality are multiplied (or divided) by a positive number, inequality does not change. When both of its sides are multiplied (or divided) by a negative number, inequality gets reversed.

i.e., \(a < b \Rightarrow ka < kb\) if \(k > 0\) and \(ka > kb\) if \(k < 0\)

(vi) \(0 < a < b\) \(\Rightarrow a' < b'\) if \(r > 0\) and \(a' > b'\) if \(r < 0\)

(vii) \(a + \frac{1}{a} \geq 2\) for \(a > 0\) and equality holds for \(a = 1\)

(viii) \(a + \frac{1}{a} \leq -2\) for \(a < 0\) and equality holds for \(a = -1\)

(ix) Squaring an inequality:
If \(a < b\), then \(a^2 < b^2\) does not follow always:
Consider the following illustrations:
\[ 2 < 3 \Rightarrow 4 < 9, \text{ but } -4 < 3 \Rightarrow 16 > 9 \]
Also if \(x > 2\) \(\Rightarrow x^2 > 4\), but for \(x < 2\) \(\Rightarrow x^2 \geq 0\)
If \(2 < x < 4\) \(\Rightarrow 4 < x^2 < 16\)
If \(-2 < x < 4\) \(\Rightarrow 0 \leq x^2 < 16\)

If \(-5 < x < 4\) \(\Rightarrow 0 \leq x^2 < 25\)

In fact \(a < b\) \(\Rightarrow a^2 < b^2\) follows only when absolute value of \(a\) is less than the absolute value of \(b\) or distance of \(a\) from zero is less than the distance of \(b\) from zero on real number line.

(x) Law of reciprocal:
If both sides of inequality have same sign, while taking its reciprocal the sign of inequality gets reversed. i.e., \(a > b > 0\) \(\Rightarrow \frac{1}{a} < \frac{1}{b}\) and \(a < b < 0\) \(\Rightarrow \frac{1}{a} > \frac{1}{b}\)

But if both sides of inequality have opposite sign, then while taking reciprocal sign of inequality does not change, i.e.
\[ a < 0 < b \Rightarrow \frac{1}{a} > \frac{1}{b} \]
If \(x \in [a, b]\) \(\Rightarrow \frac{1}{x} < \frac{1}{a}, \frac{1}{x} > \frac{1}{b}\), if \(a\) and \(b\) have same sign
If \(x \in (a, b]\) \(\Rightarrow \frac{1}{x} > \frac{1}{a}, \frac{1}{x} < \frac{1}{b}\), if \(a\) and \(b\) have opposite signs

Example 1.6
Find the values of \(x^2\) for the given values of \(x\).

(i) \(x < 2\)  (ii) \(x > -1\)  (iii) \(x \geq 2\)  (iv) \(x < -1\)

Sol.
(i) \(x < 2\) \(\Rightarrow x \in (-\infty, 0) \cup [0, 2)\)
For \(x \in [0, 2)\), \(x^2 \in [0, 4)\)
For \(x \in (-\infty, 0)\), \(x^2 \in (0, \infty)\)
\(\Rightarrow x \in [0, 2) \cup (-\infty, 0)\)
\(\Rightarrow x \in (-\infty, 0) \cup [0, 2)\)

(ii) \(x > -1\) \(\Rightarrow x \in (-1, 0) \cup [0, \infty)\)
For \(x \in (-1, 0)\), \(x^2 \in (0, 1)\)
For \(x \in [0, \infty)\), \(x^2 \in [0, \infty)\)
\(\Rightarrow x \in [0, \infty) \cup (-1, 0)\)
\(\Rightarrow x \in (-1, 0) \cup [0, \infty)\)

(iii) \(x \in [2, \infty)\)
\(\Rightarrow x^2 \in [4, \infty)\)

(iv) \(x \in (-\infty, -1)\)
\(\Rightarrow x^2 \in (1, \infty)\)

Example 1.7
Find the values of \(1/x\) for the given values of \(x\).

(i) \(x > 3\)  (ii) \(x \leq -2\)  (iii) \(x \in (-1, 3) - \{0\}\)

Sol.
(i) \(x > 3\) \(\Rightarrow x \in \left(\frac{1}{3}, \infty\right)\)
\(\Rightarrow 1/x \to \infty\)
\(\Rightarrow 0 < 1/x < \infty\)

(ii) \(x \leq -2\)
\( \Rightarrow \quad \frac{1}{x^2 + 2} \geq \frac{1}{x^2 - 2} \geq \frac{1}{x - 2} \)
\( \Rightarrow \quad \frac{1}{x^2 + 2} \geq \frac{1}{x^2 - 2} \geq \frac{1}{x - 2} \)
\( \Rightarrow \quad 0 > \frac{1}{x} > \frac{1}{2} \)

(iii) \( x \in (-1, 3) \)
\( \Rightarrow \quad x \in (-1, 0) \cup (0, 3) \)
For \( x \in (-1, 0) \)
\( \frac{1}{x} > -1 \quad \Rightarrow \quad -1 > \frac{1}{x} \quad \Rightarrow \quad \infty < x < 0 \)
(here \( \rightarrow 0^+ \) means value of \( x \) approaches 0 from its left hand side or negative side)
\( \Rightarrow \quad x < \frac{1}{3} \quad \Rightarrow \quad \frac{1}{x} > \frac{1}{3} \)
For \( x \in (0, 3) \)
\( \frac{1}{x} > 0 \quad \Rightarrow \quad 0 < \frac{1}{x} < \frac{1}{3} \)
(here \( \rightarrow 0^+ \) means value of \( x \) approaches 0 from its right hand side or positive side)
\( \Rightarrow \quad \frac{1}{x} > \frac{1}{3} \)
\( \Rightarrow \quad \frac{1}{3} < x < \infty \)

From (1) and (2), \( \frac{1}{x} \in (-\infty, -1) \cup \left(\frac{1}{3}, \infty\right) \)

\( \frac{1}{x^2 - x - 1} = \frac{1}{x^2 - \frac{5}{4}x - \frac{5}{4}} \geq 0, \forall x \in R \)
\( \Rightarrow \quad \left(\frac{x - \frac{1}{2}}{2}\right)^2 - \frac{5}{4} \geq 0 \)
\( \Rightarrow \quad \left(\frac{x - \frac{1}{2}}{2}\right) \geq \frac{\sqrt{5}}{2} \quad \Rightarrow \quad x \geq \frac{1}{2} \quad \Rightarrow \quad \frac{1}{x} \in \left[0, \frac{1}{2}\right] \)

Example 1.9 Find all possible values of the following expressions:
(i) \( \sqrt{x^2 - 4} \)
(ii) \( \sqrt{9 - x^2} \)
(iii) \( \sqrt{x^2 - 2x + 10} \)

Sol.
(i) \( \sqrt{x^2 - 4} \)
Least value of square root is 0 when \( x^2 = 4 \) or \( x = \pm 2 \). Also \( x^2 - 4 \geq 0 \)
Hence, \( \sqrt{x^2 - 4} \in [0, \infty) \).

(ii) \( \sqrt{9 - x^2} \)
Least value of square root is 0 when \( 9 - x^2 = 0 \) or \( x = \pm 3 \).
Also, the greatest value of \( 9 - x^2 \) is 9 when \( x = 0 \).
Hence, we have \( 0 \leq x^2 - 9 \leq 0 = \sqrt{9 - x^2} \in [0, 3] \).

(iii) \( \sqrt{x^2 - 2x + 10} = \sqrt{(x-1)^2 + 9} \)
Here, the least value of \( \sqrt{(x-1)^2 + 9} \) is 3 when \( x = 1 \).
Also \( (x-1)^2 + 9 \geq 9 \Rightarrow \sqrt{(x-1)^2 + 9} \geq 3 \)
Hence, \( \sqrt{x^2 - 2x + 10} \in [3, \infty) \).
1.6 Algebra

**GENERALIZED METHOD OF INTERVALS FOR SOLVING INEQUALITIES**

Let \( F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2} \cdots (x-a_n)^{k_n} (x-a)^{k_n} \)

where \( k_1, k_2, \ldots, k_n \in \mathbb{Z} \) and \( a, a_1, a_2, \ldots, a_n \) are fixed real numbers satisfying the condition

\[
a_1 < a_2 < a_3 < \cdots < a_n < a
\]

For solving \( F(x) > 0 \) or \( F(x) < 0 \), consider the following algorithm:

- We mark the numbers \( a, a_1, a_2, \ldots, a_n \) on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e., on the right of \( a \).
- Then we put the plus sign in the interval on the left of \( a \) if \( k \) is an even number and the minus sign if \( k \) is an odd number. In the next interval, we put a sign according to the following rule:
  - When passing through the point \( a \), the polynomial \( F(x) \) changes sign if \( k \) is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality \( F(x) > 0 \) is the union of all intervals in which we have put the plus sign and the solution of the inequality \( F(x) < 0 \) is the union of all intervals in which we have put the minus sign.

**Frequently used inequalities**

(i) \( (x-a)(x-b) < 0 \Rightarrow x \in (a, b) \), where \( a < b \)
(ii) \( (x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty) \), where \( a < b \)
(iii) \( x^2 \geq a^2 \Rightarrow x \in [-a, a] \)
(iv) \( x^2 \leq a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty) \)
(v) \( a^2 + bx + c < 0 \), \( a > 0 \) \( \Rightarrow x \in (a, \beta) \), where \( \alpha, \beta (\alpha < \beta) \) are roots of the equation \( ax^2 + bx + c = 0 \)
(vi) \( a^2 + bx + c > 0 \), \( a > 0 \) \( \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty) \), where \( \alpha, \beta (\alpha < \beta) \) are roots of the equation \( ax^2 + bx + c = 0 \)

**Example 1.10**

Solve \( x^2 - x - 2 > 0 \).

**Sol.**

\[
x^2 - x - 2 > 0
\]

\[
\Rightarrow (x - 2)(x + 1) > 0
\]

Now \( x^2 - x - 2 = 0 \) \( \Rightarrow x = 1, 2 \).

Now on number line \( (x\text{-axis}) \) mark \( x = 1 \) and \( x = 2 \).

Now when \( x > 2 \), \( x + 1 > 0 \) and \( x - 2 > 0 \)

\[
\Rightarrow (x + 1)(x - 2) > 0
\]

When \( -1 < x < 2 \), \( x + 1 > 0 \) but \( x - 2 < 0 \)

\[
\Rightarrow (x + 1)(x - 2) < 0
\]

When \( x < -1, x + 1 < 0 \) and \( x - 2 < 0 \)

\[
\Rightarrow (x + 1)(x - 2) > 0
\]

Hence, sign scheme of \( x^2 - x - 2 \) is

[Diagram of sign scheme]

From the figure, \( x^2 - x - 2 > 0, x \in (-\infty, -1) \cup (2, \infty) \).

**Example 1.11**

Solve \( x^2 - x - 1 < 0 \).

**Sol.**

Let's first factorize \( x^2 - x - 1 \).

For that let \( x^2 - x - 1 = 0 \)

\[
\Rightarrow x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}
\]

Now on number line \( (x\text{-axis}) \) mark \( x = \frac{1 \pm \sqrt{5}}{2} \)

[Diagram of number line]

From the sign scheme of \( x^2 - x - 1 \) which shown in the given figure.

\[
x^2 - x - 1 < 0 \Rightarrow x \in \left(\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right)
\]

**Example 1.12**

Solve \( (x - 1)(x - 2)(1 - 2x) > 0 \).

**Sol.**

We have \( (x - 1)(x - 2)(1 - 2x) > 0 \)

or \( (x - 1)(x - 2)(2x - 1) > 0 \)

or \( (x - 1)(x - 2)(2x - 1) < 0 \)

On number line mark \( x = 1/2, 1, 2 \)

[Diagram of number line]

When \( x > 2 \), all factors \((x - 1), (2x - 1)\) and \((x - 2)\) are positive.

Hence, \((x - 1)(x - 2)(2x - 1) > 0\) for \( x > 2 \).

Now put positive and negative sign alternatively as shown in figure.

Hence, solution set of \((x - 1)(x - 2)(1 - 2x) > 0\) or \((x - 1)(x - 2)(2x - 1) < 0\) is \((-\infty, 1/2) \cup (1, 2)\).

**Example 1.13**

Solve \( (2x + 1)(x - 3)(x + 7) < 0 \).

**Sol.**

\((2x + 1)(x - 3)(x + 7) < 0\)

Sign scheme of \((2x + 1)(x - 3)(x + 7)\) is as follows:

[Diagram of sign scheme]

Hence, solution is \((-\infty, -7) \cup (-1/2, 3)\).

**Example 1.14**

Solve \( \frac{2}{x} < 3 \).

**Sol.**

\[
\frac{2}{x} < 3
\]

\[
\Rightarrow \frac{2}{x} - 3 < 0
\]

(We cannot cross multiply with \( x \) as \( x \) can be negative or positive)
Number System, Inequalities and Theory of Equations

Example 1.15 Solve \( \frac{2x - 3}{3x - 5} \geq 3. \)

Sol. \[
\frac{2x - 3}{3x - 5} \geq 3
\]
\[
\Rightarrow \frac{2x - 3}{3x - 5} - 3 \geq 0
\]
\[
\Rightarrow \frac{2x - 3 - 9x + 15}{3x - 5} \geq 0
\]
\[
\Rightarrow \frac{-7x + 12}{3x - 5} \geq 0
\]
\[
\Rightarrow \frac{7x - 12}{3x - 5} \leq 0
\]
Sign scheme of \( \frac{7x - 12}{3x - 5} \) is as follows:

\[
\begin{array}{c|c|c}
& + & - \\
0 & 5/3 & 12/7 \\
\end{array}
\]
\( x = 5/3 \) is not included in the solutions as at \( x = 5/3 \) denominator becomes zero.

Example 1.16 Solve \( x > \sqrt{(1-x)}. \)

Sol. Given inequality can be solved by squaring both sides.

But sometimes squaring gives extraneous solutions which do not satisfy the original inequality. Before squaring we must restrict \( x \) for which terms in the given inequality are well defined.

\[ x > \sqrt{(1-x)} \] Here \( x \) must be positive.

Here \( \sqrt{1-x} \) is defined only when \( 1-x \geq 0 \) or \( x \leq 1 \) \hspace{1cm} (1)

Squaring given inequality but sides \( x^2 > 1-x \)

\[
\Rightarrow x^2 + x - 1 > 0
\]
\[
\Rightarrow (x - \frac{-1 - \sqrt{5}}{2})(x - \frac{-1 + \sqrt{5}}{2}) > 0
\]
\[
\Rightarrow x < \frac{-1 - \sqrt{5}}{2} \text{ or } x > \frac{-1 + \sqrt{5}}{2}
\] \hspace{1cm} (2)

\[
\text{From (1) and (2), } x \in \left[ \frac{-1 + \sqrt{5}}{2} \right], \text{ (as } x \text{ is +ve)}
\]

Example 1.17 Solve \( \frac{2}{x^2 - x + 1} - \frac{1}{x + 1} \leq 0. \)

Sol. \[
\frac{2}{x^2 - x + 1} - \frac{1}{x + 1} \leq 0
\]
\[
\Rightarrow \frac{2}{x^2 - x + 1} - \frac{1}{x + 1} - \frac{1}{x + 1} \leq 0
\]
\[
\Rightarrow \frac{2(x+1) - (x^2 - x + 1) - (2x + 1)}{(x+1)(x^2 - x + 1)} \leq 0
\]
\[
\Rightarrow \frac{-x^2 - 2x + 1}{(x+1)(x^2 - x + 1)} \leq 0
\]
\[
\Rightarrow \frac{-x^2 - 2x + 1}{(x+1)(x^2 - x + 1)} \geq 0
\]
\[
\Rightarrow \frac{x^2 + 2x - 1}{(x+1)(x^2 - x + 1)} \geq 0
\]
\[
\Rightarrow \frac{2 - x}{x^2 - x + 1} \geq 0, \text{ where } x \neq 1
\]
\[
\Rightarrow 2 - x \geq 0, x \neq 1, \text{ (as } x^2 - x + 1 > 0 \text{ for } \forall x \in \mathbb{R})
\]
\[
\Rightarrow x \leq 2, x \neq 1
\]

Example 1.18 Solve \( x(x+2)^4(x-1)^3(2x-3)(x-3)^4 \geq 0. \)

Sol. Let \( E = x(x+2)^4(x-1)^3(2x-3)(x-3)^4. \)

Here for \( x, (x+2), (2x-3) \) exponents are odd, hence sign of \( E \) changes while crossing \( x = 0, 1, 3/2 \). Also for \( (x+2), (x-3) \) exponents are even, hence sign of \( E \) does not change while crossing \( x = -2 \) and \( x = 3. \)

Further for \( x > 3, \) all factors are positive, hence sign of the expression starts with positive sign from the right hand side.

The sign scheme of the expression is as shown in the following figure.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
& + & + & + & + & + & + & + & + & + \\
0 & 1 & 1/2 & 3/2 & 2 & 3 & \infty & \infty & \infty & \infty \\
\end{array}
\]

Hence, for \( E \geq 0, \) we have \( x \in [0, 1] \cup [3/2, \infty) \)

Example 1.19 Solve \( x(2x-1)(3x-9)(x-3) < 0. \)

Sol. Let \( E = x(2x-1)(3x-9)(x-3). \)

Here \( 2x - 1 = 0 \Rightarrow x = \frac{1}{2} \) and when \( 3x - 9 = 0 \Rightarrow x = 3 \)

Now mark \( x = 0, 2, 3 \) on real number line.

Sign of \( E \) starts with positive sign from right hand side.

Also at \( x = 0, \) two factors are \( 0, x \) and \( 2x - 1, \) hence sign of \( E \) does not change while crossing \( x = 0. \)

Sign scheme of \( E \) is as shown in the following figure.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
& + & + & + & + & + & + & + & + & + \\
0 & 1/2 & 3 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\end{array}
\]

From the figure, we have \( E < 0 \) for \( x \in (2, 3). \)
Example 1.20  Find all possible values of \( \frac{x^2 + 1}{x^2 - 2} \).

Sol. Let \( y = \frac{x^2 + 1}{x^2 - 2} \)

\[ yx^2 - 2y = x^2 + 1 \]
\[ x^2 = \frac{2y + 1}{y - 1} \]

Now \( x^2 \geq 0 \Rightarrow \frac{2y + 1}{y - 1} \geq 0 \)

Now \( x^2 \geq 0 \Rightarrow \frac{2y + 1}{y - 1} \geq 0 \)

\[ y \leq -1/2 \text{ or } y > 1 \]

Solving Irrational Inequalities

Example 1.21  Solve \( \sqrt{x - 2} \geq 1 \).

Sol. We must have \( x - 2 \geq 0 \) for \( \sqrt{x - 2} \) to get defined, thus \( x \geq 2 \).

Now \( \sqrt{x - 2} \geq 1 \), as square roots are always non-negative.

Hence, \( x \geq 2 \).

Note: Some students solve it by squaring both sides for which \( x - 2 \geq 1 \) or \( x \geq 3 \) which cause loss of interval \([2, 3)\).

Example 1.22  Solve \( \sqrt{x - 1} \geq \sqrt{3 - x} \).

Sol. \( \sqrt{x - 1} \geq \sqrt{3 - x} \) is meaningful if \( x - 1 \geq 0 \) and \( 3 - x \geq 0 \), or \( 1 \leq x \leq 3 \).

Also \( \sqrt{x - 1} > \sqrt{3 - x} \) if \( x > 2 \).

Squaring, we have \( x - 1 > 3 - x \)

\[ x > 2 \]

From (1) and (2), we have \( 2 < x \leq 3 \).

Example 1.23  Solve \( x + \sqrt{x} \geq \sqrt{x - 3} \).

Sol. \( x + \sqrt{x} \geq \sqrt{x - 3} \) is meaningful only when \( x \geq 0 \).

Now \( x + \sqrt{x} \geq \sqrt{x - 3} \)

\[ x \geq 3 \]

From (1) and (2), we have \( x \geq 0 \).

Example 1.24  Solve \( (x^2 - 4)\sqrt{x^2 - 1} < 0 \).

Sol. \( (x^2 - 4)\sqrt{x^2 - 1} < 0 \)

We must have \( x^2 - 1 > 0 \)

or \( (x - 1)(x + 1) \geq 0 \)

or \( x \geq 1 \) or \( x \leq -1 \)

Also \( (x^2 - 4)\sqrt{x^2 - 1} < 0 \)

\[ x^2 - 4 < 0 \]

\[ -2 < x < 2 \]

From (1) and (2), we have \( x \in (-2, -1) \cup (1, 2) \)

ABSOLUTE VALUE OF \( x \)

Absolute value of any real number \( x \) is denoted by \( |x| \) (read as modulus of \( x \)).

The absolute value is closely related to the notions of magnitude, distance, and norm in various mathematical and physical contexts.

From an analytic geometry point of view, the absolute value of a real number is that number's distance from zero along the real number line, and more generally the absolute value of the difference of two real numbers is the distance between them.

Let's look at the number line:

Fig. 1.14

The absolute value of \( x \), denoted \(|x|\) (and which is read as "the absolute value of \( x \)"), is the distance of \( x \) from zero. This is why absolute value is never negative; absolute value only asks "how far?", not "in which direction?". This means not only that \(|3| = 3\), because \( 3 \) is three units to the right of zero, but also that \(|-3| = 3\), because \(-3\) is three units to the left of zero.

When the number inside the absolute value (the "argument" of the absolute value) was positive anyway, we did not change the sign when we took the absolute value. But when the argument was negative, we did change the sign.

If \( x > 0 \) (that is, if \( x \) is positive), then the value would not change when you take the absolute value. For instance, if \( x = 2 \), then you have \(|2| = 2 = x\). In fact, for any positive value of \( x \) (or if \( x \) equals zero), the sign would be unchanged, so:

For \( x \geq 0 \), \(|x| = x\)

On the other hand, if \( x < 0 \) (that is, if \( x \) is negative), then it will change its sign when you take the absolute value. For instance, if \( x = -4 \), then \(|-4| = 4 = 4 = 4 = -x\). In fact, for any negative value of \( x \), the sign would have to be changed, so:

For \( x < 0 \), \(|x| = -x\)

Thus \(|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\)

Also \( \sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\)

i.e., \( 2 = \sqrt{2^2} = \sqrt{(-2)^2} = \left[(-2)^2\right]^{1/2} = -2 \) is absurd as \( \sqrt{x^2} = |x| \)

\[ \Rightarrow \sqrt{(-2)^2} = \left| -2 \right| = 2 \]

Thus square root exists only for non-negative numbers and its value is also non-negative.

Some students consider \( \sqrt{4} = \pm 2 \), which is wrong.

In fact \( \sqrt{(-4)^2} = |\sqrt{(-4)^2}| = 4 \)
\[ \sqrt{1 - \sqrt{2}} = 1 - \sqrt{2} \cdot \sqrt{1 - 2} = 1 - \sqrt{2} - 1 \text{ etc.} \]

Also, some students write \( \sqrt{x^2} = \pm x \) which is wrong, in fact,
\[ \sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \]

Also \( a^2 < b^2 \Rightarrow \sqrt{a^2} < \sqrt{b^2} \Rightarrow |a| < |b| \)

Graph of function \( f(x) = y = |x| \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>±1</th>
<th>±2</th>
<th>±3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

![Fig. 1.15](image)

We can see that graph of \( y = |x| \) is in 1st and 2nd quadrant only where \( y \geq 0 \), hence \( |x| \geq 0 \).

Example 1.25

Solve the following:

(i) \( |x| = 5 \)
(ii) \( x^2 - |x| - 2 = 0 \)

Sol.

(i) \( |x| = 5 \), i.e., those points on real number line which are at distance 5 units from ‘0’, which are -5 and 5.

Thus, \( |x| = 5 \Rightarrow x = \pm 5 \)

(ii) \( x^2 - |x| - 2 = 0 \)
\[ \Rightarrow |x|^2 - |x| - 2 = 0 \]
\[ \Rightarrow (|x| - 2)(|x| + 1) = 0 \]
\[ \Rightarrow |x| = 2 \quad (\because |x| + 1 \neq 0) \]
\[ \Rightarrow x = \pm 2 \]

Example 1.26

Find the value of \( x \) for which following expressions are defined:

(i) \( \frac{1}{\sqrt{x - |x|}} \)
(ii) \( \frac{1}{\sqrt{ax + |x|}} \)

Sol.

(i) \( x - |x| = \begin{cases} x - x = 0, \text{if } x \geq 0 \\ x + x = 2x, \text{if } x < 0 \end{cases} \)

\[ \Rightarrow x - |x| \leq 0 \text{ for all } x \]
\[ \Rightarrow \frac{1}{\sqrt{x - |x|}} \text{ does not take real values for any } x \in \mathbb{R} \]
\[ \Rightarrow \frac{1}{\sqrt{x + |x|}} \text{ is not defined only when } x \in \mathbb{R} \]

(ii) \[ x + |x| = \begin{cases} x + x = 2x, \text{if } x \geq 0 \\ x - x = 0, \text{if } x < 0 \end{cases} \]
\[ \Rightarrow \frac{1}{\sqrt{x + |x|}} \text{ is defined only when } x > 0 \]

What is the geometric meaning of \( |x - y| \)?
\( |x - y| \) is the distance between \( x \) and \( y \) on the real number line.

Example 1.27

Solve the following:

(i) \( |x - 2| = 1 \)
(ii) \( |2x + 1|^2 - |x + 1| = 3 \)

Sol.

(i) \( |x - 2| = 1 \), i.e., those points on real number line which are distance 1 units from 2.

\[ |x - 2| = 1 \]
\[ \Rightarrow x - 2 = \pm 1 \]
\[ \Rightarrow x = 1 \text{ or } x = 3 \]

(ii)
\[ 2|x + 1|^2 - |x + 1| = 3 \]
\[ \Rightarrow 2|x + 1|^2 - |x + 1| - 3 = 0 \]
\[ \Rightarrow 2|x + 1|^2 - 3|x + 1| + 2|x + 1| - 3 = 0 \]
\[ \Rightarrow (2|x + 1| - 3)(|x + 1| + 1) = 0 \]
\[ \Rightarrow 2|x + 1| - 3 = 0 \]
\[ \Rightarrow |x + 1| = 3/2 \]
\[ \Rightarrow x + 1 = \pm 3/2 \]
\[ \Rightarrow x = 1/2 \text{ or } x = -5/2 \]

In general, \( |f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases} \) where \( y = f(x) \) is any real-valued function.

Example 1.28

Solve the following:

(i) \( |x - 2| = (x - 2) \)
(ii) \( |x + 3| = -x - 3 \)
(iii) \( |x^2 - x| = x^2 - x \)
1.10 Algebra

(iv) \(|x^2 - x - 2| = 2 + x - x^2\)

Sol.
(i) \(|x - 2| = (x - 2), \text{if } x - 2 \geq 0 \text{ or } x \geq 2\)
(ii) \(|x + 3| = -x - 3, \text{if } x + 3 \leq 0 \text{ or } x \leq -3\)
(iii) \(|x^2 - x| = x^2 - x, \text{if } x^2 - x \geq 0\)
   \[x(x - 1) \geq 0\]
   \[x \in (-\infty, 0) \cup [1, \infty)\]
(iv) \(|x^2 - x - 2| = 2 + x - x^2\)
   \[x^2 - x - 2 \leq 0\]
   \[(x - 2)(x + 1) \leq 0\]
   \[-1 \leq x \leq 2\]

Example 1.29 Solve \(1 - x = \sqrt{x^2 - 2x + 1}\).

Sol.
\[1 - x = \sqrt{x^2 - 2x + 1}\]
\[\Rightarrow 1 - x = \sqrt{(x - 1)^2}\]
\[\Rightarrow 1 - x = x - 1\]
\[\Rightarrow 1 - x = 0\]
\[\Rightarrow x = 1\]

Example 1.30 Solve \(|3x - 2| = x\).

Sol. \(|3x - 2| = x\)
Case (i)
When \(3x - 2 \geq 0 \text{ or } x \geq \frac{2}{3}\)
For which we have \(3x - 2 = x \text{ or } x = 1\).
Case (ii)
When \(3x - 2 < 0 \text{ or } x < \frac{2}{3}\)
For which we have \(2 - 3x = x \text{ or } x = \frac{1}{2}\).
Hence, solution set is \(\{1/2, 1\}\).

Example 1.31 Solve \(|x| = x^2 - 1\).

Sol. \(x^2 - 1 = |x|\)
\[\Rightarrow x^2 - 1 = x \text{ when } x \geq 0\]
or \(x^2 - 1 = -x \text{ when } x < 0\)
\[x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2} \text{ (as } x \geq 0)\]
\[x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2} \text{ (as } x < 0)\]

Example 1.32 Solve
\[\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1\]

Sol. \(\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1\)
\[\Rightarrow \sqrt{x - 1 - 4\sqrt{x - 1} + 4} + \sqrt{x - 1 - 6\sqrt{x - 1} + 9} = 1\]
\[\Rightarrow \sqrt{(\sqrt{x - 1} - 2)^2} + \sqrt{(\sqrt{x - 1} - 3)^2} = 1\]
\[\Rightarrow |\sqrt{x - 1} - 2| + |\sqrt{x - 1} - 3| = 1\]

Example 1.33 Prove that
\[\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1} = 2x, \quad -1 \leq x \leq 1\]

Sol. \(\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1} = \sqrt{(x + 1)^2} - \sqrt{(x - 1)^2}\)
\[= |x + 1| - |x - 1|\]
\[= \begin{cases} -2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}\]

Example 1.34

(i) For \(2 < x < 4\), find the values of \(|x|\).
(ii) For \(-3 \leq x \leq -1\), find the values of \(|x|\).
(iii) For \(-3 \leq x < 1\), find the values of \(|x|\).
(iv) For \(-5 \leq x < 7\), find the values of \(|x - 2|\).
(v) For \(1 \leq x \leq 5\), find the values of \(|2x - 7|\).

Sol.
(i) \(2 < x < 4\)
Here values on real number line whose distance lies between 2 and 4.
Here values of \(x\) are positive \(\Rightarrow |x| \in (2, 4)\)
(ii) \(-3 \leq x \leq -1\)
Here values on real number line whose distance lies between 1 and 3 or at distance 1 or 3.
\(\Rightarrow 1 \leq |x| \leq 3\)
(iii) \(-3 \leq x < 1\)
For \(-3 \leq x < 0\), \(x \in (0, 3)\)
For \(0 \leq x < 1\), \(|x| \in [0, 1]\)
So for \(-3 \leq x < 1\), \(x \in [0, 1) \cup (0, 3)\) or \(|x| \in [0, 3]\)
(iv) \(-5 \leq x < 7\)
\(\Rightarrow -7 < x - 2 < 5\)
\(\Rightarrow 0 \leq |x - 2| < 7\)
Example 1.35: For \( x \in R \), find all possible values of
(i) \( |x - 3| - 2 \)
(ii) \( 4 - |2x + 3| \)

**Sol.**
(i) We know that \( |x - 3| \geq 0 \) \( \forall x \in R \)
\[ \Rightarrow |x - 3| - 2 \geq -2 \]
\[ \Rightarrow |x - 3| - 2 \in [-2, \infty) \]
(ii) We know that \( |2x + 3| \geq 0 \) \( \forall x \in R \)
\[ \Rightarrow -|2x + 3| \leq 0 \]
\[ \Rightarrow 4 - |2x + 3| \leq 4 \]
\[ \text{or} \quad 4 - |2x + 3| \in (-\infty, 4] \]

Example 1.36: Find all possible values of
(i) \( \sqrt{x^2 - 2} \)
(ii) \( \sqrt{3 - |x - 1|} \)
(iii) \( \sqrt{4 - \sqrt{x^2}} \)

**Sol.**
(i) \( \sqrt{x^2 - 2} \)
We know that square roots are defined for non-negative values only.
It implies that we must have \( |x| - 2 \geq 0 \).
\[ \Rightarrow |x| - 2 \geq 0 \]
(ii) \( \sqrt{3 - |x - 1|} \) is defined when \( 3 - |x - 1| \geq 0 \)
But the maximum value of \( 3 - |x - 1| \) is 3 when \( |x - 1| = 0 \).
Hence, for \( \sqrt{3 - |x - 1|} \) to get defined, \( 0 \leq 3 - |x - 1| \leq 3 \).
\[ \Rightarrow 3 - |x - 1| \in [0, 3] \]
Alternatively, \( |x - 1| \geq 0 \)
\[ \Rightarrow -|x - 1| \leq 0 \]
\[ \Rightarrow |x - 1| \leq 0 \]
\[ \Rightarrow |x - 1| \leq 3 \]
But for \( \sqrt{3 - |x - 1|} \) to get defined, we must have
\[ 0 \leq 3 - |x - 1| \leq 3 \Rightarrow 0 \leq |x - 1| \leq \sqrt{3} \]
(iii) \( \sqrt{4 - \sqrt{x^2}} = \sqrt{4 - x^2} \)
\[ |x| \geq 0 \]
\[ \Rightarrow -|x| \leq 0 \]
\[ \Rightarrow 4 - |x| \leq 4 \]
But for \( \sqrt{4 - x^2} \) to get defined \( 0 \leq 4 - |x| \leq 4 \)
\[ \Rightarrow 0 \leq |x| \leq 2 \]

Example 1.37: Solve \( |x - 3| + |x - 2| = 1 \).

**Sol.**
\[ |x - 3| + |x - 2| = 1 \]
\[ \Rightarrow |x - 3| + |x - 2| = (3 - x) + (x - 2) \]
\[ \Rightarrow x - 3 \leq 0 \text{ and } x - 2 \geq 0 \]
\[ \Rightarrow x \leq 3 \text{ and } x \geq 2 \]
\[ \Rightarrow 2 \leq x \leq 3 \]

Example 1.38: Solve \( x^2 - 4|x| + 3 < 0 \).

**Sol.**
\[ x^2 - 4|x| + 3 < 0 \]
\[ \Rightarrow (|x| - 1)(|x| - 3) < 0 \]
\[ \Rightarrow 1 < |x| < 3 \]
\[ \Rightarrow -3 < x < -1 \text{ or } 1 < x < 3 \]
\[ \Rightarrow x \in (-3, -1) \cup (1, 3) \]

Example 1.39: Solve \( 0 < |x| < 2 \).

**Sol.** We know that \( |x| \geq 0 \), \( \forall x \in R \)
But given \( |x| > 0 \) \( \Rightarrow x \neq 0 \)
Now \( 0 < |x| < 2 \)
\[ \Rightarrow x \in (-2, 2), x \neq 0 \]

### Inequalities Involving Absolute Value

(i) \( |x| \leq a \) (where \( a > 0 \))
It implies those values of \( x \) on real number line which are
at distance \( a \) or less than \( a \).

![Fig. 1.17](image)

\[ \Rightarrow -a \leq x \leq a \]
e.g., \( |x| \leq 2 \Rightarrow -2 \leq x \leq 2 \)
\[ |x| < 3 \Rightarrow -3 < x < 3 \]
In general, \( |f(x)| \leq a \) (where \( a > 0 \)) \( \Rightarrow -a \leq f(x) \leq a \).

(ii) \( |x| \geq a \) (where \( a > 0 \))
It implies those values of \( x \) on real number line which are
at distance \( a \) or more than \( a \).

![Fig. 1.18](image)

\[ \Rightarrow x \leq -a \text{ or } x \geq a \]
e.g., \( |x| \geq 3 \Rightarrow x \leq -3 \text{ or } x \geq 3 \).
\[ |x| > 2 \Rightarrow x < -2 \text{ or } x > 2 \]
In general, \( |f(x)| > a \) \( \Rightarrow f(x) \leq -a \text{ or } f(x) \geq a \).

(iii) \( a \leq |x| \leq b \) (where \( a, b > 0 \))
It implies those value of \( x \) on real number line which are
at distance equal \( a \) or \( b \) or between \( a \) and \( b \).

![Fig. 1.19](image)

\[ \Rightarrow [-b, -a] \cup [a, b] \]
e.g., \( 2 \leq |x| \leq 4 \Rightarrow x \in [-4, -2] \cup [2, 4] \)

(iv) \( |x + y| < |x| + |y| \) if \( x \) and \( y \) have opposite signs.
\[ |x - y| < |x| + |y| \) if \( x \) and \( y \) have same sign.
\[ |x + y| = |x| + |y| \) if \( x \) and \( y \) have same sign or at least one of \( x \) and \( y \) are zero.
\[ |x - y| = |x| + |y| \) if \( x \) and \( y \) have opposite signs or at least one of \( x \) and \( y \) are zero.
Example 1.40  Solve $|3x - 2| < 4$.

Sol.  $|3x - 2| < 4$
$\Rightarrow -4 < 3x - 2 < 4$
$\Rightarrow -2 < 3x < 6$
$\Rightarrow -2/3 < x < 2$

Example 1.41  Solve $1 \leq |x - 2| \leq 3$.

Sol.  $1 \leq |x - 2| \leq 3$
$\Rightarrow -3 \leq x - 2 \leq -1$ or $1 \leq x - 2 \leq 3$
$\Rightarrow -1 \leq x - 1 \leq 3$ or $3 \leq x \leq 5$
$\Rightarrow x \in [-1, 1] \cup [3, 5]$

Example 1.42  Solve $0 < |x - 3| \leq 5$.

Sol.  $0 < |x - 3| \leq 5$
$\Rightarrow -5 < x - 3 < 0$ or $0 < x - 3 \leq 5$
$\Rightarrow -2 < x < 3$ or $3 < x \leq 8$
$\Rightarrow x \in [-2, 3) \cup (3, 8]$

Example 1.43  Solve $|x - 1| - 2 < 5$.

Sol.  $|x - 1| - 2 < 5$
$\Rightarrow -5 < |x - 1| - 2 < 5$
$\Rightarrow -3 < |x - 1| < 7$
$\Rightarrow |x - 1| < 7$
$\Rightarrow -7 < x - 1 < 7$
$\Rightarrow -6 < x < 8$

Example 1.44  Solve $|x - 3| \geq 2$.

Sol.  $|x - 3| \geq 2$
$\Rightarrow x - 3 \leq -2$ or $x - 3 \geq 2$
$\Rightarrow x \leq 1$ or $x \geq 5$

Example 1.45  Solve $|x| - 3 > 1$.

Sol.  $|x| - 3 > 1$
$\Rightarrow |x| > 4$
$\Rightarrow x > 4$ or $x < -4$

Example 1.46  Solve $|x - 1| + |x - 2| \geq 4$.

Sol.  Let $f(x) = |x - 1| + |x - 2|$

<table>
<thead>
<tr>
<th>A</th>
<th>B, $f(x)$</th>
<th>C, $f(x) \geq 4$</th>
<th>D, $A \cap C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; 1$</td>
<td>$1 - x + 2 - x$</td>
<td>$3 - 2x \geq 4 \Rightarrow x \leq -1/2$</td>
<td>$x \leq -1/2$</td>
</tr>
<tr>
<td>$1 \leq x \leq 2$</td>
<td>$x - 1 + 2 - x$</td>
<td>$1 \geq 4$, not possible</td>
<td></td>
</tr>
<tr>
<td>$x &gt; 2$</td>
<td>$x - 1 + x - 2$</td>
<td>$2x - 3 \geq 4 \Rightarrow x \geq 7/2$</td>
<td>$x \geq 7/2$</td>
</tr>
</tbody>
</table>

Hence, solutions is $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Example 1.47  Solve $|x + 1| + |2x - 3| = 4$.

Sol.  Let $f(x) = |x + 1| + |2x - 3|$

<table>
<thead>
<tr>
<th>A, $x &lt; -1$</th>
<th>B, $f(x)$</th>
<th>C, $f(x) \geq 4$</th>
<th>D, $A \cap C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -1$</td>
<td>$- 1 - x + 3 - 2x$</td>
<td>$2 - 3x \geq 4$</td>
<td>$\Rightarrow -1 - x = -1/2$</td>
</tr>
<tr>
<td>$-1 \leq x \leq 3/2$</td>
<td>$x + 1 + 3 - 2x$</td>
<td>$4 - x \geq 4$</td>
<td>$\Rightarrow x = 0$</td>
</tr>
<tr>
<td>$x &gt; 3/2$</td>
<td>$x + 1 + 2x - 3$</td>
<td>$3x - 2 \geq 4$</td>
<td>$\Rightarrow x = 2$</td>
</tr>
</tbody>
</table>

Hence, solutions set is $\{0, 2\}$.

Example 1.48  Solve $|x| + |x - 2| = 2$.

Sol.  We have $|x| + |x - 2| = 2$
$\Rightarrow |x| + |x - 2| = -x + (x - 2)$
$\Rightarrow 0 \leq x \leq 2$

Example 1.49  Solve $|2x - 3| + |x - 1| = |x - 2|$.

Sol.  $|2x - 3| + |x - 1| = |2x - 3| - (x - 1)|$
$\Rightarrow (2x - 3) - (x - 1) \leq 0$
$\Rightarrow 1 \leq x \leq 3/2$

Example 1.50  Solve $|x^2 + x - 4| = |x^2 - 4| + |x|$.

Sol.  $|x^2 + x - 4| = |x^2 - 4| + |x|$
$\Rightarrow x(x^2 - 4) \geq 0$
$\Rightarrow x(x + 2)(x - 2) \geq 0$
$\Rightarrow x \in [-2, 0] \cup (2, \infty)$

Example 1.51  If $|\sin x + \cos x| = |\sin x| + |\cos x|$ (since $\sin x, \cos x \neq 0$), then in which quadrant does $x$ lie?

Sol.  Here we have $|\sin x + \cos x| = |\sin x| + |\cos x|$. It implies that $\sin x$ and $\cos x$ must have the same sign. Therefore, $x$ lies in the first or third quadrant.

Example 1.52  Is $|\tan x + \cot x| < |\tan x| + |\cot x|$ true for any $x$? If it is true, then find the values of $x$.

Sol.  Since $\tan x$ and $\cot x$ have always the same sign, $|\tan x + \cot x| < |\tan x| + |\cot x|$ does not hold true for any value of $x$.

Example 1.53  Solve $\left|\frac{x - 3}{x + 1}\right| \leq 1$.

Sol.  $\left|\frac{x - 3}{x + 1}\right| \leq 1$
$\Rightarrow -1 \leq \frac{x - 3}{x + 1} \leq 1$
$\Rightarrow \frac{x - 3}{x + 1} \leq 0$ and $0 \leq \frac{x - 3}{x + 1}$
$\Rightarrow \frac{-4}{x + 1} \leq 0$ and $0 \leq \frac{2x - 2}{x + 1}$
Example 1.54  Solve $|x^2 - 2x| + |x - 4| > |x^2 - 3x + 4|.$

**Sol.** We have $|x^2 - 2x| + |x - 4| > |x^2 - 3x + 4|$
\[
\Rightarrow (x^2 - 2x)(4 - x) < 0
\]
\[
\Rightarrow x (x - 2)(x - 4) > 0
\]
\[
\Rightarrow x \in (0, 2) \cup (4, \infty)
\]

Concept Application Exercise 1.1

1. If $f(x) = x^2, 1 \leq x \leq 3$, then which of the following is the greatest?
   \[
   2 - 3x, x > 3
   \]
   \[f(0), f(3), f(4), f(2)\]

2. If $f(x)$ is a quadratic function such that $f(0) = -4, f(1) = -5$ and $f(-1) = -1$, then find the value of $f(3)$.

3. Find the value of $x^2$ for the following values of $x$:
   (i) $[-5, -1]$  (ii) $(3, 6)$
   (iii) $(-2, 3)$  (iv) $(-3, \infty)$  (v) $(\infty, 4)$

4. Find the values of $\frac{1}{x}$ for the following values of $x$:
   (i) $(2, 5)$  (ii) $[-5, -1]$  (iii) $(3, \infty)$  (iv) $(-\infty, -2)$
   (v) $[-3, 4]$

5. Which of the following is always true?
   (a) If $a > b$, then $a^2 > b^2$
   (b) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$
   (c) If $a < b$, then $|a| < |b|$

6. Find the values of $x$ which satisfy the inequalities simultaneously:
   (i) $-3 < 2x - 1 < 9$  (ii) $-1 \leq \frac{2x + 3}{5} \leq 3$

7. Find all the possible values which the following expressions take.
   (i) $2 - 5x$
   (ii) $\sqrt{x^2 - 7x + 5}$
   (iii) $x^2 - 6x$

8. Solve $x(3 - 4x)(x + 1) < 0$.

9. Solve $\frac{(x^2 + 3)(x^2 - 3x^2)(x^2 - 4)}{(x - 2)^2 x^3} \leq 0$.

10. Solve $\frac{(x - 3)(x + 5)(x - 7)}{|x - 4| (x + 6)} \leq 0$.

11. Find all possible values of $f(x) = \frac{1 - x^2}{x^2 + 3}$.

12. Solve $\frac{2x}{2x^2 + 5x + 2} > \frac{1}{x + 1}$.

13. Solve (i) $\frac{x - 1}{x - 2} < 0$  (ii) $\frac{x - 2}{x - 3} \leq 3$

14. Which of the following equations has maximum number of real roots?
   (i) $x^2 - |x| - 2 = 0$
   (ii) $x^2 - 2|x| + 3 = 0$
   (iii) $x^2 - |x| - 2 = 0$
   (iv) $x^2 - 3|x| + 2 = 0$

15. Find the number of solutions of the system of equation $x + 2y = 6$ and $|x - 3| = y$.

16. Find the values of $x$ for which $f(x) = \sqrt{1 - 5(1-x)}$ is defined.

17. Find all values of $x$ for which $f(x) = x + \sqrt{3}$.

18. Solve $|\frac{x + 2}{x - 1}| = 2$.

19. If $|x + 7| \leq 9$, then find the values of $x$.

20. Find the values of $x$ for which $\sqrt{x^2 - 12x - 3} = 5$.

21. Solve $|x - 2| > x < 3$.

22. Which of the following is/are true?
   (a) If $|x + y| = |x| + |y|$, then points $(x, y)$ lie in 1st or 3rd quadrant of any of the x-axis or y-axis.
   (b) If $|x + y| < |x| + |y|$, then points $(x, y)$ lie in 2nd or 4th quadrant.
   (c) If $|x - y| = |x| + |y|$, then points $(x, y)$ lie in 2nd or 4th quadrant.

23. Solve $|x^2 - 2x - 4| + |x + 6| = |x^2 + 2x - 8|$.

24. Solve $|x| = 2x - 1$.

25. Solve $|2x - 1| + |2x + 1| = 2$.

26. Solve $|x^2 - 4x + 3| = x + 1$.

27. Solve $|x^2 - 1| + |x^2 - 4| > 3$.

28. Solve $|x - 1| - |2x - 5| = 2x$.

**SOME DEFINITIONS**

**Real Polynomial**

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and $x$ is a real variable. Then, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a real polynomial of real variable $x$ with real coefficients.

**Complex Polynomial**

If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and $x$ is a varying complex number, then $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a complex polynomial or a polynomial of complex coefficients.

**Rational Expression or Rational Function**

An expression of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials in $x$ is called a rational expression.
In the particular case when \( Q(x) \) is a non-zero constant,
\[
P(x) \quad Q(x)
\]
reduces to a polynomial. Thus every polynomial is a rational expression but the converse is not true. Some of the examples are as follows:

1. \[ \frac{x^2 + x + 4}{x - 2} \]
2. \[ x^2 - 5x + 4 \]
3. \[ \frac{1}{x} \]
4. \[ \frac{x^2 + 1}{x} \]

**Degree of a Polynomial**

A polynomial \( f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \), real or complex, is a polynomial of degree \( n \), if \( a_n \neq 0 \).

The polynomials \( 2x^3 - 7x^2 + 5x + 3 \) and \( (3 - 2i)x^2 - ix + 5 \) are polynomials of degree 3 and 2, respectively.

A polynomial of second degree is generally called a quadratic polynomial, and polynomials of degree 3 and 4 are known as cubic and bi-quadratic polynomials, respectively.

**Polynomial Equation**

If \( f(x) \) is a polynomial, then \( f(x) = 0 \) is called a polynomial equation.

If \( f(x) \) is a quadratic polynomial, then \( f(x) = 0 \) is called a quadratic equation. The general form of a quadratic equation is \( ax^2 + bx + c = 0 \), \( a \neq 0 \). Here, \( x \) is the variable and \( a, b, \) and \( c \) are called coefficients, real or imaginary.

**Roots of an Equation**

The values of the variable satisfying a given equation are called its roots.

Thus, \( x = a \) is a root of the equation \( f(x) = 0 \), if \( f(a) = 0 \).

For example, \( x = 1 \) is a root of the equation \( x^3 - 6x^2 + 11x - 6 = 0 \), because \( 1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0 \).

Similarly, \( x = \omega \) and \( x = \omega^2 \) are roots of the equation \( x^2 + x + 1 = 0 \) as they satisfy it (where \( \omega \) is the complex cube root of unity).

**Solution Set**

The set of all roots of an equation, in a given domain, is called the solution set of the equation.

For example, the set \( \{1, 2, 3\} \) is the solution set of the equation \( x^3 - 6x^2 + 11x - 6 = 0 \).

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining all its roots.

**Example 1.55** If \( x = 1 \) and \( x = 2 \) are solutions of the equation \( x^2 + ax^2 + bx + c = 0 \) and \( a + b = 1 \), then find the value of \( b \).

**Sol.** Since \( x = 1 \) is a root of the given equation it satisfies the equation.

Hence, putting \( x = 1 \) in the given equation, we get \( a + b + c = -1 \).

But given that \( a + b = 1 \)

\[ b = -2 \]

Now put \( x = 2 \) in the given equation, we have
\[ 8 + 4a + 2b - 2 = 0 \]
\[ 6 + 2a + 2(b + a) = 0 \]
\[ 6 + 2a + 2 = 0 \]
\[ a = -4 \]
\[ b = 5 \]

**Example 1.56** Let \( f(x) = ax^2 + bx + c \), where \( a, b, c \in R \) and \( a \neq 0 \). It is known that \( f(5) = -3f(2) \) and that 3 is a root of \( f(x) = 0 \), then find the other root of \( f(x) = 0 \).

**Sol.** \( f(x) = ax^2 + bx + c \)

Given that \( f(5) = -3f(2) \)
\[ 25a + 5b + c = -3(4a + 2b + c) \]
\[ 37a + 11b + 4c = 0 \]

Or \( x = 3 \) satisfies \( f(x) = 0 \)
\[ 9a + 3b + c = 0 \]

Or \( 36a + 12b + 4c = 0 \)

[Multiplying Eq. (2) by 4]

Subtracting (3) from (1), we have
\[ a - b = 0 \]
\[ a = b \]
\[ \therefore 12a + c = 0 \]
\[ a = c \]
\[ a = -12a \]

Hence, equation \( f(x) = 0 \) becomes
\[ ax^2 + ax - 12a = 0 \]
\[ x^2 + x - 12 = 0 \]
\[ (x - 3)(x + 4) = 0 \]
\[ x = -4, 3 \]

**Example 1.57** A polynomial in \( x \) of degree three vanishes when \( x = 1 \) and \( x = -2 \), and has the values 4 and 28 when \( x = -1 \) and \( x = 2 \), respectively. Then find the value of polynomial when \( x = 0 \).

**Sol.** From the given data \( f(x) = (x - 1)(x + 2)(ax + b) \)

Now \( f(-1) = 4 \) and \( f(2) = 28 \)
\[ (-1 - 1)(-1 + 2)(-a + b) = 4 \]
\[ (2 - 1)(2 + 2)(2a + b) = 28 \]
\[ a - b = 2 \text{ and } 2a + b = 7 \]

Solving, \( a = 3 \) and \( b = 1 \)
\[ f(x) = (x - 1)(x + 2)(3x + 1) \]
\[ f(0) = -2 \]

**Example 1.58** If \( (1 - p) \) is a root of quadratic equation \( x^2 + px + (1 - p) = 0 \), then find its roots.

**Sol.** Since \( (1 - p) \) is the root of quadratic equation
\[ x^2 + px + (1 - p) = 0 \]

So \( (1 - p) \) satisfies the above equation
\[ (1 - p)^2 + p(1 - p) + (1 - p) = 0 \]
\[ (1 - p)(1 - p + p + 1) = 0 \]
\[ (1 - p)(2) = 0 \]
\[ p = 1 \]

On putting this value of \( p \) in Eq. (1), we get
\[ x^2 + x = 0 \]
\[ x(x + 1) = 0 \]
\[ x = 0, -1 \]
Example 1.59: The quadratic polynomial \( p(x) \) has the following properties:

- \( p(x) \) can be positive or zero for all real numbers
- \( p(1) = 0 \) and \( p(2) = 2 \).

Then find the quadratic polynomial.

Sol. \( p(x) \) is positive or zero for all real numbers also \( p(1) = 0 \). then we have \( p(x) = k(x - 1)^2 \), where \( k > 0 \).

Now \( p(2) = 2 \) \[ \Rightarrow k = 2 \]

\[ \therefore p(x) = 2(x - 1)^2 \]

GEOMETRICAL MEANING OF ROOTS (ZEROS) OF AN EQUATION

We know that a real number \( k \) is a zero of the polynomial \( f(x) \) if \( f(k) = 0 \). But why are the zeroes of a polynomial so important? To answer this, first we will see the geometrical representations of polynomials and the geometrical meaning of their zeroes.

We know that graph of the linear function \( y = f(x) = ax + b \) is a straight line.

Consider the function \( f(x) = x + 3 \).

![Fig. 1.20](image1.png)

Now we can see that this graph cuts the \( x \)-axis at \( x = -3 \), where value of \( y = 0 \) or we can say \( x + 3 = 0 \) (or \( y = 0 \)) when value of \( x = -3 \). Thus, \( x = -3 \) which is a root (zero) of equation \( x + 3 = 0 \) is actually the value of \( x \) where graph of \( y = f(x) = x + 3 \) intersects the \( x \)-axis.

Consider the function \( f(x) = x^2 - x - 2 \), now for \( f(x) = 0 \) or \( x^2 - x - 2 = 0 \), we have \( (x - 2)(x + 1) = 0 \) or \( x = -1 \) or \( x = 2 \). Then

![Fig. 1.21](image2.png)

graph of \( f(x) = x^2 - x - 2 \) cuts the \( x \)-axis at two values of \( x, x = -1 \) and \( x = 2 \). Following is the graph of \( y = f(x) \).

Consider the function \( f(x) = x^3 - 6x^2 + 11x - 6 \), now for \( f(x) = 0 \) we have \( (x - 1)(x - 2)(x - 3) = 0 \) or \( x = 1, 2, 3 \). Then graph of \( y = f(x) \) cuts \( x \)-axis at three values of \( x, x = 1, 2, 3 \).

Following is the graph of \( y = f(x) \).

![Fig. 1.22](image3.png)

Consider the function \( f(x) = (x^3 - 3x^2 + 2)(x^2 - x + 1) \), now for \( f(x) = 0 \) we have \( x = 1 \) or \( x = 2 \), as \( x^3 - x + 1 = 0 \) is not possible for any real value of \( x \). Hence, \( f(x) = 0 \) has only two real roots and cuts \( x \)-axis for only two values of \( x, x = 1 \) and \( x = 2 \).
Following is the graph of $y = f(x)$.

Thus, roots of equation $f(x) = 0$ are actually those values of $x$ where graph $y = f(x)$ meets $x$-axis.

**Roots (Zeros) of the Equation $f(x) = g(x)$**

Now we know that zeros of the equation $f(x) = 0$ are the $x$-coordinates of the points where graph of $y = f(x)$ intersect the $x$-axis, where $y = 0$ or zeros are $x$-coordinate of the point of intersection of $y = f(x)$ and $y = 0$ ($x$-axis).

Consider the equation $x + 5 = 2$.

Let's draw the graph of $y = x + 5$ and $y = 2$, which are as shown in the following figure.

Graph of $y = 2$ is a line parallel to $x$-axis at height 2 unit above $x$-axis. Now in the figure, we can see that graphs of $y = x + 5$ and $y = 2$ intersect at point $( -3, 2 )$ where value of $x = -3$.

Also from $x + 5 = 2$, we have $x = 2 - 3$ or $x = -3$, which is a root of the equation $x + 2 = 5$. Thus root of the equation $x + 5 = 2$ occurs at point of intersection of graphs $y = x + 5$ and $y = 2$.

Consider the another example $x^2 - 2x = 2 - x$. Let's draw the graph of $y = x^2 - 2x$ and $y = 2 - x$ as shown in the following figure.

Now in the figure, we can see that graphs of $y = x^2 - 2x$ and $y = 2 - x$ intersect at points $(-1, 3)$ and $(2, 0)$ or where values of $x$ are $x = -1$ and $x = 2$, which are in fact zeros or roots of the equation $x^2 - 2x = 2 - x$ or $x^2 - x - 2 = 0$.

The given equation simplifies to $x^2 - x - 2 = 0$. So one can also locate the roots of the same equation by plotting the graph of $y = x^2 - x - 2$, then the roots of equation are $x$-coordinates of points where graph of $y = x^2 - x - 2$ intersects with the $x$-axis (where $y = 0$), as shown in the following figure.

From the above discussion we understand that roots of the equation $f(x) = g(x)$ are the $x$-coordinate of the points of intersection of graphs $y = f(x)$ and $y = g(x)$.

**Example 1.60** In how many points graph of $y = x^3 - 3x^2 + 5x - 3$ intersect $x$-axis?

**Sol.** Number of point in which $y = x^3 - 3x^2 + 5x - 3$ intersect the $x$-axis is same as number of real roots of the equation $x^3 - 3x^2 + 5x - 3 = 0$. 
Now we can see that $x = 1$ satisfies the equation, hence one root of the equation is $x = 1$.

Now dividing $x^3 - 3x^2 + 5x - 3$ by $x - 1$, we have quotient $x^2 - 2x + 3$.

Hence, equation reduces to $(x - 1)(x^2 - 2x + 3) = 0$.

Now $x^2 - 2x + 3 = 0$ or $(x - 1)^2 + 2 = 0$ is not true for any real value of $x$.

Hence, the only root of the equation is $x = 1$.

Therefore, the graph of $y = x^3 - 3x^2 + 5x - 3$ cuts the x-axis in one point only.

**Example 1.61** In the following diagram, the graph of $y = f(x)$ is given.

![Fig. 1.27](image)

**Answer the following questions:**

(a) what are the roots of the $f(x) = 0$?

(b) what are the roots of the $f(x) = 4$?

(c) what are the roots of the $f(x) = 10$?

**Sol.**

(a) The root of the equation $f(x) = 0$ occurs for the values of $x$ where the graphs of $y = f(x)$ and $y = 0$ intersect.

From the diagram, for these point of intersection $x = -1$ and $x = 2$. Hence, roots of the equation $f(x) = 0$ are $x = -1$ and $x = 2$.

(b) The root of the equation $f(x) = 4$ occurs for the values of $x$ where the graphs of $y = f(x)$ and $y = 4$ intersect.

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From the diagram, for these point of intersection $x = -2$ and $x = 3$. Hence, roots of the equation $f(x) = 0$ are $x = -2$ and $x = 3$.

(c) Also roots of the equation $f(x) = 10$ are $-3$ and $4$.

**Example 1.62** Which of the following pair of graphs intersect?

(i) $y = x^2 - x$ and $y = 1$

(ii) $y = x^2 - 2x + 3$ and $y = \sin x$

(iii) $y = x^2 - x + 1$ and $y = x - 4$

**Sol.** $y = x^2 - x$ and $y = 1$ intersect if $x^2 - x = 1 \Rightarrow x^2 - x - 1 = 0$, which has real roots.

$y = x^2 - 2x + 3$ and $y = \sin x$ intersect if $x^2 - 2x + 3 = \sin x$ or $(x - 1)^2 + 2 = \sin x$, which is not possible as L.H.S. has the least value 2, while R.H.S. has the maximum value 1.

$y = x^2 - x + 1$ and $y = x - 4$ intersect if $x^2 - x + 1 = x - 4$ or $x^2 - 2x + 5 = 0$, which has non-real roots. Hence, graphs do not intersect.

**Example 1.63** Prove that graphs $y = 2x - 3$ and $y = x^2 - x$ never intersect.

**Sol.** $y = 2x - 3$ and $y = x^2 - x$ intersect only when $2x - 3 = x^2 - x$ or $x^2 - 3x + 3 = 0$.

Now discriminant $D = (-3)^2 - 4(3) = -3 < 0$

Hence, roots of the equation are not real, or we can say that there is no real number for which $2x - 3$ and $x^2 - x$ are equal (or $y = 2x - 3$ and $y = x^2 - x$ intersect).

Hence, proved.

**KEY POINTS IN SOLVING AN EQUATION**

**Domain of Equation**

It is a set of the values of independent variables $x$ for which each function used in the equation is defined, i.e., it takes up finite real values. In other words, the final solution obtained while solving any equation must satisfy the domain of the expression of the parent equation.

**Example 1.64** Solve $\frac{x^2 - 2x - 3}{x + 1} = 0$.

**Sol.** Equation $\frac{x^2 - 2x - 3}{x + 1} = 0$ is solvable over $R - \{-1\}$

Now $\frac{x^2 - 2x - 3}{x + 1} = 0$

$\Rightarrow \frac{x^2 - 2x - 3}{x + 1} = 0$ or $(x - 3)(x + 1) = 0$

$\Rightarrow x = 3$ (as $x \in R - \{-1\}$)

**Example 1.65** Solve $(x^2 - 4x)^{\frac{1}{2}} = x^2 - 1 = 0$.

**Sol.** Given equation is solvable for $x^2 - 1 \geq 0$

or $x \in (-\infty, -1] \cup [1, \infty)$

$(x^2 - 4x)^{\frac{1}{2}} = x^2 - 1 = 0$

$\Rightarrow (x - 2)(x + 2)\sqrt{x^2 - 1} = 0$

$\Rightarrow x = 0, -2, 2, -1, 1$
But \( x \in (-\infty, -1) \cup [1, \infty) \)
\[ \Rightarrow x = \pm 1, \pm 2 \]

**Example 1.66** Solve \( \frac{2x - 3}{x - 1} + 1 = \frac{6x - x^2 - 6}{x - 1} \).

**Sol.**
\[
\frac{2x - 3}{x - 1} + 1 = \frac{6x - x^2 - 6}{x - 1}, \quad x \neq 1
\]
\[ \Rightarrow \frac{3x - 4}{x - 1} = \frac{6x - x^2 - 6}{x - 1}, \quad x \neq 1 \]
\[ \Rightarrow 3x - 4 = 6x - x^2 - 6, \quad x \neq 1 \]
\[ \Rightarrow x^2 - 3x + 2 = 0, \quad x \neq 1 \]
\[ \Rightarrow x = 2 \]

**Extraneous Roots**

While simplifying the equation, the domain of the equation may expand and give the extraneous roots. For example, consider the equation \( \sqrt{x} = x - 2 \).

For solving, we first square it
so \( \sqrt{x} = x - 2 \)
\[ \Rightarrow x = (x - 2)^2 \quad \text{[on squaring both sides]} \]
\[ \Rightarrow x^2 - 5x + 4 = 0 \]
\[ \Rightarrow (x - 1)(x - 4) = 0 \]
\[ \Rightarrow x = 1, 4 \]

We observe that \( x = 4 \) satisfies the given equation but \( x = 1 \) does not satisfy it. Hence, \( x = 4 \) is the only solution of the given equation. The domain of actual equation is \([2, \infty)\).

While squaring the equation, domain expands to \( \mathbb{R} \), which gives extra root \( x = 1 \).

**Loss of Root**

Cancellation of common factors from both sides of equation leads to loss of root.

For example, consider an equation \( x^2 - 2x = x - 2 \)
\[ \Rightarrow x(x - 2) = x - 2 \]
\[ \Rightarrow x = 1 \]

Here we have cancelled factor \( x - 2 \) which causes the loss of root, \( x = 2 \).

The correct way of solving is
\[ x^2 - 2x = x - 2 \]
\[ \Rightarrow x^2 - 3x + 2 = 0 \]
\[ \Rightarrow (x - 1)(x - 2) = 0 \]
\[ \Rightarrow x = 1 \text{ and } x = 2. \]

**Graphs of Polynomial Functions**

When the polynomial function is written in standard form, \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad (a_n \neq 0) \), the leading term is \( a_nx^n \). In other words, the leading term is the term that the variable has its highest exponent. The degree of a term of a polynomial function is the exponent on the variable. The degree of the polynomial is the largest degree of all of its terms.

For drawing the graph of the polynomial function, we consider the following tests.

---

**Test 1: Leading Co-efficient**

If \( n \) is odd and the leading coefficient \( a_n \) is positive, then the graph falls to the left and rises to the right:

![Graph](image)

**Fig. 1.28**

If \( n \) is odd and the leading coefficient \( a_n \) is negative, the graph rises to the left and falls to the right.

![Graph](image)

**Fig. 1.29**

If \( n \) is even and the leading coefficient \( a_n \) is positive, the graph rises to the left and to the right.

![Graph](image)

**Fig. 1.30**
If \( n \) is even and the leading coefficient \( a_n \) is negative, the graph falls to the left and to the right.

![Image of graph](image1.png)

**Fig. 1.31**

**Test 2: Roots (Zeros) of Polynomial**

In other words, when a polynomial function is set equal to zero and has been completely factored and each different factor is written with the highest appropriate exponent, depending on the number of times that factor occurs in the product, the exponent on the factor that the zero is a solution for it gives the multiplicity of that zero.

The exponent indicates how many times that factor would be written out in the product, this gives us a multiplicity.

**Multiplicity of Zeros and the x-intercept**

If \( r \) is a zero of even multiplicity:

This means the graph touches the x-axis at \( r \) and turns around.

This happens because the sign of \( f(x) \) does not change from one side to the other side of \( r \).

See the graph of \( f(x) = (x - 2)^2 (x - 1)(x + 1) \).

![Image of graph](image2.png)

**Fig. 1.32**

If \( r \) is a zero of odd multiplicity:

This means the graph crosses (also touches if exponent is more than 1) the x-axis at \( r \). This happens because the sign of \( f(x) \) changes from one side to the other side or \( r \).

See the graph of \( f(x) = (x - 1)(3x - 2)(x - 3)^2 \).

![Image of graph](image3.png)

**Fig. 1.33**

Thus, in general, polynomial function graphs consist of a smooth line with a series of hills and valleys. The hills and valleys are called turning points. The maximum possible number of turning points is one less than the degree of the polynomial.

The point where graph has turning point, derivative of function \( f(x) \) becomes zero, which provides point of local minima or local maxima. Knowledge of derivative provides great help in drawing the graph of the function, hence finding its point of intersection with x-axis or roots of the equation \( f(x) = 0 \). Also we know that geometrically the derivative of function at any point of the graph of the function is equal to the slope of tangent at that point to the curve.

Consider the following graph of the function \( y = f(x) \) as shown in the following figure.

![Image of graph](image4.png)

**Fig. 1.34**
In the figure, we can see that tangent to the curve at point for which \( x < -1 \) and \( x > 1 \) makes acute angle with the positive direction of x-axis, hence derivative is positive for these points. For \( -1 < x < 1 \), tangent to the curve makes obtuse angle with the positive direction of x-axis, hence derivative is negative at these points. At \( x = -1 \) and \( x = 1 \), tangent is parallel to x-axis, where derivative is zero.

Here \( x = -1 \) is called point of maxima, where derivative changes sign from positive to negative (from left to right), and \( x = 1 \) is called point of minima, where derivative changes sign from negative to positive (from left to right).

At point of maxima and minima, derivative of the function is zero.

**Example 1.67** Using differentiation method check how many roots of the equation \( x^3 - x^2 + x - 2 = 0 \) are real?

**Sol.** Let \( y = f(x) = x^3 - x^2 + x - 2 \)

\[
\frac{dy}{dx} = 3x^2 - 2x + 1
\]

Let \( 3x^2 - 2x + 1 = 0 \), now this equation has non-real roots, i.e., derivative never becomes zero or graph of \( y = f(x) \) has no turning point.

Also when \( x \to \infty, f(x) \to \infty \) and when \( x \to -\infty, f(x) \to -\infty \)

Further \( 3x^2 - 2x + 1 > 0 \) \( \forall \ x \in \mathbb{R} \)

Thus graph of the function is as shown in the following figure.

Also \( f(0) = -2 \); hence graph cuts the x-axis for some positive value of \( x \).

Hence, the only root of the equation is positive.

Thus we can see that differentiation and then graph of the function is much important in analyzing the equation.

**Example 1.68** Analyze the roots of the following equations:

(i) \( 2x^3 - 9x^2 + 12x - (9/2) = 0 \)

(ii) \( 2x^3 - 9x^2 + 12x - 3 = 0 \)

**Sol.**

(i) Let \( f(x) = 2x^3 - 9x^2 + 12x - (9/2) \)

Then \( f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2) \)

Now \( f(x) = 0 \Rightarrow x = 1 \) and \( x = 2 \).

Hence, graph has turn at \( x = 1 \) and at \( x = 2 \).

Also, \( f(1) = 2 - 9 + 12 - (9/2) > 0 \)

and \( f(2) = 16 - 36 + 24 - (9/2) < 0 \)

Hence, graph of the function \( y = f(x) \) is as shown in the following figure.

**Fig. 1.36(a)**

As shown in the figure, graph cuts x-axis at three distinct points.

Hence, equation \( f(x) = 0 \) has three distinct roots.

(ii) For \( 2x^3 - 9x^2 + 12x - 3 = 0, f(x) = 2x^3 - 9x^2 + 12x - 3 \)

\( f(x) = 0 \Rightarrow x = 1 \) and \( x = 2 \)

Also \( f(1) = 2 - 9 + 12 - 3 = 2 \) and \( f(2) = 16 - 36 + 24 - 3 = 1 \)

Hence, graph of \( y = f(x) \) is as shown in the following figure.
Thus from the graph, we can see that \( f(x) = 0 \) has only one real root, though \( y = f(x) \) has two turning points.

**Example 1.69**  Find how many roots of the equation \( x^4 + 2x^2 - 8x + 3 = 0 \) are real.

**Sol.** Let \( f(x) = x^4 + 2x^2 - 8x + 3 \)
\[ \Rightarrow f'(x) = 4x^3 + 4x - 8 = 4(x - 1)(x^2 + x + 2) \]
Now \( f'(x) = 0 \Rightarrow x = 1 \)
Hence graph of \( y = f(x) \) has only one turn (maxima/minima).
Now \( f(1) = 1 + 2 - 8 + 3 < 0 \)
Also when \( x \to \pm \infty, f(x) \to \infty \)
Then graph of the function is as shown in the following figure.

**Fig. 1.37**

Hence, equation \( f(x) = 0 \) has only two real roots.

**EQUATIONS REDUCIBLE TO QUADRATIC**

**Example 1.70**  Solve \( \sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1 \).

**Sol.** Let \( 5x^2 - 6x = y \). Then,
\[ \sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1 \]

\[ \Rightarrow \sqrt{y + 8} - \sqrt{y - 7} = 1 \]
\[ \Rightarrow (\sqrt{y + 8} - \sqrt{y - 7})^2 = 1 \]
\[ \Rightarrow y + 8 - 2(y + 8)(y - 7) + y - 7 = 1 \]
\[ \Rightarrow 2y - 2(y + 8)(y - 7) = 0 \]
\[ \Rightarrow y(y - 1) = 0 \]
\[ \Rightarrow y = 0, 1 \]

Now,
\[ y = 0 \]
\[ \Rightarrow x^2 - 5x + 7 = 0 \]
\[ \Rightarrow x = \frac{5 \pm \sqrt{25 - 28}}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5 \pm i\sqrt{3}}{2} \]
where \( i = \sqrt{-1} \)
and
\[ y = 1 \]
\[ \Rightarrow x^2 - 5x + 6 = 0 \]
\[ \Rightarrow (x - 3)(x - 2) = 0 \]
\[ \Rightarrow 3, 2 \]

Hence, the roots of the equation are 2, 3, \((5 + i\sqrt{3})/2\) and \((5 - i\sqrt{3})/2\).

**Example 1.72**  Solve the equation \( 4^x - 5 \times 2^x + 4 = 0 \).

**Sol.** We have,
\[ 4^x - 5 \times 2^x + 4 = 0 \]
\[ \Rightarrow (2^x)^2 - 5(2^x) + 4 = 0 \]
\[ \Rightarrow y^2 - 5y + 4 = 0, \text{ where } y = 2^x \]
\[ \Rightarrow (y - 4)(y - 1) = 0 \]
\[ \Rightarrow y = 1, 4 \]
\[ \Rightarrow 2^x = 1, 2^x = 4 \]
\[ \Rightarrow 2^x = 2, 2^x = 2^2 \]
\[ \Rightarrow x = 0, 2 \]
Hence, the roots of the given equation are 0 and 2.

**Example 1.73**  Solve the equation \( 12x^3 - 56x^2 + 89x^2 - 56x + 12 = 0 \).

**Sol.** The given equation is
\[ 12x^3 - 56x^2 + 89x^2 - 56x + 12 = 0 \]
Dividing by $x^2$, we get
\[ 12x^2 - 56x + 89 - \frac{56}{x} + \frac{12}{x^2} = 0 \]
\[ \Rightarrow 12 \left( x + \frac{1}{x} \right)^2 - 56 \left( x + \frac{1}{x} \right) + 89 = 0 \]
\[ \Rightarrow 12 \left( x + \frac{1}{x} \right)^2 - 2 \cdot 56 \left( x + \frac{1}{x} \right) + 89 = 0 \]
\[ \Rightarrow 12y^2 - 56y + 65 = 0, \text{ where } y = x + \frac{1}{x} \]
\[ \Rightarrow 12y^2 - 26y - 30y + 65 = 0 \]
\[ \Rightarrow (6y - 13)(2y - 5) = 0 \]
\[ \Rightarrow y = \frac{13}{6} \text{ or } y = \frac{5}{2} \]
If $y = \frac{13}{6}$, then
\[ x + \frac{1}{x} = \frac{13}{6} \]
\[ \Rightarrow 6x^2 - 13x + 6 = 0 \]
\[ \Rightarrow (3x - 2)(2x - 3) = 0 \]
\[ \Rightarrow x = \frac{3}{2} \text{ or } x = \frac{2}{3} \]
If $y = \frac{5}{2}$, then
\[ x + \frac{1}{x} = \frac{5}{2} \]
\[ \Rightarrow 2x^2 - 5x + 2 = 0 \]
\[ \Rightarrow (x - 2)(2x - 1) = 0 \]
\[ \Rightarrow x = 2, \frac{1}{2} \]
Hence, the roots of the given equation are 2, 1/2, 2/3, 3/2.

**Example 1.74** Solve the equation $3x^2 + 4x^2 - x = 25$.

**Sol.** We have,
\[ 3x^2 + 4x^2 - x = 25 \]
\[ \Rightarrow 7x^2 - x = 25 \]
\[ \Rightarrow x^2 - \frac{x}{7} = \frac{25}{7} \]
\[ \Rightarrow (x - 2)(x + 3) = 0 \]
\[ \Rightarrow x = 2, -1 \]
Hence, the roots of the given equation are $-1$ and 2.

**Example 1.75** Solve the equation $(x - 1)^4 + (x - 5)^4 = 82$. 

**Sol.** Let 
\[ y = \frac{(x - 1) + (x - 5)}{2} = x - 3 \]
\[ \Rightarrow x = y + 3 \]
Putting $x = y + 3$ in the given equation, we obtain
\[ (y + 2)^4 + (y - 2)^4 = 82 \]
\[ \Rightarrow (y^2 + 4y + 4)^2 + (y^2 - 4y + 4)^2 = 82 \]
\[ \Rightarrow \{ (y^2 + 4y + 4) + (y^2 - 4y + 4) \}^2 = 82 \]
\[ \Rightarrow 2(y^2 + 4y + 4)^2 + 16y^2 = 82 \]
\[ \Rightarrow y^4 + 8y^2 + 16 + 16y^2 = 41 \]
\[ \Rightarrow y^4 + 24y^2 - 25 = 0 \]
\[ \Rightarrow (y^2 + 5) y^2 - 1 = 0 \]
\[ \Rightarrow y = \pm \sqrt{5}, y = \pm 1 \]
\[ \Rightarrow x = 3 \pm 5i, x = -3 \pm i \]
\[ \Rightarrow x = 3 \pm 5i, \text{ or } x = 4, -2 \]
Hence, the roots of the given equation are $3 \pm 5i, 2$ and 4.

**Example 1.76** Solve the equation $(x + 2)(x + 3)(x + 8) x(x + 12) = 4x^4$.

**Sol.** 
\[ (x + 2)(x + 3)(x + 8)(x + 12) = 4x^4 \]
\[ \Rightarrow (x+2)(x+3)(x+8) = 4x^3 \]
Dividing throughout by $x^2$, we get
\[ \left( x + \frac{14}{x} \right) \left( x + \frac{11}{x} \right) = 4 \]
\[ \Rightarrow (y + 14)(y + 11) = 4, \text{ where } y = \frac{24}{x} \]
\[ \Rightarrow y^2 + 25y + 154 = 4 \]
\[ \Rightarrow y^2 + 25y + 150 = 0 \]
\[ \Rightarrow (y + 15)(y + 10) = 0 \]
\[ \Rightarrow y = -15, -10 \]
If $y = -15$, then
\[ x + \frac{24}{x} = -15 \]
\[ \Rightarrow x^2 + 15x + 24 = 0 \]
\[ \Rightarrow x = \frac{-15 \pm \sqrt{129}}{2} \]
If $y = -10$, then
\[ x + \frac{24}{x} = -10 \]
\[ \Rightarrow x^2 + 10x + 24 = 0 \]
\[ \Rightarrow (x + 4)(x + 6) = 0 \]
\[ \Rightarrow x = -4, -6 \]
Hence, the roots of the given equation are $-4, -6, (-15 \pm \sqrt{129})/2$.

**Example 1.77** Evaluate $\sqrt{6} + \sqrt{6} + \sqrt{6} + \ldots \infty$.

**Sol.** Let $x = \sqrt{6} + \sqrt{6} + \sqrt{6} + \ldots \infty$. Then,
\[ x = \sqrt{6 + x} \]
\[ \Rightarrow x^2 = 6 + x \]
\[ \Rightarrow x^2 - x - 6 = 0 \]
\[ \Rightarrow (x - 3)(x + 2) = 0 \]
\[ \Rightarrow x = 3 \text{ or } x = -2 \]
But, the given expression is positive. So, $x = 3$. Hence, the value of the given expression is 3.
Example 1.78: Solve \( \sqrt{x+5} + \sqrt{x+21} = \sqrt{6x+40} \).

Sol. \[
\begin{align*}
\sqrt{x+5} + \sqrt{x+21} &= \sqrt{6x+40} \\
\implies (\sqrt{x+5} + \sqrt{x+21})^2 &= 6x+40 \\
\implies (x+5) + (x+21) + 2\sqrt{(x+5)(x+21)} &= 6x+40 \\
\implies 2x + 26 + 2\sqrt{(x+5)(x+21)} &= 6x+40 \\
\implies \sqrt{(x+5)(x+21)} &= 2x+7 \\
\implies (x+5)(x+21) &= (2x+7)^2 \\
\implies 3x^2 + 2x - 56 &= 0 \\
\implies (3x+14) (x-4) &= 0 \\
\implies x = 4 \text{ or } x = -14/3.
\end{align*}
\]
Clearly, \( x = -14/3 \) does not satisfy the given equation. Hence, \( x = 4 \) is the only root of the given equation.

Concept Application Exercise 1.2

1. Prove that the graph of \( y=x^2+2 \) and \( y=3x-4 \) never intersect.
2. In how many points does the line \( y+14=0 \) cut the curve whose equation is \( x(x^2+x+1)+y=0 \)?
3. Consider the following graphs:

\[\text{Fig. 1.38}\]

Answer the following questions:
(i) sum of roots of the equation \( f(x) = 0 \)
(ii) product of roots of the equation \( f(x) = 4 \)
(iii) the absolute value of the difference of the roots of equation \( f(x) = x + 2 \)

4. Solve \( x^2 + 3x + 2 = 0 \).
5. Solve \( \sqrt{x-2} + \sqrt{4-x} = 2 \).
6. Solve \( \sqrt{x-2} (x^2 - 4x - 5) = 0 \).
7. Solve the equation \( x(x+2) (x^2-1) = 0 \).

REMAINDER AND FACTOR THEOREMS

Remainder Theorem

The remainder theorem states that if a polynomial \( f(x) \) is divided by a linear function \( x - k \), then the remainder is \( f(k) \).

Proof:
In any division,
\[
\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}
\]
Let \( Q(x) \) be the quotient and \( R \) be the remainder. Then,
\[
f(x) = (x-k) Q(x) + R
\]
\[
f(k) = (k-k) Q(x) + R = 0 + R = R
\]

Note: If a \( n \)-degree polynomial is divided by a \( m \)-degree polynomial, then the maximum degree of the remainder polynomial is \( m-1 \).

Example 1.79: Find the remainder when \( x^3 + 4x^2 - 7x + 6 \) is divided by \( x - 1 \).

Sol. Let \( f(x) = x^3 + 4x^2 - 7x + 6 \). The remainder when \( f(x) \) is divided by \( x - 1 \) is
\[
f(1) = 1^3 + 4 \times 1^2 - 7 \times 1 + 6 = 4
\]

Example 1.80: If the expression \( ax^4 + bx^3 - x^2 + 2x + 3 \) has remainder \( 4x + 3 \) when divided by \( x^2 + x - 2 \), find the value of \( a \) and \( b \).

Sol. Let \( f(x) = ax^4 + bx^3 - x^2 + 2x + 3 \).
Now, \( x^2 + x - 2 = (x+2)(x-1) \).
Given, \( f(-2) = a(-2)^4 + b(-2)^3 - (-2)^2 + 2(-2) + 3 \)
\[ 4(-2) + 3 \]
\[ 16a - 8b - 4 + 3 = -5 \]

\(2a - b = 0\)

Also,

\[ f(1) = a + b - 1 + 2 + 3 = 4(1) + 3 \]

\(a + b = 3\)

From (1) and (2), \(a = 1, b = 2\).

**Factor Theorem**

*Factor Theorem Is a Special Case of Remainder Theorem*

Let,

\[ f(x) = x - k \quad Q(x) + R \]

\[ f(x) = x - k \quad Q(x) + f(k) \]

When \(f(k) = 0\), \(f(x) = (x - k) \quad Q(x)\). Therefore, \(f(x)\) is exactly divisible by \(x - k\).

**Example 1.81** Given that \(x^3 + x - 6\) is a factor of \(2x^4 + x^3 - ax^2 + bx + a + b - 1\), find the values of \(a\) and \(b\).

**Sol.** We have,

\[ x^3 + x - 6 = (x + 3)(x - 2) \]

Let,

\[ f(x) = 2x^4 + x^3 - ax^2 + bx + a + b - 1 \]

Now,

\[ f(-3) = 2(-3)^3 + (-3)^2 - a(-3)^2 - 3b + a + b - 1 = 0 \]

\[ 134 - 8a - 2b = 0 \]

\[ 4a + b = 67 \] (1)

\[ f(2) = 2(2)^4 + 2^3 - a(2)^2 + 2b + a + b - 1 = 0 \]

\[ 39 - 3a + 3b = 0 \]

\[ a - b = 13 \] (2)

From (1) and (2), \(a = 16, b = 3\).

**Example 1.82** Use the factor theorem to find the value of \(k\) for which \((a + 2b)\), where \(a, b \neq 0\) is a factor of \(a^4 + 32b^4 + a^2b(k + 3)\).

**Sol.** Let \(f(a) = a^4 + 32b^4 + a^2b(k + 3)\). Now,

\[ f(-2b) = (-2b)^4 + 32b^4 + (-2b)^2b(k + 3) = 0 \]

\[ 48b^4 - 8b^2(k + 3) = 0 \]

\[ 8b^2(6 - k) = 0 \]

Since \(b \neq 0\), so, \(3 - k = 0\) or \(k = 3\).

**Example 1.83** If \(c, d\) are the roots of the equation \((x - a) (x - b) = k \), prove that \(a, b\) are the roots of the equation \((x - c) (x - d) + k = 0\).

**Sol.** Since \(c\) and \(d\) are the roots of the equation \((x - a) (x - b) = k\), therefore,

\[ (x - a)(x - b) - k = (x - c)(x - d) \]

\[ (x - a)(x - b) = (x - c)(x - d) + k \]

\[ (x - c)(x - b) + k = (x - a)(x - b) \]

Clearly, \(a\) and \(b\) are roots of the equation \((x - a)(x - b) = 0\).

Hence, \(a, b\) are roots of \((x - c)(x - d) + k = 0\).

**Concept Application Exercise 1.3**

1. Given that the expression \(2x^2 + 3p - 4x + p\) has a remainder of 5 when divided by \(x + 2\), find the value of \(p\).
2. Determine the value of \(k\) for which \(x + 2\) is a factor of \((x + 1)^2 + (2x + 3)^2\).
3. Find the value of \(p\) for which \(x + 1\) is a factor of \(x^4 + (p - 3)x^3 + (3p - 5)x^2 + (2p - 9)x + 6\).

Find the remaining factors for this value of \(p\).

4. If \(x^2 + ax + 1\) is a factor of \(ax^2 + bx + c\), then find the conditions.

5. If \(f(x) = x^3 - 3x^2 + 2x + a\) is divisible by \(x - 1\), then find the remainder when \(f(x)\) is divided by \(x - 2\).

6. If \(f(x) = x^3 - x^2 + ax + b\) is divisible by \(x - 3\), then find the value of \(f(2)\).

**Identity**

A relation which is true for every value of the variable is called an identity.

**Example 1.84** If \((a^2 - 1)x^2 + (a - 1)x + 4a + 3 = 0\) be an identity in \(x\), then find the value of \(a\).

**Sol.** The given relation is satisfied for all real values of \(x\), so all the coefficients must be zero. Then,

\[ a^2 - 1 = 0 \Rightarrow a = \pm 1 \]

\[ a - 1 = 0 \Rightarrow a = 1 \]

\[ a^2 - 4a + 3 = 0 \Rightarrow a = 1, 3 \]

**Example 1.85** Show that \(\frac{(x + b)(x + c) + (x + c)(x + a)}{(b - a)(c - a)} + \frac{(x + a)(x + b)}{(a - c)(b - c)} = 1\) is an identity.

**Sol.** Given relation is

\[ \frac{(x + b)(x + c) + (x + c)(x + a)}{(b - a)(c - a)} + \frac{(x + a)(x + b)}{(a - c)(b - c)} = 1 \]

When \(x = -a\),

L.H.S. = \(\frac{(b - a)(c - a)}{(b - a)(c - a)} = 1 = R.H.S.\)

Similarly, when \(x = -b\),

L.H.S. = \(\frac{(c - b)(a - b)}{(c - b)(a - b)} = 1 = R.H.S.\)

When \(x = -c\),

L.H.S. = \(\frac{(a - c)(b - c)}{(a - c)(b - c)} = 1 = R.H.S.\)
Thus, the highest power of \( x \) occurring in relation (1) is 2 and this relation is satisfied by three distinct values \( a, b \) and \( c \) of \( x \); therefore, it is an equation but an identity.

**Example 1.86.** A certain polynomial \( P(x), x \in R \) when divided by \( x - a, x - b \) and \( x - c \) leaves remainders \( a, b \) and \( c \), respectively. Then find the remainder when \( P(x) \) is divided by \( (x - a)(x - b)(x - c) \) where \( a, b, c \) are distinct.

**Sol.** By remainder theorem, \( P(a) = a, P(b) = b \) and \( P(c) = c \).

Let the required remainder be \( R(x) \). Then,

\[
P(x) = (x - a)(x - b)(x - c)Q(x) + R(x)
\]

where \( R(x) \) is a polynomial of degree at most \( 2 \). We get \( R(a) = a, R(b) = b \) and \( R(c) = c \). So, the equation \( R(x) = x \) has three roots \( a, b, c \). But its degree is at most \( 2 \). So, \( R(x) = x \) must be zero polynomial (or identity). Hence \( R(x) = x \).

**QUADRATIC EQUATION**

**Quadratic Equation with Real Coefficients**

Consider the quadratic equation

\[
a x^2 + b x + c = 0
\]

where \( a, b, c \in R \) and \( a \neq 0 \).

Roots of the equation are given by

\[
x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]

Now, we observe that the nature of the roots depend upon the value of the quantity \( b^2 - 4ac \). This quantity is generally denoted by \( D \) and is known as the discriminant of the quadratic equation [Eq. (1)].

We also observe the following results:

- \( D = 0 \) Roots are equal: \( a = b = -b/2a \)
- \( D > 0 \) Roots are unequal
- \( a, b, c \in R \) and \( D > 0 \) Roots are real and distinct
- \( a, b, c \in R \) and \( D < 0 \) Roots are imaginary: \( a = p + iq, b = p - iq \) where \( i = \sqrt{-1} \)
- \( a \in Z \) and \( D \) is a perfect square \( \Rightarrow \) Roots are integral
- \( a \in Z \) and \( D \) is not a perfect square \( \Rightarrow \) Roots are irrational, i.e., \( a = p + \sqrt{q}, b = p - \sqrt{q} \)

**Fig. 1.39**

**Number System, Inequalities and Theory of Equations**

**Note:**

- If \( a, b, c \in Q \) and \( b^2 - 4ac \) is positive but not a perfect square, then roots are irrational and they always occur in conjugate pairs like \( 2 + \sqrt{3} \) and \( 2 - \sqrt{3} \). However, if \( a, b, c \) are irrational numbers and \( b^2 - 4ac \) is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation \( x^2 - (5 + \sqrt{2})x + 5\sqrt{2} = 0 \) are \( 5 \) and \( \sqrt{2} \), which do not form a conjugate pair.

- If \( b^2 - 4ac < 0 \), then roots of equations are complex. If \( a, b, c \) are real then complex roots occur in conjugate pairs such as the form \( p + iq \) and \( p - iq \). If all the coefficients are not real then complex roots may not conjugate.

**Example 1.87.** If \( a, b, c \in R^* \) and \( 2b = a + c \), then check the nature of roots of equation \( ax^2 + 2bx + c = 0 \).

**Sol.** Given equation is \( ax^2 + 2bx + c = 0 \). Hence,

\[
D = 4b^2 - 4ac = (a + c)^2 - 4ac = (a - c)^2 > 0
\]

Thus, the roots are real and distinct.

**Example 1.88.** If the roots of the equation \( a(b - c)x^2 + b(c - a)x + c(a - b) = 0 \) are equal, show that \( 2b = 1/a + 1/c \).

**Sol.** Since the roots of the given equations are equal, therefore its discriminant is zero, i.e.,

\[
b^2 - b(c - a)^2 - 4ac = (b - c)(c - a)(a - b) = 0
\]

\[

\Rightarrow b^2 - b^2c^2 + 2b^2c - 4ac^2 - 4a^2bc - 4abc = 0
\]

\[

\Rightarrow ab + bc + 2ac = 0
\]

\[

\Rightarrow \frac{1}{c} + \frac{1}{a} = \frac{2}{b}
\]

[Dividing both sides by \( abc \)]

\[
\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}
\]

**Example 1.89.** Prove that the roots of the equation \( (a+b)x^2 + 4abdx + (c^4 + d^4) = 0 \) cannot be different, if real.

**Sol.** The discriminant of the given equation is

\[
D = 16ab^2c^4d^4 - 4(a^4 + b^4)(c^4 + d^4)
\]

\[
= -4[(a^4 + b^4)(c^4 + d^4) - 4a^2b^2c^2d^2]
\]

\[
= -4[(a^4c^4 + a^4d^4 + b^4c^4 + b^4d^4) - 4a^2b^2c^2d^2]
\]

\[
= -4[(a^4c^4 + b^4d^4 - 2a^2b^2c^2d^2) + (a^4d^4 - b^4c^4)]
\]

Since roots of the given equation are real, therefore

\[
D \geq 0
\]

\[
\Rightarrow -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \geq 0
\]
\[ (a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 \leq 0 \]
\[ (a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 = 0 \quad (2) \]
(since sum of two positive quantities cannot be negative)

From (1) and (2), we get \( D = 0 \). Hence, the roots of the given quadratic equation are not different, if real.

**Example 1.90** If the roots of the equation \( x^2 - 8x + a^2 - 6a = 0 \) are real distinct, then find all possible values of \( a \).

**Sol.** Since the roots of the given equation are real and distinct, we must have
\[ D > 0 \]
\[ 64 - 4(a^2 - 6a) > 0 \]
\[ 4(16 - a^2 + 6a) > 0 \]
\[ -4(a^2 - 6a - 16) < 0 \]
\[ a^2 - 6a + 16 < 0 \]
\[ (a - 8)(a + 2) < 0 \]
\[ -2 < a < 8 \]

Hence, the roots of the given equation are real if \( a \) lies between -2 and 8.

**Example 1.91** Find the quadratic equation with rational coefficients whose one root is \( 1/(2 + \sqrt{5}) \).

**Sol.** If the coefficients are rational, then irrational roots occur in conjugate pair. Given that one root is \( a = 1/(2 + \sqrt{5}) = 2 - \sqrt{5} \), the other root is \( b = 1/(2 - \sqrt{5}) = 2 + \sqrt{5} \).

Sum of roots \( a + b = -4 \) and product of roots \( ab = -1 \). Thus, required equation is \( x^2 + 4x - 1 = 0 \).

**Example 1.92** If \( f(x) = ax^2 + bx + c, g(x) = -ax^2 + bx + c \), where \( ac \neq 0 \), then prove that \( f(x)g(x) = 0 \) has at least two real roots.

**Sol.** Let \( D_1 \) and \( D_2 \) be discriminants of \( ax^2 + bx + c = 0 \) and \( -ax^2 + bx + c = 0 \), respectively. Then,
\[ D_1 = b^2 - 4ac, \quad D_2 = b^2 + 4ac \]
Now, if \( ac \neq 0 \), then \( D_1 > 0 \). Therefore, roots of \( -ax^2 + bx + c = 0 \) are real.

If \( ac < 0 \), then \( D_1 > 0 \). Therefore, roots of \( ax^2 + bx + c = 0 \) are real.

Thus, \( f(x)g(x) \) has at least two real roots.

**Example 1.93** If \( a, b, c \in R \) such that \( a + b + c = 0 \) and \( a \neq c \), then prove that the roots of \( (b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0 \) are real and distinct.

**Sol.** Given equation is
\[ (b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0 \]

or
\[ (-2a)x^2 + (-2b)x + (-2c) = 0 \]

or
\[ ax^2 + bx + c = 0 \]
\[ D = b^2 - 4ac \]
\[ = (-c - a)^2 - 4ac \]
\[ > 0 \]

Hence, roots are real and distinct.

**Example 1.94** If \( \cos \theta, \sin \phi, \sin \theta \) are in G.P., then check the nature of roots of \( x^2 + 2 \cot \phi \cdot x + 1 = 0 \).

**Sol.** We have,
\[ \sin^2 \phi = \cos \theta \sin \theta \]
The discriminant of the given equation is
\[ D = 4 \cot^2 \phi - 4 \]
\[ = 4 \left( \cos^2 \theta - \sin^2 \phi \right) \]
\[ = \frac{4(1 - 2 \sin^2 \theta)}{\sin^2 \theta} \]
\[ = \frac{4(1 - 2 \sin \theta \cos \theta)}{\sin^2 \phi} \]
\[ = \frac{2(\sin \theta - \cos \theta)^2}{\sin \phi} \geq 0 \]

**Example 1.95** If \( a, b, c \) are odd integers, then prove that roots of \( ax^2 + bx + c = 0 \) cannot be rational.

**Sol.** Discriminant \( D = b^2 - 4ac \). Suppose the roots are rational. Then, \( D \) will be a perfect square.

Let \( b^2 - 4ac = d^2 \). Since \( a, b \) and \( c \) are odd integers, \( d \) will be odd. Now,
\[ b^2 - d^2 = 4ac \]
Let \( b = 2k + 1 \) and \( d = 2m + 1 \). Then
\[ b^2 - d^2 = (b - d)(b + d) = 2(k - m)(2(k + m + 1)) \]
Now, either \( (k - m) \) or \( (k + m + 1) \) is always even. Hence \( b^2 - d^2 \) is always a multiple of 8. But, \( 4ac \) is only a multiple of 4 (not of 8), which is a contradiction. Hence, the roots of \( ax^2 + bx + c = 0 \) cannot be rational.

**Quadratic Equations with Complex Coefficients**

Consider the quadratic equation \( ax^2 + bx + c = 0 \), where \( a, b, c \) are complex numbers and \( a \neq 0 \). Roots of equation are given by
\[ \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]
\[ \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

Here nature of roots should not be analyzed by sign of \( b^2 - 4ac \).

**Note:** In case of quadratic equations with real coefficients, imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation \( 4x^2 - 4x - 1 = 0 \) has both roots equal to \( 1/(2i) \).
Number System, Inequalities and Theory of Equations  1.27

Now α is a root of the equation \( ax^2 + bx + c = 0 \)
\[
\Rightarrow \quad \alpha x^2 + \beta x + \gamma = 0
\]
\[
\begin{align*}
\alpha & = \frac{a}{y^2} + \frac{b}{y} + \frac{c}{1} = 0 \\
\Rightarrow \quad \alpha y^2 + \beta y + \gamma & = 0
\end{align*}
\]
Hence, the required equation is \( ax^2 + bx + c = 0 \).
We get same equation if we start with \( 1/\beta \).

(ii) \ Let \( -\alpha = y \Rightarrow \alpha = -y \)
Now α is root of the equation \( ax^2 + bx + c = 0 \)
\[
\Rightarrow \quad \alpha x^2 + \beta x + \gamma = 0
\]
\[
\begin{align*}
\alpha & = \frac{a(-y)}{y^2} + \frac{b(-y)}{y} + \frac{c}{1} = 0 \\
\Rightarrow \quad \alpha y^2 + \beta y + \gamma & = 0
\end{align*}
\]
Hence, the required equation is \( ax^2 + bx + c = 0 \).

(iii) \ Let \( \frac{1-\alpha}{1+\alpha} = y \Rightarrow \frac{1}{1+y} = \alpha = \frac{1}{1+y} \)
Now α is root of the equation \( ax^2 + bx + c = 0 \)
\[
\Rightarrow \quad \alpha x^2 + \beta x + \gamma = 0
\]
\[
\begin{align*}
\alpha & = \frac{a(1-y)}{1+y} + \frac{b(1-y)}{1+y} + \frac{c}{1+y} = 0 \\
\Rightarrow \quad \alpha y^2 + \beta y + \gamma & = 0
\end{align*}
\]
Hence required equation is \( a(1-x)^2 + b(1-x^2) + c(1+x)^2 = 0 \).

Example 1.99 \ If \( a, b \) and \( c \) are in \( \text{A.P.} \) and one root of the equation \( ax^2 + bx + c = 0 \) is \( 2 \), then find the other root.
Sol. \ Let \( \alpha \) be the other root. Then,
\[
4\alpha + 2b + c = 0 \quad \text{and} \quad 2b = a + c
\]
\[
\Rightarrow \quad 5\alpha + 2c = 0
\]
\[
\therefore \quad \frac{c}{\alpha} = -\frac{5}{2}
\]
Now,
\[
2\alpha = \frac{c}{\alpha} = \frac{-5}{\frac{2}{\alpha}}
\]
\[
\therefore \quad \alpha = \frac{-5}{1+\alpha}
\]

Example 1.100 \ If the roots of the quadratic equation \( ax^2 + px + q = 0 \) are \( \tan 30^\circ \) and \( \tan 15^\circ \), respectively, then find the value of \( 2 + q - p \).
Sol. \ The equation \( x^2 + px + q = 0 \) has roots \( \tan 30^\circ \) and \( \tan 15^\circ \). Therefore,
\[
\tan 30^\circ + \tan 15^\circ = -p
\]
\[
\tan 30^\circ \tan 15^\circ = q
\]
Now,
\[
\tan 45^\circ = \tan(30^\circ + 15^\circ)
\]
\[
\Rightarrow \quad \frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ} = 1
\]
\[
\Rightarrow \quad -\frac{p}{1-q} \quad \text{[Using (1) and (2)]}
\]
\[
\Rightarrow \quad 1 - q = -p \Rightarrow q - p = 1
\]
\[ 2 + q - p = 3 \]

**Example 1.101** If the sum of the roots of the equation \(\frac{1}{x + a} + \frac{1}{x + b} = \frac{1}{c}\) is zero, then prove that the product of the roots is \((-\frac{1}{2})(a^2 + b^2)\).

**Sol.** We have,
\[
\frac{1}{x + a} + \frac{1}{x + b} = \frac{1}{c}
\]
\[
\Rightarrow \quad x^2 + (a + b - 2c)x + (ab - bc - ca) = 0
\]
Let \(\alpha, \beta\) be the roots of this equation. Then,
\[
\alpha + \beta = -(a + b - 2c) \quad \text{and} \quad \alpha \beta = ab - bc - ca
\]
It is given that
\[
\alpha + \beta = 0
\]
\[
\Rightarrow \quad -(a + b - 2c) = 0
\]
\[
\Rightarrow \quad c = \frac{a + b}{2}
\]
\[
\therefore \quad \alpha \beta = ab - bc - ca
\]
\[
= ab - \left(\frac{a + b}{2}\right)(a + b)
\]
\[
= \frac{2ab - (a + b)^2}{2} = -\frac{1}{2}(a^2 + b^2)
\]

**Example 1.102** Solve the equation \(x^2 + px + 45 = 0\). It is given that the squared difference of its roots is equal to 144.

**Sol.** Let \(\alpha, \beta\) be the roots of the equation \(x^2 + px + 45 = 0\). Then,
\[
\alpha + \beta = -p
\]
\[
\alpha \beta = 45
\]
It is given that
\[
(\alpha - \beta)^2 = 144
\]
\[
\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha \beta = 144
\]
\[
\Rightarrow \quad p^2 - 4 \times 45 = 144
\]
\[
\Rightarrow \quad p^2 = 324
\]
\[
\Rightarrow \quad p = \pm 18
\]
Substituting \(p = 18\) in the given equation, we obtain
\[
x^2 + 18x + 45 = 0
\]
\[
\Rightarrow \quad (x + 3)(x + 15) = 0
\]
\[
\Rightarrow \quad x = -3, -15
\]
Substituting \(p = -18\) in the given equation, we obtain
\[
x^2 + 18x + 45 = 0
\]
\[
\Rightarrow \quad (x - 3)(x - 15) = 0
\]
\[
\Rightarrow \quad x = 3, 15
\]
Hence, the roots of the given equation are \(-3, -15\) or \(3, 15\).

**Example 1.103** If the ratio of the roots of the equation \(x^2 + px + q = 0\) are equal to the ratio of the roots of the equation \(x^2 + bx + c = 0\), then prove that \(p^2 = b^2q\).

**Sol.** Let \(\alpha, \beta\) be the roots of \(x^2 + px + q = 0\) and \(\gamma, \delta\) be the roots of the equation \(x^2 + bx + c = 0\). Then,
\[
\alpha + \beta = -p, \quad \alpha \beta = q \quad \text{(1)}
\]
\[
\gamma + \delta = b, \quad \gamma \delta = c \quad \text{(2)}
\]
We have,
\[
\frac{\alpha - \gamma}{\beta - \delta} = \frac{\alpha + \beta}{\gamma + \delta} = \frac{-p}{b}
\]
[Using compenendo and dividendo]
\[
\Rightarrow \quad \frac{(\alpha - \gamma)^2}{(\beta - \delta)^2} = \frac{(\alpha + \beta)^2}{(\gamma + \delta)^2} = \frac{\alpha + \beta}{\gamma + \delta} = \frac{-p}{b}
\]
\[
\Rightarrow \quad \frac{(\alpha - \gamma)^2}{(\beta - \delta)^2} = \frac{\alpha + \beta}{\gamma + \delta} = \frac{\alpha + \beta}{\gamma + \delta} = \frac{-p}{b}
\]
\[
\Rightarrow \quad \frac{\alpha - \gamma}{\beta - \delta} = \frac{\alpha + \beta}{\gamma + \delta} = \frac{-p}{b}
\]
\[
\Rightarrow \quad \frac{p^2}{b^2} = \frac{q}{c}
\]
\[
\Rightarrow \quad p^2 = b^2q
\]

**Example 1.104** If \(\sin \theta, \cos \theta\) be the roots of \(ax^2 + bx + c = 0\), then prove that \(b^2 = a^2 + 2ac\).

**Sol.** We have,
\[
\sin \theta + \cos \theta = \frac{b}{a}, \quad \sin \theta \cos \theta = \frac{c}{a}
\]
Now, we know that
\[
\sin^2 \theta + \cos^2 \theta = 1
\]
\[
\Rightarrow \quad (\sin \theta + \cos \theta)^2 - 2\sin \theta \cos \theta = 1
\]
\[
\Rightarrow \quad \frac{b^2}{a^2} - 2 \times \frac{c}{a} = 1
\]
\[
\Rightarrow \quad b^2 = a^2 + 2ac
\]

**Example 1.105** If \(a\) and \(b\) \((\neq 0)\) are the roots of the equation \(x^2 + ax + b = 0\), then find the least value of \(x^2 + ax + b\) \((x \in R)\).

**Sol.** Since \(a\) and \(b\) are the roots of the equation \(x^2 + ax + b = 0\), so
\[
a + b = -a, \quad ab = b
\]
Now,
\[
ab = b \Rightarrow (a - 1)b = 0 \Rightarrow a = 1 \quad (\because b \neq 0)
\]
Putting \(a = 1\) in \(a + b = -a\), we get \(b = -2\). Hence, \(a + b = -a, a = 1\) \((\because b \neq 0)\),
\[
\Rightarrow \quad x^2 + ax + b = x^2 + x - 2 = (x + 1/2)^2 - 1/4 - 2
\]
\[
= (x + 1/2)^2 - 9/4
\]
which has a minimum value \(-9/4\).

**Example 1.106** If the sum of the roots of the equation \((a + 1)x^2 + (2a + 3)x + (3a + 4) = 0\) is \(-1\), then find the product of the roots.

**Sol.** Let \(\alpha, \beta\) be roots of the equation \((a + 1)x^2 + (2a + 3)x + (3a + 4) = 0\). Then,
\[
\alpha + \beta = -1 \Rightarrow \left(\frac{2a + 3}{a + 1}\right) = -1 \Rightarrow a = -2
\]
Now, product of the roots is \((3a + 4)(a + 1) = (-6 + 4)(-2 + 1) = 2\).
**Example 1.107** Find the value of \(a\) for which one root of the quadratic equation \((a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0\) is twice as large as the other.

**Sol.** Let the roots be \( \alpha \) and \(2\alpha\). Then,

\[
\alpha + 2\alpha = \frac{1 - 3a}{a^2 - 5a + 3}, \quad \alpha \times 2\alpha = \frac{2}{a^2 - 5a + 3}
\]

\[
\Rightarrow 2 \cdot \frac{1 - 3a}{9 - 5a + 3} = \frac{2}{a^2 - 5a + 3}
\]

\[
\Rightarrow \frac{(1 - 3a)^2}{\alpha^2 - 5a + 3} = 9 \Rightarrow 9\alpha^2 - 6a + 1 = 9a^2 - 45a + 27
\]

\[
\Rightarrow 39a = 26 \Rightarrow a = \frac{2}{3}
\]

**Example 1.108** If the difference between the roots of the equation \(x^2 + ax + 1 = 0\) is less than \(\sqrt{5}\), then find the set of possible values of \(a\).

**Sol.** If \(\alpha, \beta\) are roots of \(x^2 + ax + 1 = 0\), then

\[
1 - \beta < \sqrt{5}
\]

\[
\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta < \sqrt{5}
\]

\[
\Rightarrow \sqrt{\alpha^2 - 4} < \sqrt{5}
\]

\[
\Rightarrow \alpha^2 - 4 < 5
\]

\[
\Rightarrow \alpha^2 < 9
\]

\[
\Rightarrow -3 < \alpha < 3
\]

\[
\therefore \alpha \in (-3, 3)
\]

**Example 1.109** Find the values of the parameter \(a\) such that the roots \(\alpha, \beta\) of the equation \(2x^2 - 6x + a = 0\) satisfy the inequality \(a/\beta + \beta/a < 2\).

**Sol.** We have \(\alpha + \beta = 3\) and \(a\beta = \alpha/2\). Now,

\[
\frac{\alpha + \beta}{\alpha\beta} < 2
\]

\[
\Rightarrow \frac{(\alpha + \beta)^2}{2a\beta} < 2
\]

\[
\Rightarrow \frac{9 - 2a}{2a} < 2
\]

\[
\Rightarrow \frac{9 - 2a}{a} < 1
\]

\[
\Rightarrow \frac{9 - 2a}{a} < 0
\]

\[
\Rightarrow 9 - 2a > 0
\]

\[
\Rightarrow a < 0 \text{ or } a > 9/2
\]

**Example 1.110** If the harmonic mean between roots of \((5 + \sqrt{2})x^2 - bx + 8 + 2\sqrt{5} = 0\) is 4, then find the value of \(b\).

**Sol.** Let \(\alpha, \beta\) be the roots of the given equation whose H.M. is 4. Then,

\[
4 = \frac{2\alpha\beta}{\alpha + \beta}
\]

\[
\Rightarrow \frac{8 + 2\sqrt{5}}{5 + \sqrt{2}} = \frac{5 + \sqrt{2}}{b}
\]

\[
\Rightarrow 2 = \frac{8 + 2\sqrt{5}}{b} \Rightarrow b = 4 + \sqrt{5}
\]

**Example 1.111** If \(\alpha, \beta\) are the roots of the equation \(2x^2 - 3x - 6 = 0\), find the equation whose roots are \(\alpha^2 + 2\) and \(\beta^2 + 2\).

**Sol.** Since \(\alpha, \beta\) are roots of the equation \(2x^2 - 3x - 6 = 0\), so

\[
\alpha + \beta = 3/2 \text{ and } \alpha\beta = -3
\]

\[
\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{9}{4} + 6 = \frac{33}{4}
\]

Now,

\[
(a^2 + 2) + (\beta^2 + 2) = (a^2 + \beta^2) + 4 = \frac{33}{4} + 4 = \frac{49}{4}
\]

and

\[
(a^2 + 2)(\beta^2 + 2) = a^2\beta^2 + 2(a^2 + \beta^2) + 4
\]

\[
= (3)^2 + 2\left(\frac{33}{4}\right) + 4
\]

\[
= 9 + \frac{33}{2} + 4
\]

\[
= \frac{59}{2}
\]

So, the equation whose roots are \(\alpha^2 + 2\) and \(\beta^2 + 2\) is

\[
x^2 - x[(\alpha^2 + 2) + (\beta^2 + 2)] + (a^2 + 2)(\beta^2 + 2) = 0
\]

\[
\Rightarrow x^2 - \frac{49}{4}x + \frac{59}{2} = 0
\]

\[
\Rightarrow 4x^2 - 49x + 118 = 0
\]

**Example 1.112** If \(\alpha \neq \beta\) and \(a^2 = 5\alpha - 3\) and \(b^2 = 5\beta - 3\), find the equation whose roots are \(a/\beta\) and \(b/\alpha\).

**Sol.** We have \(a^2 = 5\alpha - 3\) and \(b^2 = 5\beta - 3\). Hence, \(\alpha, \beta\) are roots of \(x^2 = 5x - 3\), i.e., \(x^2 - 5x + 3 = 0\). Therefore,

\[
\alpha + \beta = 5 \text{ and } \alpha\beta = 3
\]

Now,

\[
\frac{a + \beta}{\alpha\beta} = \frac{a^2 + \beta^2}{\alpha\beta}
\]

\[
= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}
\]

\[
= \frac{25 - 6}{3} = \frac{19}{3}
\]
and 
\[ P = \frac{\alpha}{\beta} \frac{\beta}{\alpha} = 1 \]

So, the required equation is 
\[ x^2 - Sx + P = 0 \]
\[ \Rightarrow x^2 - \frac{19}{3}x + 1 = 0 \]
\[ \Rightarrow 3x^2 - 19x + 3 = 0 \]

**Example 1.1.3** If \( \alpha, \beta \) are the roots of the equation \( ax^2 + bx + c = 0 \), then find the roots of the equation \( ax^2 - bx(x - 1) + c(x - 1)^2 = 0 \) in terms of \( \alpha \) and \( \beta \).

**Sol.** \( ax^2 - bx(x - 1) + c(x - 1)^2 = 0 \)
\[ \Rightarrow \frac{ax^2}{a} + \frac{bx}{c} + \frac{c(x - 1)^2}{a} = 0 \]  

(1)

Now, \( \alpha \) is a root of \( ax^2 + bx + c = 0 \). Then let
\[ \alpha = \frac{x}{1 - x} \]
\[ \Rightarrow x = \frac{\alpha}{\alpha + 1} \]

Hence, the roots of (1) are \( \alpha/(1+\alpha), \beta/(1+\beta) \).

**Concept Application Exercise 1.5**

1. If the product of the roots of the equation \((a+1)x^2 + (2a+3)x + (3a+4) = 0 \) is 2, then find the sum of roots.
2. Find the value of \( a \) for which the sum of the squares of the roots of the equation \( x^2 - (a-2)x - a - 1 = 0 \) assumes the least value.
3. If \( x_1 \) and \( x_2 \) are the roots of \( x^2 + \sin \theta - 1)x - 1/2 \cos \theta = 0 \), then find the maximum value of \( x_1^2 + x_2^2 \).
4. If \( \tan \theta \) and \( \sec \theta \) are the roots of \( ax^2 + bx + c = 0 \), then prove that \( \alpha^2 = b^2(4ac-b^2) \).
5. If the roots of the equation \( x^2 - bx + c = 0 \) be two consecutive integers, then find the value of \( b^2 - 4c \).
6. If the roots of the equation \( 12x^2 - mx + 5 = 0 \) are in the ratio 2:3, then find the value of \( m \).
7. If \( \alpha, \beta \) are the roots of \( x^2 + px + 1 = 0 \) and \( \gamma, \delta \) are the roots of \( x^2 + qx + 1 = 0 \), then prove that \( \alpha - \gamma = (\alpha - \gamma) (\frac{\beta}{\gamma} - \frac{\beta}{\delta}) \times (\frac{\alpha}{\beta} - \frac{\alpha}{\delta}) \).
8. If the equation formed by decreasing each root of \( ax^2 + bx + c = 0 \) by 1 is \( 2x^2 + 8x + 2 = 0 \), find the condition.
9. If \( \alpha, \beta \) be the roots of \( x^2 - (a-1)x + b = 0 \), then find the value of \( (a/\alpha)^2 + (b/\beta)^2 + 2(a/\alpha + b/\beta) \).
10. If \( \alpha, \beta \) are roots of \( 375x^2 - 25x - 2 = 0 \) and \( \gamma, \delta \) are \( \alpha + \beta \) and \( \alpha^2 + \beta^2 \), then find the value of \( \lim_{n \to \infty} \sum_{r=1}^{n} s_r \).
11. If \( \alpha \) and \( \beta \) are the roots of the equation \( 2x^2 + 2(a+b)x + a^2 + b^2 = 0 \), then find the equation whose roots are \( (\alpha + \beta)^2 \) and \( (\alpha - \beta)^2 \).
12. If the sum of the roots of an equation is 2 and sum of their cubes is 98, then find the equation.
13. Let \( \alpha, \beta \) be the roots of \( x^2 + bx + 1 = 0 \). Then find the equation whose roots are \( -\alpha + 1/\beta \) and \(-\beta + 1/\alpha \).

**COMMON ROOT(S)**

**Condition for One Common Root**

Let us find the condition that the quadratic equations \( ax^2 + bx + c = 0 \) and \( a_2x^2 + b_2x + c_2 = 0 \) may have a common root. Let \( \alpha \) be the common root of the given equations. Then,
\[ a\alpha^2 + b\alpha + c = 0 \]
and
\[ a_2\alpha^2 + b_2\alpha + c_2 = 0 \]

Solving these two equations by cross-multiplication, we have
\[ \frac{\alpha^2}{\beta(c_2 - c)} = \frac{\alpha}{\beta(b_2 - b)} = \frac{1}{\beta(a_2 - a)} \]
\[ \Rightarrow \alpha^2 = \frac{b_2c_1 - b_1c_2}{a_2b_1 - a_1b_2} \]

(From first and third)

and
\[ \alpha = \frac{c_2c_1 - c_1c_2}{a_2b_1 - a_1b_2} \]

(From second and third)

This condition can easily be remembered by cross-multiplication method as shown in the following figure.

![Fig. 1.40](image)

(3) Product of the two smaller crosses

This is the condition required for a root to be common to two quadratic equations. The common root is given by
\[ \alpha = \frac{c_2c_1 - c_1c_2}{a_2b_1 - a_1b_2} \]

or
\[ \alpha = \frac{b_2c_1 - b_1c_2}{a_2b_1 - a_1b_2} \]

**Note:** The common root can also be obtained by making the coefficient of \( x^2 \) common to the two given equations and then subtracting the two equations. The other roots of the given equations can be determined by using the relations between their roots and coefficients.

**Condition for Both the Common Roots**

Let \( \alpha, \beta \) be the common roots of the quadratic equations \( ax^2 + bx + c = 0 \) and \( a_2x^2 + b_2x + c_2 = 0 \). Then, both the equations are identical, hence,
Example 1.114 Determine the values of $m$ for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

Sol. Let $\alpha$ be the common root of the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$. Then, $\alpha$ must satisfy both the equations. Therefore,

\[
\begin{align*}
3\alpha^2 + 4m\alpha + 2 &= 0 \\
2\alpha^2 + 3\alpha - 2 &= 0
\end{align*}
\]

Using cross-multiplication method, we have

\[
\frac{(6 - 4)\alpha}{(9 - 8m)} = \frac{(9 - 8m) - (8m - 6)}{8m - 9}
\]

\[
\Rightarrow \frac{\alpha}{m} = \frac{(9 - 8m) - (8m - 6)}{8m - 9} = \frac{15}{8m - 9}
\]

\[
\Rightarrow m = \frac{15}{8m - 9}
\]

Example 1.115 If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have common roots and $a, b, c \in \mathbb{N}$, then find the minimum value of $a + b + c$.

Sol. The roots of $x^2 + 3x + 5 = 0$ are non-real. Thus given equations will have two common roots. We have,

\[
\begin{align*}
a &= \frac{b}{c} = \frac{c}{a} \\
\Rightarrow a + b + c &= 9a
\end{align*}
\]

Thus minimum value of $a + b + c$ is 9.

Example 1.116 If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and $a, b$ and $c$ are non-zero real numbers then find the value of $(a^2 + b^2 + c^2)/abc$.

Sol. Given that $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Hence,

\[
\begin{align*}
(bx - a)^2 &= (ab - c) (ac - b^2) \\
\Rightarrow b^2c^2 + a^2 - 2ab^2c &= a^2bc - ab^2 - ac^2 + b^2c^2 \\
\Rightarrow a^2 + ab^2 + ac^2 &= 3abc
\end{align*}
\]

Example 1.117 If $a, b, c$ are positive real numbers forming a G.P. If $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then prove that $da, eb, fc$ are in A.P.

Sol. For first equation $D = 4b^2 - 4ac = 0$ (as given $a, b, c$ are in G.P.). The equation has equal roots which are equal to $-b/a$ each. Thus, it should also be the root of the second equation. Hence,

\[
\begin{align*}
d\left(\frac{-b}{a}\right)^2 - 2\frac{be}{a} + f &= 0 \\
\Rightarrow d\frac{b^2}{a^2} - 2\frac{be}{a} + f &= 0
\end{align*}
\]

Example 1.118 If the equations $x^2 + ax + 12 = 0$ and $x^2 + bx + 15 = 0$ have a common positive root, then find the values of $a$ and $b$.

Sol. We have,

\[
\begin{align*}
x^2 + ax + 12 &= 0 \\
x^2 + bx + 15 &= 0
\end{align*}
\]

Adding (1) and (2), we get

\[
x^2 + (a + b)x + 27 = 0
\]

Now subtracting it from the third given equation, we get

\[
x^2 + 9x - 3 = 0 \
\Rightarrow x^2 - 9 = 0 \
\Rightarrow x = 3, -3
\]

Thus, common positive root is 3. Hence,

\[
9 + 12 + 3a = 0 \\
\Rightarrow a = -7 \
\Rightarrow 9 + 3b + 15 = 0 \\
\Rightarrow b = -8
\]

Example 1.119 The equations $ax^2 + bx + a = 0$ and $x^2 - 2x^2 + 2x - 1 = 0$ have two roots common. Then find the value of $a + b$.

Sol. By observation, $x = 1$ is a root of equation $x^2 - 2x^2 + 2x - 1 = 0$. Thus we have

\[
(x - 1)(x^2 - x + 1) = 0
\]

Thus roots of $x^2 - x + 1 = 0$ are non-real.

Then equation $ax^2 + bx + a = 0$ has both roots common with $x^2 - x + 1 = 0$. Hence, we have

\[
\begin{align*}
a &= b \\
1 &= -1 \\
\Rightarrow a + b &= 0
\end{align*}
\]

Concept Application Exercise 1.6

1. If $x^2 + ax + b = 0$ and $x^2 + bx + ca = 0 (a \neq b)$ have a common root, then prove that their other roots satisfy the equation $x^2 + cx + ab = 0$. 

Proof: Let $\alpha$ be a common root.

\[
\begin{align*}
\alpha^2 + a\alpha + b &= 0 \\
\alpha^2 + b\alpha + ca &= 0
\end{align*}
\]

Subtracting, we get

\[
\alpha^2 - c\alpha + ab = 0
\]

or

\[
\alpha^2 + cx + ab = 0
\]
2. Find the condition that the expressions \( ax^2 + bxy + cy^2 \) and \( a, b, x, y, z \) may have factors \( y - mx \) and \( m - x \), respectively.

3. If \( a, b, c \in \mathbb{R} \) and equations \( ax^2 + bx + c = 0 \) and \( x^2 + 2x + 9 = 0 \) have a common root, then find \( a, b, c \).

4. Find the condition on \( a, b, c, d \) such that equations \( 2ax^2 + bx^2 + cx + d = 0 \) and \( 2ax^3 + 3bx + 4c = 0 \) have a common root.

5. Let \( f(x), g(x) \) and \( h(x) \) be the quadratic polynomials having positive leading coefficients and real and distinct roots. If each pair of them has a common root, then find the roots of \( f(x) + g(x) + h(x) = 0 \).

**RELATION BETWEEN COEFFICIENT AND ROOTS OF \( n \)-DEGREE EQUATIONS**

- Let \( \alpha \) and \( \beta \) be roots of quadratic equation \( ax^2 + bx + c = 0 \). Then by factor theorem
  \[ ax^2 + bx + c = a(x - \alpha)(x - \beta) \]
  Comparing coefficients, we have
  \[ a + \beta = -\frac{b}{a} \text{ and } a\beta = \frac{c}{a} \]

- Let \( \alpha, \beta, \gamma \) be roots of cubic equation \( ax^3 + bx^2 + cx + d = 0 \). Then,
  \[ ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma) \]
  Comparing coefficients, we have
  \[ a + \beta + \gamma = \frac{b}{a} \]
  \[ a\beta + a\gamma + \beta\gamma = \frac{c}{a} \]
  \[ a\beta\gamma = \frac{d}{a} \]

- If \( \alpha, \beta, \gamma, \delta \) are roots of \( ax^4 + bx^3 + cx^2 + dx + e = 0 \), then
  \[ a + \beta + \gamma + \delta = -\frac{b}{a} \]
  \[ a\beta + a\gamma + a\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} \]
  \[ a\beta\gamma + a\beta\delta + a\gamma\delta + \beta\gamma\delta = -\frac{d}{a} \]
  \[ a\beta\gamma\delta = \frac{e}{a} \]

In general, if \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \) are the roots of equation \( \alpha x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0 = 0 \), then sum of the roots is
\[ \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = -\frac{\alpha_{n-1}}{\alpha_0} \]

Sum of the product taken two at a time is
\[ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n = \frac{\alpha_{n-2}}{\alpha_0} \]

Sum of the product taken three at a time is \( -\alpha_1 a_2 \alpha_3 \) and so on. Product of all the roots is
\[ \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n = (-1)^n \frac{\alpha_0}{\alpha_0} \]

**Note:**

- A polynomial equation of degree \( n \) has \( n \) roots (real or imaginary).
- If all the coefficients are real then the imaginary roots occur in conjugate pairs, i.e., the number of imaginary roots is always even.
- If the degree of a polynomial equation is odd, then the number of real roots will also be odd. It follows that at least one of the roots will be real.

**SOLVING CUBIC EQUATION**

By using factor theorem together with some intelligent guessing, we can factorise polynomials of higher degree.

In summary, to solve a cubic equation of the form \( ax^3 + bx^2 + cx + d = 0 \):

1. Obtain one factor \( (x - \alpha) \) by trial and error
2. Factorise \( ax^3 + bx^2 + cx + d = 0 \) as \( (x - \alpha)(bx^2 + kx + s) = 0 \)
3. Solve the quadratic expression for other roots

**Example 1.120** If \( \alpha, \beta, \gamma \) are the roots of the equation \( x^3 + 4x + 1 = 0 \), then find the value of \( (\alpha + \beta + \gamma)^3 + (\beta + \gamma)^3 + (\gamma + \alpha)^3 + (\alpha + \beta)^3 \).

**Sol.** For the given equation \( \alpha + \beta + \gamma = 0 \),
\[ \alpha\beta + \beta\gamma + \gamma\alpha = 4, \quad \alpha\beta\gamma = -1 \]
Now,
\[ (\alpha + \beta + \gamma)^3 + (\beta + \gamma)^3 + (\gamma + \alpha)^3 + (\alpha + \beta)^3 \]
\[ = -\alpha\beta\gamma + 3\alpha\beta\gamma \]
\[ = -1 \]
\[ = 4 \]

**Example 1.121** Let \( \alpha + i\beta \) \( (\alpha, \beta \in \mathbb{R}) \) be a root of the equation \( x^3 + qx + r = 0, q, r \in \mathbb{R} \). Find a real cubic equation, independent of \( \alpha \) and \( \beta \), whose one root is \( 2a \).

**Sol.** If \( \alpha + i\beta \) is a root then, \( \alpha - i\beta \) will also be a root. If the third root is \( \gamma \), then
\[ (\alpha + \beta) + (\alpha - i\beta) + \gamma = 0 \]
\[ \Rightarrow \gamma = -2\alpha \]
But \( \gamma \) is a root of the given equation \( x^3 + qx + r = 0 \). Hence,
\[ (2\alpha)^3 + q(2\alpha) + r = 0 \]
\[ \Rightarrow (2\alpha)^3 + q(2\alpha) - r = 0 \]
Therefore, \( 2\alpha \) is a root of \( t^3 + qt - r = 0 \), which is independent of \( \alpha \) and \( \beta \).

**Example 1.122** In equation \( x^4 - 2x^3 + 4x^2 + 6x - 21 = 0 \) if two of its roots are equal in magnitude but opposite in sign, find the roots.

**Sol.** Given that \( \alpha + \beta = 0 \) but \( \alpha + \beta + \gamma + \delta = 2 \). Hence,
\[ \gamma + \delta = 2 \]
Let \( \alpha\beta = p \) and \( \gamma\delta = q \). Therefore, given equation is equivalent to \( (x^2 + p)(x^2 - 2x + q) = 0 \). Comparing the coefficients, we get
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Example 1.123

Solve the equation \( x^3 - 13x^2 + 15x + 189 = 0 \) if one root exceeds the other by 2.

Sol. Let the roots be \( a, a + 2, b \). Sum of roots is \( 2a + b + 2 = 13 \).
\[ a + 2a + 2b + 2 = 13 \]  
\[ 3a + 2b = 11 \]  
\[ (3a + 2b) - 2a = 11 - 2 \]  
\[ a + 2b = 9 \]

From (1) and (2), we get
\[ a^2 + 2a + 2(9 - a) = 15 \]
\[ 3a^2 - 10a - 7 = 0 \]
\[ (a - 7)(3a + 1) = 0 \]
\[ a = 7 \quad 3a + 1 = 0 \]
\[ \beta = -3, \quad \frac{35}{3} \]

Out of these values, \( a = 7, \beta = -3 \) satisfy the third relation
\[ a\beta(a + 2) = -189, \] i.e., \( -2(7)(9) = -189 \). Hence, the roots are 7, 9, -3.

REPEATED ROOTS

In equation \( f(x) = 0 \), where \( f(x) \) is a polynomial function, and if it has roots \( a, a, \beta, \ldots \) or \( a \) is a repeated root, then \( f(x) = 0 \) is equivalent to \( (x - a)^r \) \( (x - b) \cdots = 0 \), from which we can conclude that \( f'(x) = 0 \) or \( 2(x - a)g(x - b) \cdots \)
\[ f''(x) = 0 \] or \( (x - a)^2 \) \( (x - b) \cdots \)
\[ f''(x) = 0 \] or \( (x - a)^2 \) \( (x - b) \cdots \)

Thus if a root occurs twice in equation, then it is common in equations \( f(x) = 0 \) and \( f'(x) = 0 \).

Similarly, if root \( a \) occurs thrice in equation, then it is common in the equations \( f(x) = 0 \) and \( f''(x) = 0 \).

Example 1.124

If \( x - c \) is a factor of order \( m \) of the polynomial \( f(x) \) of degree \( n \) \( (1 < m < n) \), then find the polynomials for which \( x = c \) is a root.

Sol. From the given information, we have \( f(x) = (x - c)^n g(x) \), where \( g(x) \) is polynomial of degree \( n - m \). Then \( x = c \) is common root for the equations \( f(x) = 0 \), \( f'(x) = 0 \), \( f''(x) = 0 \), ... \( f^{(n-m)}(x) = 0 \), where \( f^{(r)}(x) \) represents \( r \)-th derivative of \( f(x) \) w.r.t. \( x \).

Example 1.125

If \( ax^3 + bx^2 + cx + d = 0 \) and \( ax^3 + bx^2 + cx + d = 0 \) have a pair of repeated roots common, then prove that
\[
\begin{vmatrix}
3a_1 & 2b_1 & c_1 \\
3a_2 & 2b_2 & c_2 \\
a_1b_1 - a_2b_2 & c_1 - c_2 \\
d_1a_2 - d_2a_1
\end{vmatrix} = 0
\]

Concept Application Exercise 1.7

1. If \( b^2 < 2ac \), then prove that \( ax^2 + bx + c = 0 \) has exactly one real root.
2. If two roots of \( x^2 + ax + b = 0 \) are equal in magnitude but opposite in sign, then prove that \( ab = c \).
3. If \( a, b, \gamma \) are the roots of \( x^2 + 8 = 0 \), then find the equation whose roots are \( a^4, b^2, \gamma \).
4. If \( a, b, \gamma \) are the roots of the equation \( x^2 - px + q = 0 \), then find the cubic equation whose roots are \( a^2 + b^2, a\gamma, \gamma^2 \).
5. If the roots of the equations \( ax^2 + bx + c = 0 \) and \( a'x^2 + b'x + c' = 0 \) are \( a', a \), and \( a, b \), then find the equation whose roots are \( a', a, a \).

QUADRATIC EXPRESSION IN TWO VARIABLES

The general quadratic expression \( ax^2 + 2hxy + by^2 + c \) can be factorized into two linear factors. Given quadratic expression is
\[
ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0
\]

Corresponding equation is
\[
ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0
\]

or
\[
ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0
\]

\[
\frac{-2(hy + gy) \pm \sqrt{(4(hy + gy)^2 - 4(4hy - 2gy + c)}}}{2a}
\]

\[
\Rightarrow x = \pm \frac{(by - ax) \pm \sqrt{(by - ax)^2 + 2gy - 2gy + c}}{2a}
\]

\[
= a + hy + g = \pm \frac{(by - ax)^2 + 2gy - ab}{2a}
\]

Now, expression (1) can be resolved into two linear factors if \((h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac\) is a perfect square and \(h^2 - ab > 0\). But \((h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac\) will be a perfect square if
MATHEMATICS

Calculus

G. Tewani

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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

G. TeWani
# Functions

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NUMBER SYSTEM AND INEQUALITIES

Number System

Natural Numbers
The set of numbers \( \{1, 2, 3, 4, \ldots \} \) is called natural numbers, and is denoted by \( N \), i.e., \( N = \{1, 2, 3, \ldots \} \).

Integers
The set of numbers \( \{-\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \) is called integers, and the set is denoted by \( I \) or \( Z \).

Here, we represent
- Positive integers = \( \{1, 2, 3, 4, \ldots \} \) = Natural numbers
- Negative integers = \( \{-\ldots, -3, -2, -1\} \) = Whole numbers
- Non-positive integers (or \( N_0 \)) = \( \{0, 1, 2, 3, 4, \ldots \} \)
- Non-negative integers = \( \{-\ldots, -3, -2, -1, 0\} \)

Rational Numbers
A number which can be written as \( \frac{a}{b} \), where \( a \) and \( b \) are integers, \( b \neq 0 \) and H.C.F. of \( a \) and \( b \) is 1, is called a rational number, and a set of rational numbers is denoted by \( Q \).

Note:
- Every integer is a rational number as it could be written as \( Q = \frac{a}{b} \) (where \( b = 1 \)).
- All recurring decimals are rational numbers; e.g., \( n = 0.333\ldots = \frac{1}{3} \).
- "Two consecutive rational numbers" is meaningless.
- The set of rational numbers cannot be expressed in roster form.

Irrational Numbers
Those values which could be neither terminated nor expressed as recurring decimals are irrational numbers (i.e., such numbers cannot be expressed in \( \frac{a}{b} \) form). Their set is denoted by \( Q^c \) (i.e., complement of \( Q \)), e.g., \( \sqrt{2}, \pi, -\frac{1}{\sqrt{3}}, 2 + \sqrt{2}, \ldots \)

Note:
- "Two consecutive irrational numbers" is meaningless.
- The set of irrational numbers cannot be expressed in roster form.

Real Numbers
The set of numbers that contains both rational and irrational numbers is called real numbers and is denoted by \( R \). As from, the above definitions, it could be shown that real numbers can be expressed on number line with respect to origin as

Intervals
The set of numbers between any two real numbers is called interval. The following are the types of interval.

Closed Interval
\( x \in [a, b] = \{x : a \leq x \leq b\} \)

Open Interval
\( x \in (a, b) \) or \( ]a, b[ = \{x : a < x < b\} \)

Semi-Open or Semi-Closed Interval
\( x \in [a, b[ \) or \( (a, b] = \{x : a \leq x < b\} \)

Some Facts About Inequalities
The following are some very useful points to remember:

- If \( a \leq b \) then either \( a < b \) or \( a = b \)
- If \( a < b \) and \( b < c \) then \( a < c \)
- If \( a < b \) then \( -a > -b \), i.e., the inequality sign reverses if both sides are multiplied by a negative number
- If \( a < b \) and \( c < d \) then \( a + c < b + d \) and \( a - d < b - c \)
- If \( a < b \) then \( ka < kb \) if \( k > 0 \) and \( ka > kb \) if \( k < 0 \)
- If \( 0 < a < b \) then \( a^r < b^r \) if \( r > 0 \) and \( a^r > b^r \) if \( r < 0 \)
- \( a + \frac{1}{a} \geq 2 \) for \( a > 0 \) and equality holds for \( a = 1 \)
- \( a + \frac{1}{a} \leq -2 \) for \( a < 0 \) and equality holds for \( a = -1 \)
- If \( x > 2 \) then \( 0 < \frac{1}{x} < \frac{1}{2} \)
- If \( x < -3 \) then \( -\frac{1}{x} < 0 \)
- If \( x < 2 \), then we must consider \( -\infty < x < 0 \) or \( 0 < x < 2 \)

For \( x = 0 \), \( \frac{1}{x} \) is not defined.
\[
\lim_{x \to +\infty} \frac{1}{x} < \frac{1}{x} < \frac{1}{x+1} \quad \text{or} \quad \lim_{x \to -\infty} \frac{1}{x} > \frac{1}{x} > \frac{1}{x-1}
\]

\[
\Rightarrow 0 > \frac{1}{x} > \frac{1}{x+1} \quad \text{or} \quad \frac{1}{x} > 0 \frac{1}{x-1} > \frac{1}{x} > 2
\]

\[\Rightarrow \frac{1}{x} \in (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)\]

**Generalized Method of Intervals**

Let \( F(x) = (x-a_1)^{k_1} (x-a_2)^{k_2} \cdots (x-a_n)^{k_n} (x-a_{n+1})^{k_{n+1}} \).

Here \( k_1, k_2, \ldots, k_n \in \mathbb{Z} \) and \( a_1, a_2, \ldots, a_n \) are fixed real numbers satisfying the condition

\[ a_1 < a_2 < a_3 < \cdots < a_{n-1} < a_n. \]

For solving \( F(x) > 0 \) or \( F(x) < 0 \), consider the following algorithm:

- We mark the numbers \( a_1, a_2, \ldots, a_n \) on the number axis and put plus sign in the interval on the right of the largest of these numbers, i.e., on the right of \( a_n \).
- Then we put plus sign in the interval on the left of \( a_n \) if \( k_n \) is an even number and minus sign if \( k_n \) is an odd number.
- In the next interval, we put a sign according to the following rule:
  - When passing through the point \( a_{n-1} \), the polynomial \( F(x) \) changes sign if \( k_{n-1} \) is an odd number. Then we consider the next interval and put a sign in it using the same rule.

Thus we consider all the intervals: The solution of the inequality \( F(x) > 0 \) is the union of all intervals in which we put plus sign and the solution of the inequality \( F(x) < 0 \) is the union of all intervals in which we put minus sign.

**Frequently Used Inequalities**

a. \((x-a)(x-b) < 0 \Rightarrow x \in (a, b)\), where \( a < b \)

b. \((x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)\), where \( a < b \)

c. \( x^2 \geq a^2 \Rightarrow x \in (-\infty, -a) \cup [a, \infty) \)

d. \( x^2 \geq a^2 \Rightarrow x \in (-\infty, -a) \cup [a, \infty) \)

e. If \( ax^2 + bx + c < 0, (a > 0) \Rightarrow x \in (\alpha, \beta), \) where \( \alpha, \beta \) \( (\alpha < \beta) \) are the roots of the equation \( ax^2 + bx + c = 0 \)

f. If \( ax^2 + bx + c > 0, (a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty), \) where \( \alpha, \beta \) \( (\alpha < \beta) \) are the roots of the equation \( ax^2 + bx + c = 0 \)

**Example 1.1** Solve \((2x+1)(x-3)(x+7) < 0\).

**Sol.** \((2x+1)(x-3)(x+7) < 0\)

**Sign scheme of \((2x+1)(x-3)(x+7)\) is as follows:**

\[
\begin{array}{c|ccc}
\hline
& -7 & -1/2 & 3 \\
\hline
+ & + & + & + \\
\hline
\end{array}
\]

Hence, solution is \((-\infty, -7) \cup (-1/2, 3)\).

**Example 1.2** Solve \(\frac{2}{x} < 3\).

**Sol.** \(\frac{2}{x} < 3\)

\[
\Rightarrow \frac{2}{x} - 3 < 0
\]

\[
\Rightarrow \frac{2-3x}{x} < 0
\]

\[
\Rightarrow \frac{2-3x}{x} < 0
\]

\[
\Rightarrow \frac{3x-2}{x} > 0
\]

\[
\Rightarrow \frac{x}{x-2/3} > 0
\]

\[
\Rightarrow x \in (0, 2/3) \cup (2/3, \infty).
\]

**Example 1.3** Solve \(\frac{2x-3}{3x-5} \geq 3\).

**Sol.** \(\frac{2x-3}{3x-5} \geq 3\)

\[
\Rightarrow \frac{2x-3}{3x-5} - 3 \geq 0
\]

\[
\Rightarrow \frac{2x-3-9x+15}{3x-5} \geq 0
\]

\[
\Rightarrow \frac{-7x+12}{3x-5} \geq 0
\]

\[
\Rightarrow \frac{7x-12}{3x-5} \leq 0
\]

**Sign scheme of \(\frac{x-2/3}{3x-5}\) is as follows:**

\[
\begin{array}{c|ccc}
\hline
& 0 & 2/3 \\
\hline
+ & - & + \\
\hline
\end{array}
\]

**Example 1.4** Solve \((x-1)^2(x+4) < 0\).

**Sol.** \((x-1)^2(x+4) < 0\)

**Sign scheme of \((x-1)^2(x+4)\) is as follows:**
1.4 Calculus

Fig. 1.8

Sign of expression does not change at \( x = 1 \) as \((x - 1)\) factor has even power.
Hence, solution of \((1)\) is \( x \in (\infty, -4) \).

Example 1.5 Solve \( x > \sqrt{1-x} \).

Sol. Given inequality can be solved by squaring both sides.
But sometimes squaring gives extraneous solutions that do not satisfy the original inequality. Before squaring, we must restrict \( x \) for which terms in the given inequality are well-defined.
\( x > \sqrt{1-x} \). Here \( x \) must be positive.
Here \( \sqrt{1-x} \) is defined only when \( 1-x \geq 0 \) or \( x \leq 1 \)

Squaring the given inequality we get \( x^2 > 1-x \)

\( x^2 + x - 1 > 0 \) \( \Rightarrow \) \( x = \frac{-1 - \sqrt{5}}{2} \) or \( x = \frac{-1 + \sqrt{5}}{2} \)

\( \Rightarrow x < \frac{-1 - \sqrt{5}}{2} \) or \( x > \frac{-1 + \sqrt{5}}{2} \) \( \quad (2) \)

From \((1)\) and \((2)\), \( x \in \left( \frac{-1 + \sqrt{5}}{2}, 1 \right) \) (as \( x \) is +ve)

Example 1.6 Find the domain of \( f(x) = \sqrt{1-\sqrt{1-x^2}} \).

Sol. \( f(x) = \sqrt{1-\sqrt{1-x^2}} \)

\( \Rightarrow 1-\sqrt{1-x^2} \geq 0 \)

\( \Rightarrow \sqrt{1-x^2} \leq 1 \)

\( \Rightarrow \sqrt{1-x^2} \leq \sqrt{1} \)

\( \Rightarrow x \leq 1 \Rightarrow x \in [-1, 1] \).

Sign Scheme of \( f(x) = f_1(x) f_2(x) f_3(x) \ldots f_n(x) \)
Put the values of \( x \), which are roots of the equation, \( f_1(x) = 0, f_2(x) = 0, \ldots, f_n(x) = 0 \) on the number line and follow the same procedure explained in the above problems.

Example 1.7 Solve \((x-1)(x+1)\cos x > 0\), for \( x \in [-\pi, \pi] \).

Sol. Let \( f(x) = (x-1)(x+1)\cos x \)

\[ \begin{array}{c|cccc} & \pi & -\pi/2 & -1 & 1 & \pi/2 & \pi \\ \hline + & + & - & - & + & - & + \end{array} \]

FUNCTION

Roughly speaking, term function is used to define the dependence of one physical quantity on another, e.g., volume \( V \) of a sphere of radius \( r \) is given by \( V = \frac{4}{3} \pi r^3 \). This dependence of \( V \) on \( r \) would be denoted as \( V = f(r) \) and we would simply say that \( V \) is a function of \( r \). Here \( f \) is purely a symbol (for that matter, any other letter could have been used in place of \( f \)), and it is simply used to represent the dependence of one quantity on the other.
Definition of Function

Function can be easily defined with the help of the concept of mapping. Let \( A \) and \( B \) be any two non-empty sets. "A function from \( A \) and \( B \) is a rule or correspondence that assigns to each element of set \( A \), one and only one element of set \( B \)." Let the correspondence be \( f \). Then mathematically we write \( f: A \rightarrow B \) where \( y = f(x), \ x \in A \) and \( y \in B \). We say that \( y \) is the image of \( x \) under \( f \) (or \( x \) is the pre-image of \( y \)).

- A mapping \( f: A \rightarrow B \) is said to be a function if each element in the set \( A \) has a image in set \( B \). It is possible that a few elements in the set \( B \) are present which are not the images of any element in set \( A \).
- Every element in set \( A \) should have one and only one image. That means it is impossible to have more than one image for a specific element in set \( A \). Functions cannot be multi-valued (A mapping that is multi-valued is called a relation from \( A \) and \( B \)).

![Fig. 1.10](image)

Let us consider some other examples to make the above mentioned concepts clear.

a. Let \( f: R^+ \rightarrow R \) where \( y^2 = x \). This cannot be considered a function as each \( x \in R^+ \) would have two images namely \( \pm \sqrt{x} \). Hence, it does not represent a function. Thus, it would be a relation.

b. Let \( f: [-2, 2] \rightarrow R \), where \( x^2 + y^2 = 4 \). Here \( y = \pm \sqrt{4-x^2} \), that means for each \( x \in [-2, 2] \) we would have two values of \( y \) (except when \( x = \pm 2 \)). Hence, it does not represent a function.

c. Let \( f: R \rightarrow R \) where \( y = x^2 \). Here for each \( x \in R \) we would have a unique value of \( y \) in the set \( R \) (as cube of any two distinct real numbers are distinct). Hence, it would represent a function.

Function as a Set of Ordered Pairs

A function \( f: A \rightarrow B \) can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to \( A \) and second element is the corresponding element of \( B \).

As such a function \( f: A \rightarrow B \) can be considered as a set of ordered pairs \((a, f(a))\) where \( a \in A \) and \( f(a) \in B \) which is the \( f \) image of \( a \). Hence, \( f \) is a subset of \( A \times B \).

As a particular type of relation, we can define a function as follows:

A relation \( R \) from a set \( A \) to a set \( B \) is called a function if

- each element of \( A \) is associated with some element of \( B \)
- each element of \( A \) has unique image in \( B \)

Thus, a function \( f \) from a set \( A \) to a set \( B \) is a subset of \( A \times B \) in which each \( a \in A \) appears in one and only one ordered pair belonging to \( f \). Hence, a function \( f \) is a relation from \( A \) to \( B \) satisfying the following properties:

- \( f \subseteq A \times B \)
- \( \forall a \in A \Rightarrow (a, f(a)) \in f \)
- \( \forall (a, b) \in f \) and \( (a, c) \in f \Rightarrow b = c \)

Thus, the ordered pairs of \( f \) must satisfy the property that each element of \( A \) appears in some ordered pair and no two ordered pairs have same first element.

Note:

Every function is a relation but every relation is not necessarily a function.

Distinction between a Relation and a Function by Graphs (Vertical Line Test)

![Fig. 1.11](image)

The above figures show the graph of two arbitrary curves. In Fig. 1.11 (a), any line drawn parallel to \( y \)-axis would meet the curve at only one point. That means each element of \( A \) would have one and only one image. Thus, Fig. 1.11(a) represents the graph of a function.

In Fig. 1.11 (b), certain line parallel to \( y \)-axis, (e.g., line \( L \)) would meet the curve in more than one points \((A, B \) and \( C)\). Thus, element \( x_0 \) of \( A \) would have three distinct images. Thus, this curve does not represent a function.

Hence, if \( y = f(x) \) represents a function, lines drawn parallel to \( y \)-axis through different points corresponding to points of set \( X \) should meet the curve in one and only one point.

Consider the graph of following relations:

Equation of an ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is a relation, which is a combination of two functions \( y = \pm b \sqrt{1 - \frac{x^2}{a^2}} \).

The upper branch represents function \( y = b \sqrt{1 - \frac{x^2}{a^2}} \) and the lower branch represents the function \( y = -b \sqrt{1 - \frac{x^2}{a^2}} \).
**Domain, Co-Domain, Range**

Let \( f: A \rightarrow B \) be a function. In general, sets \( A \) and \( B \) could be any arbitrary non-empty sets. But at this level, we would confine ourselves only to real-valued functions. That means it would be invariably assumed that \( A \) and \( B \) are the subsets of real numbers.

Set \( A \) is called domain of the function \( f \).

Set \( B \) is called co-domain of the function \( f \).

The set of images of different elements of set \( A \) is called the range of the function \( f \). It is obvious that a range would be a subset of co-domain as we may have few elements in co-domain which are not the images of any element of set \( A \) (of course, these elements of co-domain will not be included in the range). Range is also called domain of variation. Domain of function \( f \) is normally represented as Domain \( (f) \). Range is represented as Range \( (f) \).

Note that sometimes domain of the function is not explicitly defined. In these cases, domain would mean the set of values of \( x \) for which \( f(x) \) assumes real values. For example, if \( y = f(x) \) then Domain \( (f) = \{ x : f(x) \text{ is a real number} \} \).

**Rules for the Domain of a Function**

a. Domain \((f(x) + g(x)) = \text{Domain } f(x) \cap \text{Domain } g(x)\)

b. Domain \((f(x) \times g(x)) = \text{Domain } f(x) \cap \text{Domain } g(x)\)

c. Domain \(\left(\frac{f(x)}{g(x)}\right) = \text{Domain } f(x) \cap \{ x : g(x) \neq 0 \}\)

d. Domain \(\sqrt{f(x)} = \text{Domain } f(x) \cap \{ x : f(x) \geq 0 \}\)

**Some Important Definitions**

1. **Polynomial function**: If a function \( f \) is defined by \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \), where \( n \) is a non-negative integer and \( a_n, a_{n-1}, \ldots, a_0 \) are real numbers and \( a_n \neq 0 \), then \( f \) is called a polynomial function of degree \( n \).

2. **Algebraic function**: \( y \) is an algebraic function of \( x \), if it is a function that satisfies an algebraic equation of the form
   \[ P_0(x) y^n + P_1(x) y^{n-1} + \cdots + P_{n-1}(x) y + P_n(x) = 0 \]
   where \( n \) is a positive integer and \( P_i(x) \) are polynomials in \( x \). For example, \( x^3 + y^3 - 3xy = 0 \) or \( y = x^3 + 2y \) is an algebraic function, since it satisfies the equation \( y^2 - x^3 = 0 \).

3. **Rational function**: A function that can be written as the quotient of two polynomial function is said to be a rational function.

4. **Explicit function**: A function \( y = f(x) \) is said to be an explicit function of \( x \) if the dependent variable \( y \) can be expressed in terms of independent variable \( x \) only. For example, \( i) y = x - \cos x \), (ii) \( y = x + \log x - 2x^3 \).

5. **Implicit function**: A function \( y = f(x) \) is said to be an implicit function of \( x \) if \( y \) cannot be written in terms of \( x \) only. For example, \( i) ax^2 + 2bxy + by^2 + 2gx + 2fy + c = 0 \), (ii) \( xy = \sin (x + y) \).

6. **Bounded functions**: A function is said to be bounded if \( |f(x)| \leq M \), where \( M \) is a finite positive real number.

7. **Identity function**: The function \( f : R \rightarrow R \) is called an identity function if \( f(x) = x \forall x \in R \).

**DIFFERENT TYPES OF FUNCTIONS**

**Quadratic Function**

Let \( f(x) = ax^2 + bx + c \), where \( a, b, c, \in R \) and \( a \neq 0 \).

We have \( f(x) = a \left[ x^2 + \frac{b}{a} x + \frac{c}{a} \right] \).
\[ y = f(x) = a \left[ x^2 + \frac{b}{a}x + \frac{b^2}{4a} + \frac{c}{a} \right] = a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \]

\[ y = f(x) = a \left( x + \frac{b}{2a} \right)^2 - \frac{D}{4a} \Rightarrow \left( y + \frac{D}{4a} \right) = a \left( x + \frac{b}{2a} \right)^2. \]

Thus, \( y = f(x) \) represents a parabola whose axis is parallel to the \( y \)-axis and vertex \( A\left(-\frac{b}{2a}, -\frac{D}{4a}\right)\). For some values of \( x \), \( f(x) \) may be positive, negative or zero and for \( a > 0 \), the parabola opens upwards and for \( a < 0 \), the parabola opens downwards. This gives the following cases:

a. \( a > 0 \) and \( D < 0 \), so \( f(x) > 0 \forall x \in R \), i.e., \( f(x) \) is positive for all values of \( x \).
   
   Range of function is \( \left[ -\frac{D}{4a}, \infty \right) \)

   \[ x = -\frac{b}{2a} \] is a point of minima.

b. \( a < 0 \) and \( D < 0 \) so \( f(x) < 0 \forall x \in R \), i.e., \( f(x) \) is negative for all values of \( x \).
   
   Range of function is \( \left( -\infty, -\frac{D}{4a} \right] \)

   \[ x = -\frac{b}{2a} \] is a point of maxima.

c. \( a > 0 \) and \( D = 0 \), so \( f(x) \geq 0 \forall x \in R \), i.e., \( f(x) \) is positive for all values of \( x \) except at vertex where \( f(x) = 0 \).

d. \( a > 0 \) and \( D > 0 \)
   
   Let \( f(x) = 0 \) have two real roots \( \alpha \) and \( \beta \) (where \( \alpha < \beta \)) then:
   
   \( f(x) > 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty) \) and \( f(x) < 0 \forall x \in (\alpha, \beta) \).

![Formula and Graph](image1.png)

Fig. 1.18

e. \( a < 0 \) and \( D = 0 \)
   
   so \( f(x) \leq 0 \forall x \in R \), i.e., \( f(x) \) is negative for all values of \( x \) except at vertex where \( f(x) = 0 \).

![Formula and Graph](image2.png)

Fig. 1.19

f. \( a < 0 \) and \( D > 0 \)
   
   Let \( f(x) = 0 \) have two roots \( \alpha \) and \( \beta \) (where \( \alpha < \beta \)) then:
   
   \( f(x) < 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty) \) and \( f(x) > 0 \), \( \forall x \in (\alpha, \beta) \)

![Formula and Graph](image3.png)

Fig. 1.20

Note: If \( f(x) \geq 0 \forall x \in R \Rightarrow a > 0 \) and \( D \leq 0 \)

and if \( f(x) \leq 0 \forall x \in R \Rightarrow a < 0 \) and \( D \leq 0 \).

Example 1.9 Find the range of \( f(x) = x^2 - x - 3 \).

Sol. \( f(x) = x^2 - x - 3 = (x - \frac{1}{2})^2 - \frac{1}{4} - 3 = (x - \frac{1}{2})^2 - \frac{13}{4} \)

Now \((x - \frac{1}{2})^2 \geq 0 \forall x \in R \Rightarrow (x - \frac{1}{2})^2 - \frac{13}{4} \geq -\frac{13}{4} \)

\[ x = \frac{1}{2} \]

Hence, range is \[ \left[ -\frac{13}{4}, \infty \right) \]

Example 1.10 Find the domain and range of \( f(x) = \sqrt{x^2 - 3x + 2} \).

Sol. For domain \( x^2 - 3x + 2 \geq 0 \)

\[ \Rightarrow (x - 1)(x - 2) \geq 0 \]

\[ \Rightarrow x \in (-\infty, 1) \cup (2, \infty) \]

Now, \( f(x) = \sqrt{x^2 - 3x + 2} \)
1.8 Calculus

\[ \sqrt{\frac{(x-3)^2}{2} + 2 - \frac{9}{4}} \]

Now, the least permissible value of \( \frac{(x-3)^2}{2} - \frac{1}{4} \) is 0 when \( \frac{(x-3)^2}{2} = \frac{1}{2} \). Hence, the range is \([0, \infty)\).

**Example 1.11** Find the range of the function \( f(x) = 6^x + 3^x + 6^x + 3^{-x} + 2 \).

**Sol.**

\[ f(x) = 6^x + 3^x + 6^x + 3^{-x} + 2 \]

\[ = (\sqrt{6^x} - \sqrt{3^x})^2 + (\sqrt{3^x} + 3^{-x})^2 + 6 \geq 6 \cdot 2 \]

Hence, the range is \([6, \infty)\).

**Example 1.12** Find the domain and range of

\[ f(x) = \sqrt{x^2 - 4x + 6} \]

**Sol.**

\[ x^2 - 4x + 6 = (x-2)^2 + 2 \] which is always positive.

Hence, the domain is \( R \).

Now, \( f(x) = \sqrt{(x-2)^2 + 2} \)

The least value of \( f(x) \) is \( \sqrt{2} \) when \( x = 2 \).

Hence, the range is \([\sqrt{2}, \infty)\).

**Example 1.13** Find the range of \( f(x) = \frac{x^2 - x + 1}{x^2 + x + 1} \)

**Sol.** Let \( y = \frac{x^2 - x + 1}{x^2 + x + 1} \)

\[ \Rightarrow (1-y) x^2 - (1+y) x + 1 - y = 0 \]

Now \( x \) is real, then \( D \geq 0 \)

\[ \Rightarrow (1+y)^2 - 4(1-y)^2 \geq 0 \]

\[ \Rightarrow (1+y-2+2y)(1+y-2-2y) \geq 0 \]

\[ \Rightarrow (3y-1)(3y-1) \geq 0 \]

\[ \Rightarrow \frac{1}{3} \leq y \leq \frac{1}{3} \] \( \Rightarrow \) The range is \([\frac{1}{3}, 3]\).

**Example 1.14** Find the complete set of values of \( a \) such that \( \frac{x^2 - x}{1-ax} \) attains all real values.

**Sol.**

\[ y = \frac{x^2 - x}{1-ax} \]

\[ \Rightarrow x^2 - yx - ay = 0 \]

\[ \Rightarrow x^2 + x(ay-1) - y = 0 \]

Since \( x \) is real \( \Rightarrow (ay-1)^2 + 4y \geq 0 \)

\[ \Rightarrow a^2 y^2 + 2ay(2-a) + 1 \geq 0 \ \forall y \in \mathbb{R} \]

\[ \Rightarrow a^2 > 0, 4(2-a)^2 - 4a^2 \leq 0 \Rightarrow 4 - 4a \leq 0 \Rightarrow a \in [1, \infty) \]

**Concept Application Exercise 1.2**

1. Find the range of \( f(x) = \frac{x^3 + 3x + 71}{x^2 + 2x - 7} \)

2. Find the range of \( f(x) = \frac{x-1 + \sqrt{5-x}}{x} \)

3. If \( f(x) = \sqrt{x^2 + ax + 4} \) is defined for all \( x \), then find the values of \( a \).

4. Find the domain and range of \( f(x) = \sqrt{3-2x-x^2} \).

**Modulus Function**

\[ y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \]

Domain: \( R \)

Range: \([0, \infty)\)

Nature: even function

![Graph of Modulus Function](Fig. 1.21)

**Fig. 1.21**

\[ y = |x-a| = \begin{cases} x-a, & x \geq a \\ a-x, & x < a \end{cases} \text{ where } a > 0 \]

![Graph of Modulus Function](Fig. 1.21(a))

**Fig. 1.21(a)**

**Properties of Modulus Function**

a. \( |x| = a \) \( \Rightarrow \) Points on the real number line whose distance from origin is \( a \)

\[ \Rightarrow x = \pm a \]

b. \( |x| \leq a \) \( \Rightarrow \) \( x^2 \leq a^2 \)

![Graph of Modulus Function](Fig. 1.21(b))

**Fig. 1.21(b)**
Example 1.15 Solve \(|3x-2| \leq \frac{1}{2}\).

Sol. \(|3x-2| \leq \frac{1}{2}\)

\[ \Rightarrow \frac{-1}{2} \leq 3x-2 \leq \frac{1}{2} \]

\[ \Rightarrow \frac{3}{2} \leq 3x \leq \frac{5}{2} \]

\[ \Rightarrow \frac{1}{2} \leq x \leq \frac{5}{6} \]

\[ \Rightarrow x \in \left[\frac{1}{2}, \frac{5}{6}\right]. \]

Example 1.16 Solve \(|x-1|-5| \geq 2\).

Sol. \(|x-1|-5| \geq 2\)

\[ \Rightarrow |x-1| \geq 2 \text{ or } |x-1| = 5 \]

\[ \Rightarrow x-1 \leq -2 \text{ or } x-1 \geq 7 \]

\[ \Rightarrow x \in (-\infty, 5] \cup [8, \infty) \]

Example 1.17 Solve \(-1 \leq \frac{1}{|x| - 2} \leq 1\), where \(x \in \mathbb{R}, x \neq \pm 2\) or find the domain of \(f(x) = \sqrt{1-|x|} \).

Sol. \(\frac{-1}{|x| - 2} \leq 1\)

\[ \Rightarrow \frac{-1}{|x| - 2} \leq 1 \]

\[ \Rightarrow -1 \leq |x| - 2 \]

\[ \Rightarrow -1 \leq -2 \]

\[ \Rightarrow -1 \leq 0 \]

\[ \Rightarrow \frac{1-|x|}{|x|-2} \geq 0 \]

\[ \Rightarrow \frac{1-|x|}{|x|-2} \geq 0, \text{ where } y = |x|. \]

Fig. 1.22

\[ \Rightarrow 1 \leq y < 2. \]

\[ \Rightarrow 1 \leq |x| < 2. \]

\[ \Rightarrow x \in (-2, -1) \cup (1, 2). \]

Example 1.18 Solve \(\frac{|x+3|+x}{x+2} > 1\).

Sol. We have \(\frac{|x+3|+x}{x+2} > 1\)

Clearly, L.H.S. of this inequation is meaningful for \(x \neq -2\).

Given \(\frac{|x+3|+x}{x+2} > 1\)

\[ \Rightarrow \frac{|x+3|+x-x-2}{x+2} > 0 \]

\[ \Rightarrow \frac{|x+3|-2}{x+2} > 0 \]

If \(|x+3|-2 = 0 \Rightarrow x+3 = \pm 2 \Rightarrow x = -5, -1\).

Hence, the sign scheme of the expression \(\frac{|x+3|-2}{x+2}\) is as follows:

Fig. 1.23

From the above sign scheme, \(x \in (-5, -2) \cup (-1, \infty)\).

Example 1.19 Solve \(|x-1|+|x-2| \geq 4\).

Sol. Let \(f(x) = |x-1|+|x-2|\)

<table>
<thead>
<tr>
<th>A.</th>
<th>B.</th>
<th>C.</th>
<th>D.</th>
<th>E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x &lt; 1)</td>
<td>(1-x+2-x)</td>
<td>(3-2x\geq 4)</td>
<td>(x \leq -1)</td>
<td></td>
</tr>
<tr>
<td>(1 \leq x \leq 2)</td>
<td>(x-1+2)</td>
<td>(x \geq 4), not possible</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x = 1)</td>
<td>(x = 1)</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>(x &gt; 2)</td>
<td>(x-1+x-2)</td>
<td>(2x-3\geq 4)</td>
<td>(x \geq 7/2)</td>
<td></td>
</tr>
</tbody>
</table>

Hence, the solution is \(x \in (-\infty, -1/2) \cup [7/2, \infty)\).

Example 1.20 Solve \(|\sin x + \cos x| = |\sin x| + |\cos x|\), \(x \in [0, 2\pi]\).

Sol. The given relation holds only when \(\sin x\) and \(\cos x\) have same sign or at least one of them is zero.

Hence, \(x \in [0, \pi/2) \cup [\pi, 3\pi/2) \cup (2\pi, \infty)\).
1. Solve the following:
   a. $1 \leq |x-2| \leq 3$
   b. $0 < |x-3| \leq 5$
   c. $|x-2| + |2x-3| = |x-1|$
   d. $\frac{|x-3|}{x+1} \leq 1$

2. Find the domain of
   a. $f(x) = \frac{1}{\sqrt{x-|x|}}$
   b. $f(x) = \frac{1}{\sqrt{x+|x|}}$

3. Find the set of real value(s) of $a$ for which the equation $|2x+3| + |2x-3| = ax+6$ has more than two solutions.

4. If $a < b < c$, then find the range of $f(x) = |x-a| + |x-b| + |x-c|$

5. Find the range of $f(x) = \sqrt{1-\sqrt{x^2-6x+9}}$

**Trigonometric Functions**

1. $y = f(x) = \sin x$
   - Domain $\rightarrow \mathbb{R}$, Range $\rightarrow [-1, 1]$
   - Period $\rightarrow 2\pi$
   - Nature $\rightarrow$ odd, many-one in its actual domain
     - $\sin^2 x, |\sin x| \in [0, 1]$
     - $\sin x = 0 \Rightarrow x = n\pi, n \in \mathbb{Z}$
     - $\sin x = 1 \Rightarrow x = (4n+1)\pi/2, n \in \mathbb{Z}$
     - $\sin x = -1 \Rightarrow x = (4n-1)\pi/2, n \in \mathbb{Z}$
     - $\sin x = \sin \alpha \Rightarrow x = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$
     - $\sin x \geq 0 \Rightarrow x \in \bigcup_{n \in \mathbb{Z}} [2n\pi, \pi + 2n\pi]$

2. $y = f(x) = \cos x$
   - Domain $\rightarrow \mathbb{R}$, Range $\rightarrow [-1, 1]$
   - Period $\rightarrow 2\pi$
   - Nature $\rightarrow$ even, many-one in its actual domain
     - $\cos^2 x, |\cos x| \in [0, 1]$
     - $\cos x = 0 \Rightarrow x = (2n+1)\pi/2, n \in \mathbb{Z}$
     - $\cos x = 1 \Rightarrow x = 2n\pi, n \in \mathbb{Z}$
     - $\cos x = -1 \Rightarrow x = (2n+1)\pi, n \in \mathbb{Z}$
     - $\cos x = \cos \alpha \Rightarrow x = 2n\pi \pm \alpha, n \in \mathbb{Z}$

3. $y = f(x) = \tan x$
   - Domain $\rightarrow \mathbb{R} \setminus (2n+1)\pi/2, n \in \mathbb{Z}$
   - Range $\rightarrow (-\infty, \infty)$
   - Period $\rightarrow \pi$
   - Nature $\rightarrow$ odd, many-one in its actual domain
   - Discontinuous at $x = (2n+1)\pi/2, n \in \mathbb{Z}$
     - $\tan^2 x, |\tan x| \in [0, \infty)$
     - $\tan x = 0 \Rightarrow x = n\pi, n \in \mathbb{Z}$
     - $\tan x = \tan \alpha \Rightarrow x = n\pi + \alpha, n \in \mathbb{Z}$

4. $y = f(x) = \cot x$
   - Domain $\rightarrow \mathbb{R} \setminus n\pi, n \in \mathbb{Z}$
   - Range $\rightarrow (-\infty, \infty)$
   - Period $\rightarrow \pi$
   - Nature $\rightarrow$ odd, many-one in its actual domain
   - Discontinuous at $x = n\pi, n \in \mathbb{Z}$
     - $\cot^2 x, |\cot x| \in [0, \infty)$
     - $\cot x = 0 \Rightarrow x = (2n+1)\pi/2, n \in \mathbb{Z}$
5. \( y = f(x) = \sec x \)
   Domain \( \to \mathbb{R}, 2k\pi, k \in \mathbb{Z} \)
   Range \( \to (0, \infty) \cup (-\infty, 0) \)
   Period \( \to 2\pi \)
   \( \sec^2 x, |\sec x| \in [1, \infty) \)
   Nature \( \to \) even, many-one in its actual domain
   ![Graph of sec x]

6. \( y = f(x) = \cosec x \)
   Domain \( \to \mathbb{R}, 2k\pi, k \in \mathbb{Z} \)
   Range \( \to (-\infty, -1] \cup [1, \infty) \)
   Period \( \to 2\pi \)
   \( \cosec^2 x, |\cosec x| \in [1, \infty) \)
   Nature \( \to \) odd, many-one in its actual domain
   ![Graph of cosec x]

**Important Result**

\[ f(x) = a \cos x + b \sin x = \sqrt{a^2 + b^2} \left[ \cos x\tan^{-1} \left( \frac{a}{b} \right) + \sin x \right] \]

Proof: Let \( a = r \sin \alpha, b = r \cos \alpha \)
   \( \Rightarrow a^2 + b^2 = r^2 \) and \( \tan \alpha = \frac{a}{b} \)
   Now, \( f(x) = r \left[ \cos x\sin \alpha + \sin x \cos \alpha \right] \)
   \[ = r \sin(x + \alpha) = \sqrt{a^2 + b^2} \sin \left( x + \tan^{-1} \frac{a}{b} \right) \]

Since \(-1 \leq \sin \left( x + \tan^{-1} \frac{a}{b} \right) \leq 1\)
   \( \Rightarrow -\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2} \sin \left( x + \tan^{-1} \frac{a}{b} \right) \leq \sqrt{a^2 + b^2} \)

\[ \Rightarrow \text{The range of } f(x) = a \cos x + b \sin x \text{ is } \left[ -\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2} \right] \]

**Example 1.21**
Find the domain of the functions.

\[ f(x) = \frac{1}{1 + 2 \sin x} \]

Sol. To define \( f(x) \), we must have \( 1 + 2 \sin x \neq 0 \)
   \( \Rightarrow \sin x \neq -\frac{1}{2} \Rightarrow x \neq n\pi + \frac{7\pi}{6}, n \in \mathbb{Z} \)
   Hence, the domain of the function is
   \[ R = \left\{ n\pi + \frac{7\pi}{6}, n \in \mathbb{Z} \right\} \]

**Example 1.22**
Solve \( \sin x > -\frac{1}{2} \) or find the domain of

\[ f(x) = \frac{1}{\sqrt{1 + 2 \sin x}} \]

Sol. To define \( f(x) \), we must have \( 1 + 2 \sin x > 0 \) or \( \sin x > -\frac{1}{2} \).
   The function \( \sin x \) has the least positive period \( 2\pi \). That is why it is sufficient to solve inequality of the form
   \( \sin x > a, \sin x \geq a, \sin x < a, \sin x \leq a \) first on the interval of length \( 2\pi \), and then get the solution set by adding numbers of the form \( 2n\pi, n \in \mathbb{Z} \), to each of the solutions obtained on that interval.
   Thus, let us solve this inequality on the interval
   \[ \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right] \]

From Fig. 1.30, \( \sin x > -\frac{1}{2} \) when \( -\frac{\pi}{6} < x < \frac{7\pi}{6} \)
   Thus, on generalizing the above solution, we get
   \[ 2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}, n \in \mathbb{Z} \]

**Example 1.23**
Find the number of solutions of \( \sin x = \frac{x}{10} \).

Sol. Here, let \( f(x) = \sin x \) and \( g(x) = \frac{x}{10} \). Also we know that
   \(-1 \leq \sin x \leq 1 \Rightarrow -\frac{x}{10} \leq x \leq \frac{x}{10} \Rightarrow -10 \leq x \leq 10 \).
   Thus, sketch both the curves when \( x \in [-10, 10] \).
Example 1.24 Find the number of solutions of the equation \( \sin x = x^2 + x + 1 \).

Sol. Let \( f(x) = \sin x \) and \( g(x) = x^2 + x + 1 \), as shown in Fig. 1.32, which do not intersect at any point, therefore, there is no solution.

Example 1.25 Find the range of \( f(x) = \sin^2 x - \sin x + 1 \).

Sol. \( f(x) = \sin^2 x - \sin x + 1 = \left( \sin x - \frac{1}{2} \right)^2 + \frac{3}{4} \).

Now, \(-1 \leq \sin x \leq 1 \Rightarrow -\frac{3}{4} \leq \sin x - \frac{1}{2} \leq \frac{1}{2} \).
\( \Rightarrow 0 \leq \left( \sin x - \frac{1}{2} \right)^2 \leq \frac{3}{4} \Rightarrow \frac{3}{4} \leq \left( \sin x - \frac{1}{2} \right)^2 + \frac{3}{4} \leq 3 \).

Hence, the range is \( \left[ \frac{3}{4}, 3 \right] \).

Example 1.26 Find the range of \( f(x) = \frac{1}{2 \cos x - 1} \).

Sol. \(-1 \leq \cos x \leq 1 \)
\( \Rightarrow -2 \leq 2 \cos x \leq 2 \)
\( \Rightarrow -3 \leq 2 \cos x - 1 \leq 1 \).

For \( \frac{1}{2 \cos x - 1}, -3 \leq 2 \cos x - 1 < 0 \) or \( 0 < 2 \cos x - 1 \leq 1 \).
\( \Rightarrow -\infty < \frac{1}{2 \cos x - 1} \leq -\frac{1}{3} \) or \( 1 \leq \frac{1}{2 \cos x - 1} < \infty \).

Hence, the range is \( \left( -\infty, -\frac{1}{3} \right] \cup [1, \infty) \).

Example 1.27 Find the domain for \( f(x) = \sqrt{\cos(\sin x)} \).

Sol. \( f(x) = \sqrt{\cos(\sin x)} \) is defined if \( \cos(\sin x) \geq 0 \).

Example 1.28 If \( f(x) = \frac{\sin x}{\sqrt{1 + \tan^2 x}} - \frac{\cos x}{\sqrt{1 + \cot^2 x}} \), then find the range of \( f(x) \).

Sol. \( f(x) = \frac{\sin x}{|\sec x|} - \frac{\cos x}{|\csc x|} = \sin x |\cos x| - \cos x |\sin x| \).

Clearly, the domain of \( f(x) \) is \( R \cup \{ n\pi, (2n+1)\frac{\pi}{2} | n \in I \} \)
and period of \( f(x) \) is \( 2\pi \).

Example 1.29 Find the range of \( f(x) = |\sin x| + |\cos x|, x \in R \).

Sol. \( f(x) = |\sin x| + |\cos x| \forall x \in R \).

Clearly \( f(x) > 0 \).

Also, \( f^2(x) = \sin^2 x + \cos^2 x + |2 \sin x \cos x| = 1 + |\sin 2x| \)
\( \Rightarrow 1 \leq f(x) \leq 2 \)
\( \Rightarrow 1 \leq f(x) \leq \sqrt{2} \).

Example 1.30 Find the range of \( f(\theta) = 5 \cos \theta + 3 \cos \left( \theta + \frac{\pi}{3} \right) + 3 \).

Sol. \( f(\theta) = 5 \cos \theta + 3 \cos \left( \theta + \frac{\pi}{3} \right) + 3 \)
\( = 5 \cos \theta + \frac{3}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3 \).
Thus, the range of $f(\theta)$ is $[-4, 10]$.

**Concept Application Exercise 1.4**

1. Find the domain of $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$.
2. Solve (a) $\tan x < 2$, (b) $\cos x \leq -\frac{1}{2}$.
3. Prove that the least positive value of $x$, satisfying $\tan x = x + 1$, lies in the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
4. Find the range of $f(x) = \sec \left(\frac{\pi}{4} \cos x\right)$, where $-\infty < x < \infty$.
5. If $x \in [1, 2]$, then find the range of $f(x) = \tan x$.
6. Find the range of $f(x) = \frac{1}{1 - 3\sqrt{1 - \sin^2 x}}$.

**Inverse Trigonometric Functions**

$f(x) = \sin^{-1} x$

Domain: $[-1, 1]$, Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1} (\sin x) = x$, for all $x \in [-\pi/2, \pi/2]$

$\sin (\sin^{-1} x) = x$, for all $x \in [-1, 1]$

$\sin^{-1}(-x) = -\sin^{-1} (x)$, for all $x \in [-1, 1]$

$f(x) = \cos^{-1} x$

Domain: $[-1, 1]$, Range: $[0, \pi]$

$\cos^{-1} (\cos x) = x$, for all $x \in [0, \pi]$

$\cos (\cos^{-1} x) = x$, for all $x \in [-1, 1]$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$, for all $x \in [-1, 1]$

$f(x) = \tan^{-1} x$

Domain: $R$, Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan^{-1} (\tan x) = x$, for all $x \in \mathbb{R}$

$\tan (\tan^{-1} x) = x$, for all $x \in \mathbb{R}$

$\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in \mathbb{R}$

$f(x) = \cot^{-1} x$

Domain: $R$, Range: $\left(0, \pi\right)$

$\cot^{-1} (\cot x) = x$, for all $x \in (0, \pi)$

$\cot (\cot^{-1} x) = x$, for all $x \in \mathbb{R}$

$\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in \mathbb{R}$

$f(x) = \sec^{-1} x$

Domain: $(-\infty, -1) \cup [1, \infty)$, Range: $[0, \pi] - \left\{\pi/2\right\}$

$\sec^{-1} (\sec x) = x$, for all $x \in [0, \pi] - \left\{\pi/2\right\}$

$\sec (\sec^{-1} x) = x$, for all $x \in (-\infty, -1) \cup [1, \infty)$

$\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1) \cup [1, \infty)$
Example 1.31 Find the domain of \( f(x) = \sin^{-1}\left(\frac{x^2}{2}\right) \).

Sol. \( f(x) = \sin^{-1}\left(\frac{x^2}{2}\right) \) is defined, if \(-\frac{x^2}{2} \leq 1\) or \(-2 \leq x^2 \leq 2\).

\[
0 \leq x^2 \leq 2, \\
\Rightarrow -\sqrt{2} \leq x \leq \sqrt{2}
\]

Therefore, the domain of \( f(x) \) is \([-\sqrt{2}, \sqrt{2}]\).

Example 1.32 Find the range of \( f(x) = \sin^{-1}x + \tan^{-1}x + \cos^{-1}x \).

Sol. Clearly, the domain of the function is \([-1, 1]\).

Also \( \tan^{-1}x \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right] \) for \( x \in [-1, 1] \).

Now, \( \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \) for \( x \in [-1, 1] \).

Thus, \( f(x) = \tan^{-1}x + \frac{\pi}{2} \) where \( x \in [-1, 1] \).

Hence, the range is \( \left[\frac{-\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \).

Example 1.33 Find the domain of \( f(x) = \sqrt{\cos^{-1}x - \sin^{-1}x} \).

Sol. We must have \( \cos^{-1}x \geq \sin^{-1}x \)

\[
\Rightarrow \frac{\pi}{2} - \sin^{-1}x \geq \sin^{-1}x
\]

\[
\Rightarrow \frac{\pi}{2} \geq 2\sin^{-1}x
\]

\[
\Rightarrow \sin^{-1}x \leq \frac{\pi}{4}, \text{ but } \frac{\pi}{2} \leq \sin^{-1}x
\]

\[
\Rightarrow \frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{4}
\]

\[
\Rightarrow \sin\left(\frac{\pi}{2}\right) \leq x \leq \sin\left(\frac{\pi}{4}\right)
\]

\[
\Rightarrow x \in \left[-1, \frac{1}{\sqrt{2}}\right]
\]

Example 1.34 Find the range of \( \tan^{-1}\left(\frac{2x}{1+x^2}\right) \).

Sol. First, we must get the range of \( \frac{2x}{1+x^2} = y \)

We have \( xy^2 - 2x + y = 0 \)

Since \( x \) is real, \( D \geq 0 \), \( \Rightarrow 4 - 4y^2 \geq 0 \), \( \Rightarrow -1 \leq y \leq 1 \)

\[
\Rightarrow \tan^{-1}(y) = \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]
\]

as \( \tan x \) is an increasing function.

Example 1.35 Find the domain for \( f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right) \).

Sol. \( f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right) \) is defined for \(-1 \leq \frac{1+x^2}{2x} \leq 1\), or

\[
\frac{1+x^2}{2x} \leq 1
\]

\[
\Rightarrow |x| \leq 1, \text{ for all } x.
\]

\[
\Rightarrow 1+x^2 \leq 2|x|, \text{ for all } x
\]

(as \( 1+x^2 > 0 \))

\[
\Rightarrow x^2 - 2|x| + 1 \leq 0
\]

\[
\Rightarrow |x|^2 - 2|x| + 1 \leq 0
\]

(as \( x^2 = |x|^2 \))

\[
\Rightarrow (|x|-1)^2 \leq 0
\]

But \((|x|-1)^2\) is always either positive or zero. Thus, \((|x|-1)^2=0 \Rightarrow |x|=1 \Rightarrow x=\pm 1\).

Thus, the domain for \( f(x) \) is \([-1, 1]\).

Example 1.36 Find the range of \( f(x) = \cot^{-1}\left(2x-x^2\right) \).

Sol. Let \( \theta = \cot^{-1}\left(2x-x^2\right) \), where \( \theta \in (0, \pi) \)

\[
\Rightarrow \cot\theta = 2x-x^2, \text{ where } \theta \in (0, \pi)
\]

\[
\Rightarrow \cot\theta = 1 - (1 - 2x + x^2), \text{ where } \theta \in (0, \pi)
\]

\[
\Rightarrow \cot\theta = 1 - (1-x)^2, \text{ where } \theta \in (0, \pi)
\]
\[ \Rightarrow \cot \theta \leq 1, \text{ where } \theta \in (0, \pi) \]
\[ \Rightarrow \frac{\pi}{4} \leq \theta < \frac{\pi}{2} \]
\[ \Rightarrow \text{The range of } f(x) \in \left[ \frac{\pi}{4}, \pi \right) \]

**Concept Application Exercise 1.5**

1. Find the domain of the following functions:
   
   a. \( f(x) = \frac{\sin^{-1} x}{x} \)
   
   b. \( f(x) = \sin^{-1} (x - 1) - 2 \)
   
   c. \( f(x) = \cos^{-1} (1 + 3x + 2x^2) \)
   
   d. \( f(x) = \sin^{-1} (x - 3) \)
   
   e. \( f(x) = \cos^{-1} \left( \frac{6 - 3x}{4} \right) + \cos^{-1} \left( \frac{x - 1}{2} \right) \)
   
   f. \( f(x) = \sqrt{\sec^{-1} \left( 2 - \left| \frac{x}{2} \right| \right)} \)

2. Find the range of \( f(x) = \tan^{-1} \left( \sqrt{x^2 - 2x + 2} \right) \).

3. Find the range of \( f(x) = \cos^{-1} \sqrt{1 - x^2} - \sin^{-1} x \).

4. Find the range of the function,
   
   \( f(x) = \cot^{-1} \log_3 (x^4 - 2x^2 + 3) \)

**Exponential and Logarithmic Functions**

**Exponential Function**

\( y = a^x, \ a > 1 \)

- **Domain** \( \rightarrow \mathbb{R} \)
- **Range** \( \rightarrow (0, \infty) \)
- **Nature**
  - Non-periodic
  - One-to-one
  - Neither odd nor even
  - Monotonically increasing, \( a > 1 \)
  - Monotonically decreasing, \( 0 < a < 1 \)

**Logarithmic Function**

Logarithm function is the inverse of exponential function. Hence, the domain and range of the logarithmic functions are range and domain of exponential function, respectively.

Also, the graph of function can be obtained by taking the mirror image of the graph of the exponential function in the line \( y = x \).

\( y = \log_a x, \ a > 0 \text{ and } a \neq 1 \)

- **Domain** \( \rightarrow (0, \infty) \)
- **Range** \( \rightarrow (-\infty, \infty) \)
- **Period** non-periodic
- **Nature** neither odd nor even

**Properties of Logarithmic Function**

For \( x, y > 0 \text{ and } a > 0, a \neq 2 \)

1. \( \log_a (xy) = \log_a x + \log_a y \)
2. \( \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \)
3. \( \log_a (x^b) = b \log_a x \)
Example 1.40 Find the domain of 

\[ f(x) = \log_{10} \log_{2} \log_{4}(\tan^{-1} x)^{-1}. \]

Sol. We must have \( \log_{4}(\tan^{-1} x)^{-1} > 0 \)

\[ \Rightarrow \log_{2}(\tan^{-1} x)^{-1} > 1 \]

\[ \Rightarrow 0 < (\tan^{-1} x)^{-1} < 2/\pi \]

\[ \Rightarrow \pi/2 < \tan^{-1} x < \infty, \text{which is not possible.} \]

Hence, the domain is \( \phi \).

Example 1.41 Find the domain and range of

\[ f(x) = \sqrt[3]{\log_{3}(\cos(x))}. \]

Sol. \( f(x) = \sqrt[3]{\log_{3}(\cos(x))} \).

\( f(x) \) is defined only if \( \log_{3}(\cos(x)) \geq 0 \)

\[ \Rightarrow \cos(x) \geq 1 \]

\[ \Rightarrow \cos(x) = 1 \text{ as } -1 \leq \cos(x) \leq 1 \]

\[ \Rightarrow x = 0 \Rightarrow x = n\pi, n \in \mathbb{Z}. \]

Hence, the domain consists of the multiples of \( \pi \), i.e., Domain = \( \{ n\pi; n \in \mathbb{Z} \} \).

Also, the range is \( (0) \).

Example 1.42 Solve \( \log_{2}(x^2-1) \leq 0 \).

Sol. Given \( \log_{2}(x^2-1) \leq 0 \)

If \( x > 1 \), then

\[ 0 < x^2 - 1 \leq 1 \]

\[ \Rightarrow 1 < x^2 \leq 2 \]

\[ \Rightarrow x \in [1, \sqrt{2}] \]

If \( 0 < x < 1 \) then \( x^2 - 1 \geq 1 \Rightarrow x^2 \geq 2 \)

\[ \Rightarrow x \in (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty) \]

\[ \Rightarrow x = \phi \]

Thus, \( x \in (1, \sqrt{2}] \).

Example 1.43 Find the number of solutions of \( 2^x + 3^x + 4^x - 5^x = 0 \).

Sol. \( 2^x + 3^x + 4^x = 5^x \)

\[ \Rightarrow \left( \frac{2}{5} \right)^x + \left( \frac{3}{5} \right)^x + \left( \frac{4}{5} \right)^x = 1 \]

Now, the number of solution of the equation is equal to number of times

\[ y = \left( \frac{2}{5} \right)^x + \left( \frac{3}{5} \right)^x + \left( \frac{4}{5} \right)^x \text{ and } y = 1 \text{ intersect.} \]

From the graph, the equation has only one solution.
Example 1.44 Find the domain of \( f(x) = \sin^{-1}\left( \log_9\left( \frac{x^2}{4} \right) \right) \).

Sol. We have \( f(x) = \sin^{-1}\left( \log_9\left( \frac{x^2}{4} \right) \right) \).

Since the domain of \( \sin^{-1}x \) is \([-1, 1]\).

Therefore, \( f(x) = \sin^{-1}\left( \log_9\left( \frac{x^2}{4} \right) \right) \) is defined,

if \(-1 \leq \log_9\left( \frac{x^2}{4} \right) \leq 1 \)

\[ \Rightarrow \frac{4}{9} \leq \frac{x^2}{4} \leq 9 \]

\[ \Rightarrow \frac{4}{9} \leq x^2 \leq 36 \]

\[ \Rightarrow \frac{2}{3} \leq x \leq 6 \]

\[ \Rightarrow x \in \left[ -\frac{2}{3}, \frac{2}{3} \right] \cup \left[ \frac{2}{3}, 6 \right] \]

Hence, the domain of \( f(x) \) is \( \left[ -\frac{2}{3}, \frac{2}{3} \right] \cup \left[ \frac{2}{3}, 6 \right] \).

Example 1.45 Find the domain of function \( f(x) = \log_4\left( \log_3(18x^2 - 77) \right) \).

Sol. We have \( f(x) = \log_4\left( \log_3(18x^2 - 77) \right) \).

Since \( \log_3 x \) is defined for all \( x > 0 \). Therefore, \( f(x) \) is defined if

\[ \log_3 (18x^2 - 77) > 0 \text{ and } 18x^2 - 77 > 0 \]

\[ \Rightarrow 18x^2 - 77 > 0 \text{ and } x^2 > 5 \]

\[ \Rightarrow x \in (\infty, \infty) \text{ and } x > \frac{\sqrt{77}}{3} \]

\[ \Rightarrow x > \frac{\sqrt{77}}{3} \]

\[ \Rightarrow x > \left( \frac{\sqrt{77}}{3} \right) \]

\[ \Rightarrow x \in (\infty, \infty) \]

Hence, the domain of \( f(x) \) is \((\infty, \infty)\).

Example 1.46 Find the domain of \( f(x) = \sqrt{\log_{0.4}\left( \frac{x-1}{x+5} \right)} \).

Sol. \( f(x) = \sqrt{\log_{0.4}\left( \frac{x-1}{x+5} \right)} \) exists if \( \log_{0.4}\left( \frac{x-1}{x+5} \right) \geq 0 \) and

\[ \frac{x-1}{x+5} > 0 \]

\[ \Rightarrow \frac{x-1}{x+5} \leq (0.4)^0 \text{ and } \frac{x-1}{x+5} > 0 \]

\[ \Rightarrow \frac{x-1}{x+5} \leq 1 \text{ and } \frac{x-1}{x+5} > 0 \]

\[ \Rightarrow x \in (-\infty, -5) \cup (1, \infty) \]

\[ \Rightarrow x > 1 \]

\[ \Rightarrow x > 1 \]

Thus, the domain \( f(x) \) is \((-\infty, -5) \cup (1, \infty)\).

Greatest Integer and Fractional Part Function

Greatest Integer Function (Floor Value Function)

\[ y = f(x) = [x] \] (Greatest integer \( \leq x \))

Graph of \( y = [x] \)

Properties
- Domain \( \rightarrow R; \) Range \( \rightarrow Z; \)
- \([x] = n \) \( n \in \mathbb{Z} \) \( \Rightarrow x \in [n, n+1) \).
1.18 Calculus

\[ x - 1 < \{x\} \leq x \]
\[ -\lfloor x \rfloor + \{x\} = 0, \text{ if } x \in \mathbb{Z} \]
\[ -\lfloor x \rfloor + \{x\} = -1, \text{ if } x \notin \mathbb{Z} \]
\[ \lfloor x \rfloor \geq n \Rightarrow x \geq n, n \in \mathbb{Z} \]
\[ \lfloor x \rfloor \leq n \Rightarrow x < n + 1, n \in \mathbb{Z} \]
\[ \lfloor x \rfloor > n \Rightarrow x \geq n + 1, n \in \mathbb{Z} \]

\[ \begin{align*}
\lfloor x + \{x\} \rfloor & = \lfloor x \rfloor + \{x\} \\
\lfloor x + 1 \rfloor & = \lfloor x \rfloor + 1 \\
\lfloor x + 2 \rfloor & = \lfloor x \rfloor + 2 \\
& \vdots \\
\lfloor x + n - 1 \rfloor & = \lfloor x \rfloor + n - 1 \\
\end{align*} \]

\[ \text{or } \lfloor x \rfloor + \lfloor \frac{x + 1}{n} \rfloor + \lfloor \frac{x + 2}{n} \rfloor + \ldots + \lfloor \frac{x + n - 1}{n} \rfloor = \lfloor nx \rfloor, n \in \mathbb{N} \]

**Fractional Part Function**

\[ y = f(x) = \{x\} = x - \lfloor x \rfloor \]

**Graph of \( y = \{x\} \)**

**Properties**

- **Domain**: \( \mathbb{R} \); **Range**: \( [0, 1) \); **Period**: 1.
- \( \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \), if \( 0 \leq \{x\} + \{y\} < 1 \)
- \( \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1, 1 \leq \{x\} + \{y\} < 2 \)
- \( \{x\} + \{y\} = 0 \) if \( x \in \mathbb{Z} \)
- \( \{x\} + \{y\} = 1 \) if \( x \notin \mathbb{Z} \)

**Example 1.47** Find the domain of \( f(x) = \sqrt{\lfloor x \rfloor - 1} + \sqrt{4 - \{x\}} \), where \( \lfloor \cdot \rfloor \) represents the greatest integer function.

**Solution**

Given \( f(x) = \sqrt{\lfloor x \rfloor - 1} + \sqrt{4 - \{x\}} \), then \( f(x) \) is defined when \( \lfloor x \rfloor - 1 \geq 0 \) and \( 4 - \{x\} \geq 0 \)
\[ \therefore 1 \leq \lfloor x \rfloor \leq 4 \text{ or } 1 \leq x < 5. \]

Hence, the domain of \( f(x) = D_f = [1, 5) \).

**Example 1.48** Find the domain and range of \( f(x) = \sin^{-1} \{x\} \)

(where \( \lfloor \cdot \rfloor \) represents the greatest integer function).

**Solution**

\( f(x) = \sin^{-1} \{x\} \) is defined if \( -1 \leq \{x\} \leq 1 \)
\[ \Rightarrow [x] = -1, 0, 1 \]
\[ \Rightarrow x \in [-1, 2) \]
\[ \Rightarrow \text{Range is } \{\sin^{-1}(-1), \sin^{-1} 0, \sin^{-1} 1\} = \{-\pi/2, 0, \pi/2\}. \]

**Example 1.49** Find the domain and range of \( f(x) = \log \{x\} \), where \( \{\cdot\} \) represents the fractional part function.

**Solution**

We know that \( 0 \leq \{x\} < 1 \forall x \in \mathbb{R} \)
Now when \( \{x\} = 0 \), \( \log \{x\} \) is not defined. So \( x \) cannot be integer. Hence, the domain is \( R - I \).
Now for \( 0 < \{x\} < 1 \), \( -\infty < \log \{x\} < 0 \Rightarrow \text{Range is } (-\infty, 0) \)

**Example 1.50** Find the range of \( f(x) = \lfloor \sin \{x\} \rfloor \) where \( \{\} \) represents the fractional part function, \( [\cdot] \) represents greatest the integer function.

**Solution**

\( f(x) = \lfloor \sin \{x\} \rfloor \)

Here, \( \{x\} \) can take all its possible values and sine function is defined for all real values.

Hence, \( 0 \leq \{x\} < 1 \)
\[ \Rightarrow 0 \leq \sin \{x\} < 1 \]
\[ \Rightarrow \lfloor \sin \{x\} \rfloor = 0. \]

Hence, the range is \( \{0\} \).

**Example 1.51** Solve \( 2[x] = x + \{x\} \), where \([\cdot]\) and \(\{\cdot\}\) denote the greatest integer function and fractional part, respectively.

**Solution**

Given \( 2[x] = x + \{x\} \)
\[ \Rightarrow 2[x] = [x] + \{x\} \]
\[ \Rightarrow \{x\} = \frac{2[x] - [x]}{2} \]
\[ \Rightarrow 0 \leq \frac{2[x] - [x]}{2} < 1 \]
\[ \Rightarrow 0 \leq \{x\} < 2 \]
\[ \Rightarrow \{x\} = 0, 1. \]

For \( [x] = 0 \), we get \( \{x\} = 0 \Rightarrow x = 0 \)
For \( [x] = 1 \), we get \( \{x\} = \frac{1}{2} \Rightarrow x = \frac{3}{2} \).

**Example 1.52** Find the range of \( f(x) = \frac{x - \lfloor x \rfloor}{1 - \lfloor x \rfloor + \{x\}} \), where \( \lfloor \cdot \rfloor \) represents the greatest integer function.

**Solution**

\[ f(x) = \frac{x - \lfloor x \rfloor}{1 - \lfloor x \rfloor + \{x\}} = \frac{x}{1 + \{x\}} = 1 - \frac{1}{1 + \{x\}} \]

Now, \( 0 \leq \{x\} < 1 \)
\[ \Rightarrow 1 \leq \{x\} + 1 < 2 \]
\[ \Rightarrow \frac{1}{2} < \frac{1}{1 + \{x\}} \]
\[ \Rightarrow -1 \leq -\frac{1}{1 + \{x\}} < -\frac{1}{2} \]
\[ \Rightarrow 0 \leq 1 - \frac{1}{1 + \{x\}} < \frac{1}{2} \]

**Example 1.53** Solve the system of equation in \( x, y, \) and \( z \) satisfying the following equations
\[ x + y + z = 3.1 \]
\[ \lfloor x \rfloor + y + z = 4.3 \]
\[ \lfloor x \rfloor + y + z + 5.4 \]

(where \( \lfloor \cdot \rfloor \) denotes the greatest integer function and \( \{\cdot\} \) denotes fractional part.)
Solutions

Adding all the three equations, \(2(x + y + z) = 12.8\) or \(x + y + z = 6.4\).

Adding first two equations, we get \(x + y + z + [y] + \{x\} = 7.4\).

Adding second and third equations, we get \(x + y + z + [z] + \{y\} = 9.7\).

Adding first and fourth equations, we get \(x + y + z + [x] + \{z\} = 8.5\).

From (1) and (2), \([y] + \{x\} = 1\).

From (1) and (3), \([z] + \{y\} = 3.3\).

From (1) and (4), \([x] + \{z\} = 2.1\).

\([z] = 2, [y] = 1, [x] = 3, \{y\} = 0.3 \text{ and } \{z\} = 0.1\).

\(\Rightarrow x = 2, y = 1.3, z = 3.1\).

Example 1.54

Solve \(x^2 - 4 - [x] = 0\) (where \([\cdot]\) denotes the greatest integer function).

Solution

The best method to solve such a system is graphical one.

Given equation is \(x^2 - 4 = [x]\).

Then, the solution of the equation are values of \(x\) where graph \(y = x^2 - 4\) and \(y = [x]\) intersect.

\[\text{Fig. 1.48}\]

From the graph, it is seen that graph intersects when \(x^2 - 4 = 2\) and \(x^2 - 4 = -2\).

\(\Rightarrow x = \sqrt{6} \text{ or } -\sqrt{2}\).

Example 1.55

If \(f(x) = \begin{cases} [x], & 0 \leq [x] < 0.5 \\ [x]+1, & 0.5 \leq [x] < 1 \end{cases}\), then prove that \(f(x) = -f(-x)\) (where \([\cdot]\) and \(\{\cdot\}\) represent the greatest integer function and fractional part function).

Solution

\(f(-x) = \begin{cases} [-x], & 0 \leq [-x] < 0.5 \\ [-x]+1, & 0.5 \leq [-x] < 1 \end{cases}\)

\(\Rightarrow \begin{cases} [-x], & 0 \leq [-x] < 0.5 \\ [-x]+1, & 0.5 \leq [-x] < 1 \end{cases}\)

\(\Rightarrow \begin{cases} [-x], & 0 \leq [-x] < 0.5 \\ [-x]+1, & 0.5 \leq [-x] < 1 \end{cases}\)

\(\Rightarrow \begin{cases} [-x], & [-x] = 0 \\ [-x]+1, & 0 < [-x] < 0.5 \\ [-x]+1, & 0.5 < [-x] < 1 \end{cases}\)

\(\Rightarrow \begin{cases} [-x], & [-x] = 0 \\ -[-x], & 0 < 1-[-x] < 0.5 \\ -[-x]+1, & 0.5 < 1-[-x] < 1 \end{cases}\)

\(\Rightarrow \begin{cases} [-x], & [x] = 0 \\ -[-x], & 0 < 1-[-x] < 0.5 \\ -[-x]+1, & 0.5 < 1-[-x] < 1 \end{cases}\)

Concept Application Exercise 1.7

In the following questions:

1. Solve \(|x|^2 - 5|x| + 6 = 0\).

2. If \(y = 3|x| + 1 = 4|x - 1| - 10\), then find the value of \(|x+2y|\).

3. Find the domain of

\(a. f(x) = \frac{1}{\sqrt{x-\{x\}}} \quad b. f(x) = \frac{1}{\log(x-\{x\})} \quad c. f(x) = \log(x-\{x\})\).

4. Find the domain of \(f(x) = \frac{1}{\sqrt{|x|-1}} - 5\).

5. Find the domain of \(f(x) = \frac{1}{\sqrt{1-\sin x}}\).

6. Find the range of \(f(x) = \log_2(\{x\}) - \log_2(1-\{x\})\).

7. Find the range of \(f(x) = \cos(\log_2\{x\})\).

8. Find the range of \(f(x) = \log_2(\{x\}) - \sin x\).

9. Solve: \((x-2)[x] = \{x\} - 1\), (where \([\cdot]\) and \(\{\cdot\}\) denotes the greatest integer function less than or equal to \(x\) and fractional part function respectively).

Signum Function

\(y = f(x) = \text{sgn}(x)\)

\(\text{Fig. 1.49}\)

\(\text{sgn}(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \text{ or } \{x\} = 0 \\ 0, & x = 0 \end{cases}\)

Domain \(\rightarrow \mathbb{R}\); Range \(\rightarrow \{-1, 0, 1\}\);
1.20 Calculus

Nature: Any one, odd function

In general, \( \text{sgn}(f(x)) = \begin{cases} \frac{|f(x)|}{f(x)}, & f(x) \neq 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0 \\ 0, & f(x) = 0 \\ 1, & f(x) > 0 \end{cases} \)

or \( \text{sgn}(f(x)) = \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ -1, & x > 0 \end{cases} \)

Example 1.56 Write the equivalent (piecewise) definition of \( f(x) = \text{sgn}(\sin x) \).

\[
\begin{align*}
\text{Sol.} \quad \text{sgn}(\sin x) &= \begin{cases} -1, & \sin x < 0 \\ 0, & \sin x = 0 \\ 1, & \sin x > 0 \end{cases} \\
&= \begin{cases} -1, & x \in (\pi + (2n + 1)\pi, (2n + 2)\pi), n \in \mathbb{Z} \\ 0, & x = n\pi, n \in \mathbb{Z} \\ 1, & x \in (2n\pi, (2n + 1)\pi), n \in \mathbb{Z} \end{cases}
\end{align*}
\]

Fig. 1.50

Example 1.57 Find the range of \( f(x) = \text{sgn}(x^2 - 2x + 3) \).

\[
\begin{align*}
\text{Sol.} \quad x^2 - 2x + 3 &= (x-1)^2 + 1 > 0 \quad \forall \quad x \in \mathbb{R} \\
\Rightarrow f(x) &= \text{sgn}(x^2 - 2x + 3) = 1 \\
\text{Hence, the range is} \quad \{1\}.
\end{align*}
\]

Functions of the Form \( f(x) = \max\{g_1(x), g_2(x), \ldots, g_s(x)\} \) or \( f(x) = \min\{g_1(x), g_2(x), \ldots, g_s(x)\} \)

Let's consider the function \( f(x) = \max\{x, x^2\} \).

To write the equivalent definition of the function, first draw the graph of \( y = x \) and \( y = x^2 \).

Now from the graph, we can see that

For \( x \in (-\infty, 0) \), the graph of \( y = x^2 \) lies above the graph of \( y = x \), or \( x^2 > x \).

For \( x \in (0, 1) \), the graph of \( y = x \) lies above the graph of \( y = x^2 \) or \( x > x^2 \).

For \( x \in (1, \infty) \), the graph of \( y = x^2 \) lies above the graph of \( y = x \) or \( x^2 > x \).

Hence, we have \( f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \leq x \leq 1 \\ x^2, & x > 0 \end{cases} \)

For \( f(x) = \min\{x, x^2\} \), we have \( f(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ x, & x > 0 \end{cases} \)

Example 1.58 If \( f, g : R \rightarrow R \) be the two given functions, then prove that \( 2 \min\{f(x) - g(x), 0\} = f(x) - g(x) - |f(x) - g(x)| \).

\[
\begin{align*}
\text{Sol.} \quad h(x) &= 2 \min\{f(x) - g(x), 0\} = \begin{cases} f(x) - g(x), & f(x) > g(x) \\ 0, & f(x) \leq g(x) \end{cases} = \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x), & f(x) \leq g(x) \end{cases} \\
&= \begin{cases} f(x) - g(x), & f(x) > g(x) \\ f(x) - g(x), & f(x) \leq g(x) \end{cases} = h(x) = f(x) - g(x) - |f(x) - g(x)|.
\end{align*}
\]

Example 1.59 Draw the graph of the function \( f(x) = \max\{\sin x, \cos 2x\} \), \( x \in [0, 2\pi] \). Write the equivalent definition of \( f(x) \) and find the range of the function.

\[
\begin{align*}
\text{Sol.} \quad \sin x &= \cos 2x \\
\Rightarrow \sin x &= 1 - 2\sin^2 x \\
\Rightarrow 2\sin^2 x + \sin x - 1 &= 0 \\
\Rightarrow (2\sin x - 1)(\sin x + 1) &= 0 \\
\Rightarrow \sin x &= 1/2 \text{ or } \sin x = -1 \\
\Rightarrow x &= \pi/6, 5\pi/6 \text{ or } x = \pi
\end{align*}
\]

Fig. 1.52

From the graph \( f(x) = \begin{cases} \cos 2x, & 0 \leq x < \frac{\pi}{6} \\ \sin x, & \frac{\pi}{6} \leq x < \frac{5\pi}{6} \\ \cos 2x, & \frac{5\pi}{6} < x \leq 2\pi \end{cases} \)

Also range of the function is \([-1, 1]\).
Methods to Determine One-One and Many-One

a. Let \( x_1, x_2 \in \text{domain of } f \) and if \( x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \) for every \( x_1, x_2 \) in the domain, then \( f \) is one-one else many-one.

b. Conversely, if \( f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \) for every \( x_1, x_2 \) in the domain, then \( f \) is one-one else many-one.

c. If the function is entirely increasing or decreasing in the domain, then \( f \) is one-one else many-one.

d. Any continuous function \( f(x) \) that has at least one local maxima or local minima is many-one.

e. All even functions are many-one.

f. All polynomials of even degree defined in \( R \) have at least one local maxima or minima and hence are many-one in the domain \( R \). Polynomials of odd degree can be one-one or many-one.

g. If \( f \) is a rational function, then \( f(x_1) = f(x_2) \) will always be satisfied when \( x_1 = x_2 \) in the domain. Hence, we can write \( f(x_1) = f(x_2) = (x_1 - x_2) g(x_1, x_2) \) where \( g(x_1, x_2) \) is some function in \( x_1 \) and \( x_2 \).

h. Draw the graph of \( y = f(x) \) and determine whether \( f(x) \) is one-one or many-one.

Example 1.60

Let \( f : R \to R \) where \( f(x) = \frac{x^2 + 4x + 7}{x^2 + x + 1} \). Is \( f(x) \) one-one?

Sol.

Let \( f(x_1) = f(x_2) \)

\[
\Rightarrow \frac{x^2 + 4x_1 + 7}{x^2_1 + x_1 + 1} = \frac{x^2 + 4x_2 + 7}{x^2_2 + x_2 + 1}
\]

\[
\Rightarrow (x_1 - x_2)(x_1 + 2x_1 + x_2 + x_2 + 1) = 0
\]

One solution is obviously \( x_1 = x_2 \).

Let us consider \( 2x_1 + x_2 + x_2 + 1 = 0 \).

Here, we have got a relation between \( x_1 \) and \( x_2 \) and for each value of \( x_1 \) in the domain we get a corresponding value of \( x_2 \) which may or may not be same as \( x_1 \). Let us check this out:

If \( x_1 = 0 \), we get \( x_2 = -1 \neq x_1 \) and both lie in the domain of \( f \). Hence, we have two different values \( x_1 = 0 \) and \( x_2 = -1/2 \) for which \( f(x) \) has the same value.

That is, \( f(0) = f(-1/2) = 7 \) and hence \( f \) is many-one.

Example 1.61

If \( f : X \to \{1, \infty\} \) be a function defined as \( f(x) = 1 + 3x^3 \). Find the super set of all the sets \( X \) such that \( f(x) \) is one-one.

Sol.

Note that \( f(x) \geq 1 \)

\[
\Rightarrow 1 + 3x^3 \geq 1
\]

\[
\Rightarrow x^3 \geq 0
\]

\[
\Rightarrow x \in \{0, \infty\}.
\]
Moreover, for \( x_1, x_2 \in [0, \infty), x_1 \neq x_2 \)
\[ \Rightarrow 1 + 3x_1^3 \neq 1 + 3x_2^3 \]
\[ \Rightarrow f(x_1) \neq f(x_2) \]
Thus, \( f : [1, \infty) \) is one-one for \( x \in [0, \infty) \).

### Onto and Into Functions

Let \( f : X \rightarrow Y \) be a function. If each element in the co-domain \( Y \) has at least one pre-image in the domain \( X \), that is, for every \( y \in Y \) there exists at least one element \( x \in X \) such that \( f(x) = y \), then \( f \) is onto. In other words, the range of \( f = Y \) for onto functions.

On the other hand, if there exists at least one element in the co-domain \( Y \) which is not an image of any element in the domain \( X \), then \( f \) is into.

Onto function is also called surjective function and a function which is both one-one and onto is called bijective function.

For example, \( f : R \rightarrow R \) where \( f(x) = \sin x \) is into:
\[ f : R \rightarrow R \text{ where } f(x) = ax^3 + b \text{ is onto where } a \neq 0, b \in R. \]
Note that a function will be either onto or into.

### Methods to Determine Whether a Function is Onto or Into

**a.** If range = co-domain, then \( f \) is onto. If range is a proper subset of co-domain, then \( f \) is into.

**b.** Solve \( f(x) = y \) for \( x \), say \( x = g(y) \).

Now if \( g(y) \) is defined for each \( y \in \text{co-domain} \) and \( g(y) \in \text{domain of } f \) for all \( y \in \text{co-domain} \), then \( f(x) \) is onto. If this requirement is not met by at least one value of \( y \) in the co-domain, then \( f(x) \) is into.

### Remark

a. An into function can be made onto by redefining the co-domain as the range of the original function.

b. Any polynomial function \( f : R \rightarrow R \) is onto if degree is odd, into if degree of \( f \) is even.

### Number of Functions (Mappings)

Consider set \( A \) has \( n \) different elements and set \( B \) has \( r \) different elements and function \( f : A \rightarrow B \)

<table>
<thead>
<tr>
<th>Description</th>
<th>Equivalent to number of ways in which ( n ) different balls can be distributed among ( r ) persons if</th>
<th>Number of functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Total number of functions</td>
<td>Any one can get any number of objects</td>
<td>( r )</td>
</tr>
<tr>
<td>2. Total number of one-to-one function</td>
<td>Each gets exactly one object or permutation of ( n ) different objects taken ( r ) at a time</td>
<td>( C_n \cdot n! ) for ( r \geq n ), ( 0 ) for ( r &lt; n )</td>
</tr>
<tr>
<td>3. Total number of many-one functions</td>
<td>At least one gets more than one ball</td>
<td>( r^n - C_n \cdot n! \cdot n, r \geq n ), ( r^r ) for ( r &lt; n )</td>
</tr>
<tr>
<td>4. Total number of onto functions</td>
<td>Each gets at least one ball</td>
<td>( r^r - (r-1)^n + C_2 (r-2)^n - C_3 (r-3)^n + \cdots, r &lt; n ), ( r \geq n ), ( 0 ) for ( r &gt; n )</td>
</tr>
<tr>
<td>5. Total number of into function</td>
<td>All the balls are received by any one person</td>
<td>( r )</td>
</tr>
</tbody>
</table>

### Example 1.62

Let \( f : R \rightarrow R \) where \( f(x) = \sin x \). Show that \( f \) is into.

**Sol.** Since the co-domain of \( f \) is the set \( R \), whereas the range of \( f \) is the interval \([-1, 1]\), hence \( f \) is into.

Can you make it onto?

The answer is 'yes', if you redefine the co-domain.

Let \( f \) be defined from \( R \) to another set \( Y = [-1, 1] \), i.e.,

\[ f : R \rightarrow Y \text{ where } f(x) = \sin x, \text{ then } f \text{ is onto as range } f(x) = [-1, 1] = Y. \]

### Example 1.63

Let \( f : N \rightarrow Z \) be a function defined as \( f(x) = x - 1000 \). Show that \( f \) is an into function.

**Sol.** Let \( f(x) = y = x - 1000 \)
\[ \Rightarrow x = y + 1000 = g(y) \text{ (say)} \]
Here, \( g(y) \) is defined for each \( y \in I \), but \( g(y) \notin N \) for \( y \leq -1000 \). Hence, \( f \) is into.
Example 1.64 Let \( A = \{ x : -1 \leq x \leq 1 \} = B \) be a mapping \( f : A \to B \).
Then, match the following columns:

<table>
<thead>
<tr>
<th>Column I (Function)</th>
<th>Column II (Type of mapping)</th>
</tr>
</thead>
<tbody>
<tr>
<td>p. ( f(x) =</td>
<td>x</td>
</tr>
<tr>
<td>q. ( f(x) = x</td>
<td>x</td>
</tr>
<tr>
<td>r. ( f(x) = x^3 )</td>
<td>c. onto</td>
</tr>
<tr>
<td>s. ( f(x) = [x] ) where ([\cdot]) represents greatest integer function</td>
<td>d. into</td>
</tr>
<tr>
<td>t. ( f(x) = \sin \frac{\pi x}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

Sol. p, \( f(x) = |x| \)

![Fig. 1.55](image1.png)

The graph shows that \( f(x) \) is many-one, as the straight line parallel to \( x \)-axis meets at more than one point. Hence, function is many-one-into, therefore neither injective nor surjective.

q. \( f(x) = |x| \)

\[
|\begin{align*}
&x^2, \quad x \geq 0 \\
&-x^2, \quad x < 0
\end{align*}\]

![Fig. 1.56](image2.png)

The graph shows that \( f(x) \) is one-one, as the straight line parallel to \( x \)-axis cuts only at one point.
Here, the range \( f(x) \in [-1, 1] \).
Thus, range = co-domain.
Hence, onto.
Therefore, \( f(x) \) is one-one and onto or (bijective).

r. \( f(x) = x^3 \)

![Fig. 1.57](image3.png)

The graph shows that \( f(x) \) is one-one onto (i.e., bijective) (as explained in the above example).

s. \( f(x) = [x] \)

![Fig. 1.58](image4.png)

The graph shows that \( f(x) \) is many-one, as the straight line parallel to \( x \)-axis meets at more than one point. Here, the range \( f(x) \in \{-1, 0, 1\} \), which shows into as the range \( \subset \) co-domain.
Hence, many-one-into.

t. \( f(x) = \sin \frac{\pi x}{2} \)

![Fig. 1.59](image5.png)

The graph shows that \( f(x) \) is one-one and onto as range = co-domain.
Therefore, \( f(x) \) is bijective.

Example 1.65 Show \( f : R \to R \) defined by \( f(x) = (x-1)(x-2)(x-3) \) is surjective but not injective.

Sol.

![Fig. 1.60](image6.png)
Graphically, \( y = f(x) = (x-1)(x-2)(x-3) \), which is clearly many-one and onto.

**Example 1.66** If the function \( f: R \rightarrow A \) given by \( f(x) = \frac{x^2}{x^2 + 1} \) is surjection, then find \( A \).

**Sol.** The domain of \( f(x) \) is all real numbers.
Since, \( f: R \rightarrow A \) is surjective, therefore \( A \) must be the range of \( f(x) \).
Let \( f(x) = y \)
\[
\Rightarrow y = \frac{x^2}{x^2 + 1}
\]
\[
\Rightarrow x^2 y + y = x^2
\]
\[
\Rightarrow x = \sqrt{\frac{y}{1-y}} \text{ exists if } \frac{y}{1-y} \geq 0
\]

- 
  - 
  - 

Fig. 1.61

\[
\Rightarrow 0 \leq y < 1. \text{ Hence, } A \in [0, 1).
\]

**Example 1.67** If \( f: R \rightarrow R \) be a function such that \( f(x) = x^3 + x^2 + 3x + \sin x \). Then identify the type of function.

**Sol.**
\[
f(x) = x^3 + x^2 + 3x + \sin x
\]
\[
\Rightarrow f'(x) = 3x^2 + 2x + 3 + \cos x
\]
\[
= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{8}{9}\right] - \cos x > 0 \text{ as } 3\left(\frac{x + 1}{3}\right)^2 + \frac{8}{9} \text{ min}
\]
\[
= \frac{8}{3} \text{ and } -\cos x \text{ has a maximum value } 1.
\]
\[
\Rightarrow f(x) \text{ is strictly increasing and hence one-one.}
\]
Also, \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = -\infty \). Thus, the range of \( f(x) \) is \( R \), hence onto.

**Example 1.68** If \( f: R \rightarrow R, f(x) = \begin{cases} x & \text{if } x \in Q, \\ x^2 & \text{if } x \in Q.
\end{cases}\) then identify the type of function.

**Sol.**
\[
f(2) = f\left(\frac{3}{4}\right) \Rightarrow \text{many-to-one function}
\]
and \( f(x) \neq \sqrt{3} \quad \forall x \in R \Rightarrow \text{into function.}

**Concept Application Exercise 1.9**

1. Which of the following functions from \( Z \) to itself are bijections?
   a. \( f(x) = x^3 \)
   b. \( f(x) = x + 2 \)
   c. \( f(x) = 2x + 1 \)
   d. \( f(x) = x^2 + x \)

2. If \( f: N \rightarrow Z \) \( f(n) = \begin{cases} \frac{n-1}{2} & \text{when } n \text{ is odd}, \\
\frac{n}{2} & \text{when } n \text{ is even}
\end{cases} \)

**Odd Function**
A function \( y = f(x) \) is said to be an odd function if \( f(-x) = -f(x) \forall x \in D_f \).
Graph of an odd function \( y = f(x) \) is symmetrical in opposite quadrants, i.e., if point \((x, y)\) lies on the graph then \((-x, -y)\) also lies on the graph.

**Fig. 1.62**
### Properties of Odd and Even Functions

- Sometimes it is easy to prove that \( f(x) = f(-x) = 0 \) for even functions and \( f(x) \neq f(-x) = 0 \) for odd functions.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f(x) + g(x) )</th>
<th>( f(x) - g(x) )</th>
<th>( f(x)g(x) )</th>
<th>( f(x)/g(x) )</th>
<th>( \log f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>Even</td>
<td>Even</td>
<td>Even</td>
<td>Even</td>
<td>Even</td>
<td>Even</td>
</tr>
<tr>
<td>Even</td>
<td>Odd</td>
<td>Neither even nor odd</td>
<td>Neither even nor odd</td>
<td>Odd</td>
<td>Odd</td>
<td>Even</td>
</tr>
<tr>
<td>Odd</td>
<td>Even</td>
<td>Neither even nor odd</td>
<td>Neither even nor odd</td>
<td>Odd</td>
<td>Odd</td>
<td>Even</td>
</tr>
<tr>
<td>Odd</td>
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\[ f(x) = x^2 \sin x. \]

\[ f(x) = \sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}. \]

\[ f(x) = \log \left( \frac{1-x}{1+x} \right). \]

\[ f(x) = \log \left( \frac{1+x}{1-x} \right) - f(-x), \text{ hence } f(x) \text{ is odd.} \]

\[ f(-x) = (-x)^2 \sin (-x) = -x^2 \sin x = -f(x), \text{ hence } f(x) \text{ is odd.} \]

\[ f(x) = \log \left( \frac{1-x}{1+x} \right). \]

\[ f(x) = \log \left( \frac{1+x}{1-x} \right) = -f(x), \text{ hence } f(x) \text{ is odd.} \]

\[ f(x) = \sin x - \cos x. \]

\[ f(x) = \frac{e^x + e^{-x}}{2}. \]

\[ f(x) = \log \left( x + \sqrt{1 + x^2} \right). \]

\[ f(-x) = \log \left( -x + \sqrt{1 + x^2} \right). \]

\[ f(x) = \log \left( \frac{1-x}{1+x} \right). \]

\[ f(-x) = \log \left( \frac{1+x}{1-x} \right). \]

\[ f(x) = \log \left( \frac{1+x}{1-x} \right) - f(-x), \text{ hence } f(x) \text{ is odd.} \]

\[ f(x) = \sin x - \cos x. \]

\[ f(x) = \frac{e^x + e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = f(x), \text{ hence } f(x) \text{ is even.} \]

### Example 1.69
Which of the following functions is (are) even, odd or neither

- a. \( f(x) = x^2 \sin x. \)
- b. \( f(x) = \sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}. \)
- c. \( f(x) = \log \left( \frac{1-x}{1+x} \right). \)
- d. \( f(x) = \log \left( x + \sqrt{1 + x^2} \right). \)
- e. \( f(x) = \sin x - \cos x. \)
- f. \( f(x) = \frac{e^x + e^{-x}}{2}. \)

**Solution**

- a. \( f(-x) = (-x)^2 \sin (-x) = -x^2 \sin x = -f(x), \text{ hence } f(x) \text{ is odd.} \)
- b. \( f(-x) = \sqrt{1 + (-x) + (-x)^2} - \sqrt{1 - (-x) + (-x)^2} \)
  \[= \sqrt{1 - x^2} - \sqrt{1 + x^2} + (-x)^2 = -f(x), \text{ hence } f(x) \text{ is odd.} \)
- c. \( f(-x) = \log \left( \frac{1-(x)}{1-(-x)} \right) = \log \left( \frac{1+x}{1-x} \right). \)

### Example 1.70
Prove that \( f(x) \) given by \( f(x + y) = f(x) + f(y) \) \( \forall x \in R \) is an odd function.

**Solution**

Given \( f(x+y) = f(x) + f(y) \) \( \forall x \in R \) (1)

Replace \( y \) by \(-x\), we have \( f(x-x) = f(x) + f(-x) \)

\[ f(x) + f(-x) = 0 \]

Now put \( x = y = 0 \) in (1), we have \( f(0 + 0) = f(0) + f(0) \)

\[ f(0) = 0. \]
Then from (2), \( f(x) + f(-x) = 0 \). Hence, \( f(x) \) is an odd function.

**Extension of Domain**

Let a function be defined on a certain domain which is entirely non-negative (or non-positive). The domain of \( f(x) \) can be extended to the set \( X = \{ x : x \in \text{domain of } f(x) \} \) in two ways:

a. **Even extension:** The even extension is obtained by defining a new function \( f(-x) \) for \( x \in X \), such that \( f(-x) = f(x) \).

b. **Odd extension:** The odd extension is obtained by defining a new function \( f(-x) \) for \( x \in X \), such that \( f(-x) = -f(x) \).

**Example 1.71**

If \( f(x) = \begin{cases} x^3 + x^2 & \text{for } 0 \leq x \leq 2 \\ x + 2 & \text{for } 2 < x \leq 4 \end{cases} \), then find the even and odd extension of \( f(x) \).

**Sol.** For even extension \( f(x) = f(-x) \):

\[
 f(x) = \begin{cases} 
 (x^3 + (x)^2) & 0 \leq x \leq 2 \\
 -x + 2 & 2 < -x \leq 4 \\
 -x^2 - x^3 & -2 \leq x \leq 0 
\end{cases}
\]

The odd extension of \( f(x) \) is as follows:

\[
 h(x) = \begin{cases} 
 x - 2 & -4 \leq x < -2 \\
 x^3 - x^2 & -2 \leq x \leq 0 
\end{cases}
\]

**Example 1.72**

Let the function \( f(x) = x^2 + x \sin x - \cos x + \log(1 + |x|) \) be defined on the interval \([0, 1] \). Define functions \( g(x) \) and \( h(x) \) in \([-1, 0] \) satisfying \( g(-x) = -f(x) \) and \( h(-x) = f(x) \) for all \( x \in [0, 1] \).

**Sol.** Clearly, \( g(x) \) is the odd extension of the function \( f(x) \) and \( h(x) \) is the even extension.

Since \( x^2, \cos x, \log(1 + |x|) \) are even functions and \( x, \sin x \) are odd functions.

\[
 g(x) = -x^2 + x + \sin x + \cos x - \log(1 + |x|)
\]

and \( h(x) = x^2 - x - \sin x - \cos x + \log(1 + |x|) \)

Clearly, this function satisfies the restriction of the problem.

**Concept Application Exercise 1.10**

Identify the following functions whether odd or even or neither.

1. \( f(x) = (g(x) - g(-x))^3 \)
2. \( f(x) = \log \left( \frac{x^4 + x^2 + 1}{x^4 + x^2 + 1} \right) \)
3. \( f(x) = xg(x)g(-x) + \tan (\sin x) \)
4. \( f(x) = \cos |x| + \left\lfloor \frac{\sin x}{2} \right\rfloor \)

**PERIODIC FUNCTIONS**

A function \( f : X \rightarrow Y \) is said to be a periodic function if there exists a positive real number \( T \) such that \( f(x + T) = f(x) \), for all \( x \in X \). The least of all such positive numbers \( T \) is called the principal period or simply period of \( f \). All periodic functions can be analyzed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

In other words, a function is said to be periodic function if its each value is repeated after a definite interval.

Here, the least positive value of \( T \) is called the fundamental period of the function. Clearly, \( f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots \)

**Important Facts about Periodic Functions**

- If \( f(x) \) is periodic with period \( T \), then \( af(x + \mathbf{b}) = \mathbf{a}f(x) \) and \( f(bx + c) = \mathbf{f}(bx) \) for all \( x \), \( b, c \in \mathbb{R} \).

- If \( f(x) \) is periodic with period \( T \), then \( f(ax + b) = f(x) \), where \( a, b, c \in \mathbb{R} \).

**Proof:** Consider \( a > 0 \)

- Let \( f(x + T) = f(x) \) and \( f(ax + aT) + b = f(ax + b) \)
- \( f(ax + b + aT') = f(ax + b) \)
- \( f(y + aT') = f(y + T) \)
- \( \Rightarrow T' = \frac{T}{a} \) (since \( a \) is always positive).

- Let \( f(x) \) has period \( p = m/r \) and \( q \) has period \( q = r/s \) \( (r, s, t \in \mathbb{N} \) and co-prime \() \) and \( p \) and \( q \) are also periodic with periods \( \frac{T}{a} \) and \( \frac{T}{b} \), respectively.

Then \( t \) will be the period of \( f + g \), provided there does not exist a positive number \( k \) for which \( f(x + k) + g(x + k) = f(x) + g(x) \), else \( k \) will be the period. The same rule is applicable for any positive algebraic combination of \( f(x) \) and \( g(x) \).

**LCM of** \( p \) and \( q \) exists if \( p \) and \( q \) are rational quantities. If \( p \) and \( q \) are irrational, then LCM of \( p \) and \( q \) does not exist unless they have same irrational surd. LCM of rational and irrational is not possible.

- \( \sin x, \cos x, \csc x, \sec x \) and \( \tan x \) have period \( \pi \) if \( n \) is odd and \( \pi \) if \( n \) is even.

- \( \tan x \) and \( \cot x \) have period \( \pi \) whether \( n \) is odd or even.

- A constant function is periodic but does not have a well-defined period.
• If $g$ is periodic, then $fog$ will always be a periodic function.

Period of $fog$ may or may not be the period of $g$.

• A continuous periodic function is bounded.

• If $f(x), g(x)$ are periodic functions with periods $T_1, T_2$, respectively, then, we have $h(x) = f(x) + g(x)$ has period as

a. LCM of $\{T_1, T_2\}$; if $f(x)$ and $g(x)$ cannot be interchanged by adding a least positive number less than the LCM of $\{T_1, T_2\}$.

b. $k$; if $f(x)$ and $g(x)$ can be interchanged by adding a least positive number $k (k < \text{LCM of } \{T_1, T_2\})$.

For example, consider the function $f(x) = |\sin x| + |\cos x|$, $|\sin x|$ and $|\cos x|$ have period $\pi$, hence according to the rule of LCM period of $f(x)$ is $\pi$.

But $f\left( x + \frac{\pi}{2} \right) = |\sin \left( x + \frac{\pi}{2} \right)| + |\cos \left( x + \frac{\pi}{2} \right) |
= |\cos x| + |\sin x|$. Hence, period of $f(x)$ is $\pi/2$.

**Example 1.73** Find the periods (if periodic) of the following functions, $[.]$ denotes the greatest integer function.

a. $f(x) = e^{\log(\sin x)} + \tan^2 x - \csc(3x - 5)$

Sol. $f(x) = e^{\log(\sin x)} + \tan^2 x - \csc(3x - 5)$

Period of $e^{\log(\sin x)}$ is $2\pi$, $\tan x$ is $\pi$, $\csc(3x - 5)$ is $\frac{2\pi}{3}$

.. period = LCM of $\left\{ 2\pi, \pi, \frac{2\pi}{3} \right\} = 2\pi$.

b. $f(x) = x - (x - b) = b + (x - b) \Rightarrow f(x)$ has period $1$.

c. $f(x) = |\sin x + \cos x|$

Since period of $|\sin x + \cos x|$ is $\pi$ and period of $|\sin x| + |\cos x|$ is $2\pi$.

Hence, period of $f(x) = \text{LCM of } \left\{ \frac{\pi}{2}, \pi \right\} = \pi$

d. $f(x) = \tan \frac{\pi}{2} \left[ x \right] \Rightarrow \tan \frac{\pi}{2} \left[ x + T \right] = \tan \frac{\pi}{2} \left[ x \right]$.

.. period = 2 (least positive value).

**Example 1.74** Find the period if $f(x) = \sin x + \{x\}$, where $\{x\}$ is the fractional part of $x$.

Sol. Here $\sin x$ is periodic with period $2\pi$, $\{x\}$ is periodic with $1$, LCM of $2\pi$ (irrational) and $1$ (rational) does not exist. Thus, $f(x)$ is not periodic.

**Example 1.75** If $f(x) = \sin x + \cos ax$ is a periodic function, show that $a$ is a rational number.

**Sol.** Period of $\sin x = \frac{2\pi}{1}$ and period of $\cos ax = \frac{2\pi}{|a|}$

.. period of $\sin x + \cos ax = \text{LCM of } \frac{2\pi}{1} \text{ and } \frac{2\pi}{|a|}$

$\Rightarrow \text{LCM of } \frac{2\pi}{1} \text{ and } \frac{2\pi}{|a|} = \frac{2\pi}{\frac{|a|}{\lambda}} \text{ where } \lambda$ is the HCF of $1$ and $a$.

Since $\lambda$ is the HCF of $1$ and $a$, $\frac{1}{\lambda}$ and $\frac{|a|}{\lambda}$ should be both integers.

Suppose $\frac{1}{\lambda} = p$ and $\frac{|a|}{\lambda} = q$, then $\frac{|a|}{1} = \frac{q}{p}$, where $p, q \in \mathbb{Z}$.

i.e., $|a| = \frac{q}{p}$.

Hence, $a$ is the rational number.

**Example 1.76** Discuss whether the function $f(x) = \sin(\cos x + x)$ is periodic or not, if yes then what is its period.

**Sol.** Clearly, $f(x + 2\pi) = \sin(\cos(2\pi + x) + 2\pi + x) = \sin(\cos x + (x + \cos x)) = \sin(\cos x + \cos x)$

Hence, period is $2\pi$.

**Example 1.77** Find the period of $\cos(\cos x) + \cos(\sin x)$.

**Sol.** Clearly, the domain of the function is $R$

Let $f(x) = f(x + T)$, for all $x$

.. $f(0) = f(T)$

.. $\cos 1 = \cos(\cos T) + \cos(\sin T)$

Clearly, $T = \frac{\pi}{2}$ satisfies the equation, hence period is $\frac{\pi}{2}$

**Example 1.78** Find the period of the function satisfying the relation $f(x) + f(x + 3) = 0$ for all $x \in R$.

**Sol.** Given $f(x) + f(x + 3) = 0$ for all $x \in R$

Replace $x$ by $x + 3$,

We have $f(x + 3) + f(x + 6) = 0$.

From (1) and (2), $f(x) = f(x + 6)$.

Hence, the function has period 6.

**Concept Application Exercise 1.11**

1. Match the column

<table>
<thead>
<tr>
<th>Column I (Function)</th>
<th>Column II (Period)</th>
</tr>
</thead>
<tbody>
<tr>
<td>p. $f(x) = \sin^2 x + \cos^2 x$</td>
<td>a. $\pi/2$</td>
</tr>
<tr>
<td>q. $f(x) = \cos^2 x + \sin^2 x$</td>
<td>b. $\pi$</td>
</tr>
<tr>
<td>r. $f(x) = \sin^2 x + \cos^2 x$</td>
<td>c. $2\pi$</td>
</tr>
<tr>
<td>s. $f(x) = \cos^2 x - \sin^2 x$</td>
<td>d. $3\pi/2$</td>
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</table>
2. Which of the following functions is not periodic?
   a. \( \sin 3x + \sin^2 x \)  
   b. \( \cos \sqrt{x} + \cos^2 x \)  
   c. \( \cos 4x + \tan^2 x \)  
   d. \( \cos 2x + \sin x \)  

3. Let \([x]\) denote the greatest integer less than or equal to \(x\). If the function \(f(x) = \tan \left( \sqrt{[n]} \right) x\) has period \(\frac{\pi}{3}\), then find the values of \(n\).

4. Find the period of
   a. \[ \frac{\sin 4x + |\cos 4x|}{|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x|} \]
   b. \(f(x) = \sin \frac{\pi x}{n} - \cos \frac{\pi x}{(n+1)!}\)
   c. \(f(x) = \sin x + \frac{\tan \frac{x}{2}}{2} + \sin \frac{x}{2}^2 + \tan \frac{x}{2}^2 + \ldots + \sin \frac{x}{2}^{n-1} + \frac{\tan x}{2}^n\)

5. If \(f(x) = \lambda \sin x + \lambda^2 \cos x + g(\lambda)\) has the period equal to \(\pi/2\), then find the value of \(\lambda\).

6. If \(f(x)\) satisfying the relation \(f(x) + f(x + 4) = f(x + 2) + f(x + 6)\) for all \(x\), then prove that \(f(x)\) is periodic and find its period.

**COMPOSITE FUNCTION**

Let \(A, B\) and \(C\) be three non-empty sets.

Let \(f: A \to B\) and \(g: B \to C\) be two functions, then \(gof: A \to C\).
This function is called composition of \(f\) and \(g\), given by
\[ (gof)(x) = g(f(x)) \quad \forall x \in A. \]

Thus, the image of every \(x \in A\) under the function \(gof\) is the \(g\)-image of the \(f\)-image of \(x\).

The \(gof\) is defined only if \(\forall x \in A, f(x)\) is an element of the domain of \(g\) so that we can take its \(g\)-image.

The range of \(f\) must be a subset of the domain of \(g\) in \(gof\).

**Properties of Composite Functions**

a. The composition of functions is not commutative in general, i.e., \(fg \neq gf\).

b. The composition of functions is associative, i.e., if \(h: A \to B, g: B \to C\) and \(f: C \to D\) are three functions, then \((fg)oh = f(goh)\).

c. The composition of any function with the identity function is the function itself, i.e., \(f: A \to B\) then \(fI_A = f\) where \(I_A\) and \(I_B\) are the identity functions of \(A\) and \(B\), respectively.

d. If \(f: A \to B\) and \(g: B \to C\) are one-one, then \(gof: A \to C\) is also one-one.

**Proof**

Suppose \(gof(x_1) = gof(x_2)\)
\[ \Rightarrow g(f(x_1)) = g(f(x_2)) \]
\[ \Rightarrow f(x_1) = f(x_2), \text{ as } g \text{ is one-one} \]
\[ \Rightarrow x_1 = x_2, \text{ as } f \text{ is one-one} \]
Hence, \(gof\) is one-one.

e. If \(f: A \to B\) and \(g: B \to C\) are onto, then \(gof: A \to C\) is also onto.

**Proof**

Given an arbitrary element \(z \in C\), there exists a pre-image \(y\) of \(z\) under \(g\) such that \(g(y) = z\), since \(g\) is onto. Further, for \(y \in B\), there exists an element \(x\) in \(A\) with \(f(x) = y\), since \(f\) is onto.

Therefore, \(gof(x) = g(f(x)) = g(y) = z\), showing that \(gof\) is onto.

f. If \(gof(x)\) is one-one, then \(f(x)\) is necessarily one-one but \(g(x)\) may not be one-one.

Consider the function \(f(x)\) and \(g(x)\) as shown in the following figure.

**Fig. 1.65**

Here \(f\) is one-one, but \(g\) is many-one. But \(g(f(x))\): \((1, 1), (2, 2), (3, 3), (4, 4)\) is one-one.

g. If \(gof(x)\) is onto, then \(g(x)\) is necessarily onto but \(f(x)\) may not be onto.

**Fig. 1.66**

Here, \(f\) is into and \(g\) is onto. But \(gof(x)\): \((1, 1), (2, 2), (3, 3), (4, 4)\) is onto.

Thus, it can be verified in general that \(gof\) is one-one implies that \(f\) is one-one. Similarly, \(gof\) is onto implies that \(g\) is onto.

**Example 1.79**

Let \(f: \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}\) and \(g: \{3, 4, 5, 9\} \to \{7, 11, 15\}\) be functions defined as \(f(2) = 3, f(3) = 4, f(4) = f(5) = 5\) and \(g(3) = g(4) = 7, g(5) = g(9) = 11\). Find \(gof\).

Sol. We have \(gof(2) = g(f(2)) = g(3) = 7, gof(3) = g(f(3)) = g(4) = 7, gof(4) = g(f(4)) = g(5) = 11\) and \(gof(5) = g(f(5)) = 11\).

**Example 1.80**

Let \(f(x)\) and \(g(x)\) be bijective functions where \(f: \{a, b, c, d\} \to \{1, 2, 3, 4\}\) and \(g: \{3, 4, 5, 6\} \to \{w, x, y, z\}\), respectively. Then, find the number of elements in the range set of \(g(f(x))\).
Example 1.81
Let \( f(x) = ax + b \) and \( g(x) = cx + d, a \neq 0, c \neq 0 \).
Assume \( a = 1, b = -2 \). If \((gof)(x) = (gof)(x)\) for all \( x \), what can you say about \( c \) and \( d \)?

\[ \begin{align*}
\text{Sol.} \quad (gof)(x) &= f(g(x)) = a(cx + d) + b \\
&= acx + ad + b \\
\text{and} \quad (gof)(x) &= g(f(x)) = c(ax + b) + d \\
&= cacx + cab + d \\
\text{Given that} \quad (gof)(x) &= (gof)(x)\) \text{ and at} \quad a = 1, b = 2 \\
\Rightarrow \quad cx + d + 2 = cx + 2c + d \Rightarrow c = 1 \text{ and } d \text{ is arbitrary}.
\end{align*} \]

Example 1.82
Suppose that \( g(x) = 1 + \sqrt{x} \) and \( f(g(x)) = 3 + 2\sqrt{x} + x \), then find the function \( f(x) \).

\[ \begin{align*}
\text{Sol.} \quad g(x) &= 1 + \sqrt{x} \text{ and } f(g(x)) = 3 + 2\sqrt{x} + x \\
\Rightarrow \quad f(1 + \sqrt{x}) &= 3 + 2\sqrt{x} + x \\
\text{Put} \quad 1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2 \\
\text{Then} \quad f(y) &= 3 + 2(y - 1)^2 + y^2 \\
\text{Therefore,} \quad f(x) &= 2x^2 + x.
\end{align*} \]

Example 1.83
The function \( f(x) \) is defined in \([0, 1]\). Find the domain of \((f \circ g)(x)\).

\[ \begin{align*}
\text{Sol.} \quad f(x) \text{ is defined in } [0, 1], \text{ i.e., the only value of } x \text{ that we can substitute lies between } [0, 1]. \\
\text{For } f(\tan x) \text{ to be defined, we must have } 0 \leq \tan x \leq 1 \text{ [as } x \text{ replaced by } \tan x]\) \\
i.e., n\pi \leq x \leq n\pi + \frac{\pi}{4}, n \in \mathbb{Z} \text{ [in general]} \\
\text{Thus, the domain for } f(\tan x) = [n\pi, n\pi + \frac{\pi}{4}], n \in \mathbb{Z}.
\end{align*} \]

Example 1.84
\[ \begin{align*}
f(x) &= \begin{cases} 
1, & 0 < x < 1 \\
2x, & x \geq 1
\end{cases} \quad \text{and } g(x) = \begin{cases} 
x^2, & x < 0 \\
1, & x \geq 0
\end{cases}
\end{align*} \]
then find \( (gof)(x) \) and find its domain and range.

\[ \begin{align*}
\text{Sol.} \quad (gof)(x) &= \begin{cases} 
g(x) + 1, & g(x) < 0 \\
(g(x))^2, & g(x) \geq 0
\end{cases} \\
&= \begin{cases} 
x^3 + 1, & x < 0 \\
(2x - 1)^2, & x \geq 1
\end{cases} \\
\text{For } x < 0, x^3 + 1 \in (-\infty, 1) \\
\text{For } 0 \leq x < 1, x^6 \in [0, 1) \\
\text{For } x \geq 1, (2x - 1)^2 \in [1, \infty)
\end{align*} \]
Hence, the range is \( \mathbb{R} \) and function is many-one.

**Inverse Functions**

If \( f: A \to B \) be a function defined by \( y = f(x) \) such that \( f \) is both one-one and onto, then there exists a unique function \( g: B \to A \) such that for each \( y \in B, g(y) = x \) if and only if \( y = f(x) \). The function \( g \) so defined is called the inverse of \( f \) and denoted by \( f^{-1} \). Also if \( g \) is the inverse of \( f \), then \( f \) is the inverse of \( g \) and the two functions \( f \) and \( g \) are said to be inverses of each other.

The condition for the existence of inverse of a function is that the function must be one-one and onto. Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and the domain of the original function becomes the range of the inverse function.

**Properties of Inverse Functions**

- The inverse of bijective function is unique and bijective.
- Let \( f: A \to B \) be a function such that \( f \) is bijective and \( g: B \to A \) is inverse of \( f \), then \( fog = fog = I_B \) = identity function of set \( B \). Then \( fog = I_A \) = identity function of set \( A \).
- If \( fog = gof \) then either \( f^{-1} = g \) or \( g^{-1} = f \) and \( fog \) and \( gof \) = \( fog(x) = x \).

![Fig. 1.67](image-url)
1.30 Calculus

- If \( f \) and \( g \) are two bijective functions such that \( f: A \rightarrow B \) and \( g: B \rightarrow C \), then \( g \circ f: A \rightarrow C \) is bijective. Also \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\).
- Graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) are symmetrical about \( y = x \) line and intersect on line \( y = x \) or \( f(x) = f^{-1}(x) = x \) whenever graphs intersect.

![Graphs](image)

**Fig. 1.68**

But in the case of the function \( f(x) = \begin{cases} x + 4, & x \in [1, 2] \\ -x + 7, & x \in [5, 6] \end{cases} \),

\[ f^{-1}(x) = \begin{cases} x + 4, & x \in [1, 2] \\ 7 - x, & x \in [5, 6] \end{cases} \]

\( y = f(x) \) and \( y = f^{-1}(x) \) intersect at \((3/2, 11/2)\) and \((11/2, 3/2)\) which do not lie on the line \( y = x \).

![Graph](image)

**Fig. 1.69**

**Example 1.87** Which of the following functions has inverse function?

a. \( f: Z \rightarrow Z \) defined by \( f(x) = x + 2 \)

b. \( f: Z \rightarrow Z \) defined by \( f(x) = 2x \)

c. \( f: Z \rightarrow Z \) defined by \( f(x) = x \)

d. \( f: Z \rightarrow Z \) defined by \( f(x) = |x| \)

Sol. Functions in options a and e are both one-one and have range \( Z \), i.e., onto, hence invertible. \( f: Z \rightarrow Z \) defined by \( f(x) = 2x \) is one-one but has only even integers in the range, hence not onto.

\( f: Z \rightarrow Z \) defined by \( f(x) = |x| \) is many-one and has range \( \mathbb{N} \cup \{0\} \).

Thus, both the functions are not invertible.

**Example 1.88** Let \( A = R - \{3\}, B = R - \{1\} \) and let \( f: A \rightarrow B \) defined by \( f(x) = \frac{x - 2}{x - 3} \). Is \( f \) invertible? Explain.

**Sol.** Let \( x_1, x_2 \in AC \)

\[ \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3} \]
\[ \Rightarrow x_1.x_2 - 3x_1 - 2x_2 + 6 = x_1.x_2 - 3x_2 - 3x_1 + 6 \]
\[ \Rightarrow x_1 = x_2 \]

\( f \) is one-one.

To find whether \( f \) is onto or not, first let us find the range of \( f \).

\[ \text{Let } y = f(x) = \frac{x - 2}{x - 3} \]
\[ \Rightarrow x - 3y = x - 2 \]
\[ \Rightarrow x = 3y - 2 \]
\[ \Rightarrow x = \frac{3y - 2}{y - 1} \]

\( x \) is defined if \( y \neq 1 \), i.e., the range of \( f \) is \( R - \{1\} \) which is also the co-domain of \( f \).

Also, for no value of \( y \), \( x \) can be 3, i.e., if we put

\[ 3 = x = \frac{3y - 2}{y - 1} \]
\[ \Rightarrow 3y - 3 = 3y - 2 \Rightarrow -3 = -2 \text{ not possible. Hence, } f \text{ is onto.} \]

**Example 1.89** Let \( f: R \rightarrow R \) be defined by \( f(x) = (e^x - e^{-x})/2 \). Is \( f(x) \) invertible? If so, find its inverse.

**Sol.** Let us check for inevitability of \( f(x) \)

(a) One-one

Let \( x_1, x_2 \in R \) and \( x_1 < x_2 \)

\[ e^{x_1} < e^{x_2} \quad (\because e > 1) \]

Also \( x_1 < x_2 \Rightarrow -x_2 < -x_1 \)

\[ e^{-x_2} < e^{-x_1} \quad (\because e > 1) \]

(1) + (2) \( \Rightarrow 1/2(e^{x_1} - e^{-x_1}) < 1/2(e^{x_2} - e^{-x_2}) \Rightarrow f(x_1) < f(x_2) \)

i.e. \( f \) is one-one.

(b) Onto

As \( x \rightarrow -\infty, f(x) \rightarrow \infty \)

Similarly, as \( x \rightarrow -\infty, f(x) \rightarrow -\infty \), i.e., \( -\infty < f(x) < \infty \) so long as \( x \in (-\infty, \infty) \).
Hence, the range of \( f \) is same as the set \( R \). Therefore, \( f(x) \) is onto.

Since \( f(x) \) is both one-one and onto, \( f(x) \) is invertible.

(c) To find \( f^{-1} \)

\[
y = f(x) = (e^x - e^{-x})/2
\]
\[
\Rightarrow e^x - e^{-x} = 2y
\]
\[
= e^{2x} - 2ye^x - 1 = 0
\]
\[
\Rightarrow e^x = y \pm \sqrt{y^2 + 1}
\]

(as \( y - \sqrt{y^2 + 1} < 0 \) for all \( y \) and \( e^x \) is always positive)
\[
\Rightarrow x = \log_e \left( y + \sqrt{y^2 + 1} \right)
\]
\[
f^{-1}(x) = \log_e \left( x + \sqrt{x^2 + 1} \right)
\]

Example 1.90

If \( f(x) = (ax^2 + b)^3 \), then find the function \( g \) such that \( f(g(x)) = g(f(x)) \).

Sol. \( f(g(x)) = g(f(x)) \)
\[
f(x) = (ax^2 + b)^3
\]
If \( g(x) = f^{-1}(x) \)
\[
y = (ax^2 + b)^3 \Rightarrow y^{1/3} - b = ax^2
\]
\[
g(x) = \sqrt[3]{y/b} - b
\]

Example 1.91

If \( f(x) = 3x - 2 \) and \( (gof)^{-1}(x) = x - 2 \), then find the function \( g(x) \).

Sol. \( f(x) = 3x - 2 \)
\[
\Rightarrow f^{-1}(x) = \frac{x + 2}{3}
\]
Now \( (gof)^{-1}(x) = x - 2 \)
\[
\Rightarrow f^{-1}(g^{-1}(x)) = x - 2
\]
\[
\Rightarrow g^{-1}(x) + 2 = x - 2
\]
\[
\Rightarrow g^{-1}(x) = 3x - 8
\]
\[
g(x) = \frac{x + 8}{3}
\]

Example 1.92

Find the inverse of \( f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \leq x \leq 4 \\ 8\sqrt{x}, & x > 4 \end{cases} \)

Sol. Given \( f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \leq x \leq 4 \\ 8\sqrt{x}, & x > 4 \end{cases} \)

Let \( f(x) = y \)
\[
\Rightarrow x = f^{-1}(y)
\]
\[
: x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \leq y \leq 4 \\ \sqrt{y^2/64}, & y > 4 \end{cases}
\]
\[
: y = \begin{cases} \sqrt{y}, & y < 1 \\ y, & 1 \leq y \leq 16 \\ y^2/64, & y > 16 \end{cases}
\]
\[
\Rightarrow f^{-1}(y) = \begin{cases} \sqrt{y}, & 1 \leq y \leq 16 \\ \sqrt{y^2/64}, & y > 16 \end{cases}
\]
[From (1)]

Hence, \( f^{-1}(x) = \begin{cases} \sqrt{x}, & 1 \leq x \leq 16 \\ \sqrt{x^2/64}, & x > 16 \end{cases} \)

Example 1.93

Solve the equation \( x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}} \)

where \( x \geq \frac{3}{4} \)

Sol. \( f(x) = x^2 - x + 1 \)
\[
\text{and } g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}},
\]
are inverse of one another so \( f(x) = g(x) \).
When \( f(x) = x \)
\[
\Rightarrow x^2 - x + 1 = x
\]
\[
\Rightarrow x = 1
\]

Concept Application Exercise 1.13

Find the inverse of the following functions:

1. \( f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2 \)
2. \( f: R \rightarrow (-\infty, 1) \) is given by \( f(x) = 1 - 2^x \)
3. Let \( f: (2, 3) \rightarrow (0, 1) \) be defined by \( f(x) = x - \lfloor x \rfloor \), where \( \lfloor . \rfloor \) represents greatest integer function.
4. \( f: Z \rightarrow Z \) be defined by \( f(x) = \lfloor x \rfloor \), where \( \lfloor . \rfloor \) denotes the greatest integer function.
5. \( f(x) = \begin{cases} x^2 - 1, & x < 2 \\ x^2 + 3, & x \geq 2 \end{cases} \)
6. \( f: [-1, 1] \rightarrow [-1, 1] \) defined by \( f(x) = x|x| \)
7. \( f: (-\infty, 1) \rightarrow \left[ \frac{1}{2}, \infty \right) \), where \( f(x) = 2^{x-2} \)

IDENTICAL FUNCTION

Two functions \( f \) and \( g \) are said to be identical if

a. The domain of \( f \) = the domain of \( g \), i.e., \( D_f = D_g \).
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The range of \( f \) is the range of \( g \).

c. \( f(x) = g(x), \forall x \in D_f \) or \( x \in D_g \), e.g., \( f(x) = x \) and \( g(x) = \sqrt{x^2} \) are not identical functions as \( D_f = D_g \) but \( R_f = R, R_g = [0, \infty) \).

Example 1.94 Find the values of \( x \) for which the following functions are identical.

a. \( f(x) = x \) and \( g(x) = \frac{1}{1/x} \)

b. \( f(x) = \cos x \) and \( g(x) = \frac{1}{\sqrt{1 + \tan^2 x}} \)

c. \( f(x) = \frac{\sqrt{9-x^2}}{\sqrt{x-2}} \) and \( g(x) = \frac{9-x^2}{x-2} \)

d. \( f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x} \) and \( g(x) = \tan^{-1} x + \cos^{-1} x = \frac{\pi}{2} \) for \( x \in [-1, 1] \).

Hence, the functions are identical if \( x \in (0, 1] \)

**TRANSFORMATION OF GRAPHS**

a. \( f(x) \) transforms to \( f(x) + a \) (where \( a \) is +ve)

That is, \( f(x) \rightarrow f(x) + a \) shift the given graph of \( f(x) \) upward through \( a \) units.

\( f(x) \rightarrow f(x) - a \), shift the given graph of \( f(x) \) downward through \( a \) units.

Graphically, it could be stated as

![Graph 1.71](image)

b. \( f(x) \) transforms to \( f(x-a) \)

That is, \( f(x) \rightarrow f(x-a) \); \( a \) is positive. Shift the graph of \( f(x) \) through \( a \) unit towards right.

That is, \( f(x) \rightarrow f(x+a) \); \( a \) is positive. Shift the graph of \( f(x) \) through \( a \) units towards left.

Graphically, it could be stated as

![Graph 1.72](image)

Example 1.95 Plot \( y = |x|, y = |x-2| \) and \( y = |x+2| \).

Sol. As discussed \( f(x) \rightarrow f(x-a) \); shift towards right.

\( y = |x-2| \) is shifted 2 units towards right.

Also \( y = |x+2| \) is shifted 2 units towards left.

![Graph 1.73](image)
c. \( f(x) \) transforms to \( f(ax) \)

That is, \( f(x) \rightarrow f(ax); \ a > 1 \).
Shrink (or contract) the graph of \( f(x) \) \( a \) times along the \( x \)-axis.

Again \( f(x) \rightarrow f \left( \frac{1}{a} x \right); \ a > 1 \), stretch (or expand) the graph of \( f(x) \) \( a \) times along the \( x \)-axis.
Graphically, it could be stated as shown in Fig. 1.74.

Example 1.96
Plot \( y = \sin x \) and \( y = \sin 2x \).

Sol. Here \( y = \sin 2x \), shrink (or contract) the graph of \( \sin x \) by a factor of 2 along the \( x \)-axis.

From Fig. 1.75, \( \sin x \) is periodic with period \( 2\pi \) and \( \sin 2x \) with period \( \pi \).

Example 1.97
Plot \( y = \sin x \) and \( y = \sin \frac{x}{2} \).

Sol. Here \( y = \sin \left( \frac{x}{2} \right) \); stretch (or expand) the graph of \( \sin x \), 2 times along the \( x \)-axis.

From Fig. 1.76, \( \sin x \) is periodic with period \( 2\pi \) and \( \sin \left( \frac{x}{2} \right) \) is periodic with period \( 4\pi \).

d. \( f(x) \) transforms to \( y = af(x) \)

It is clear that the corresponding points (points with same \( x \) co-ordinates) would have their ordinates in the ratio of \( 1 : a \).

e. \( f(x) \) transforms to \( f(-x) \)

That is, \( f(x) \rightarrow f(-x) \)
To draw \( y = f(-x) \), take the image of the curve \( y = f(x) \) in the \( y \)-axis as plane mirror.
Or
Turn the graph of \( f(x) \) by \( 180^\circ \) about the \( y \)-axis.
Graphically, it is shown as...
**Example 1.100** Plot the curve \( y = \log_e (-x) \).

**Sol.** Here \( y = \log_e (-x) \); take mirror image of \( y = \log_e x \) about \( y \)-axis. Graphically, it is shown as

![Graph of \( y = \log_e (-x) \)](image)

**Fig. 1.81**

**Example 1.101** Draw the graph for \( y = [\log x] \).

**Sol.** To draw graph for \( y = [\log x] \), we have to follow two steps:

a. Leave the (+ve) part of \( y = \log x \) as it is.

b. Take images of (-ve) part of \( y = \log x \), i.e., the part below \( x \)-axis in the \( x \)-axis as plane mirror. Graphically, it is shown as

![Graphs of \( y = \log x \) and \( y = [\log x] \)](image)

**Fig. 1.84**

\( y = \log_e x \) is differentiable for all \( x \in (0, \infty) \) (Fig. 1.84(a)) but \( y = [\log x] \) is clearly differentiable for all \( x \in (0, \infty) - \{1\} \) as at \( x = 1 \) there is a sharp edge (Fig. 1.84(b)).

**Example 1.102** Sketch the graph for \( y = [\sin x] \).

**Sol.** Here \( y = \sin x \) is known.

\( \therefore \) To draw \( y = [\sin x] \), we take the mirror image (in \( x \)-axis) of the part of the graph of \( \sin x \) which lies below \( x \)-axis.

![Image of the part below \( x \)-axis](image)

**Fig. 1.85**

From the above figure, it is clear

\( y = [\sin x] \) is differentiable for all \( x \in \mathbb{R} - \{n\pi; n \in \text{integer}\} \).

**i.** \( f(x) \) transforms to \( f(|x|) \)

That is, \( f(x) \rightarrow f(|x|) \)

If we know \( y = f(x) \), then to plot \( y = f(|x|) \), we would follow two steps:

a. Leave the graph lying right side of the \( y \)-axis as it is.

b. Take the image of \( f(x) \) in the right of \( y \)-axis with \( y \)-axis as the plane mirror and the graph of \( f(x) \) lying left-side of the \( y \)-axis (if it exists) is omitted.

**Or**

Neglect the curve for \( x < 0 \) and take the images of curves for \( x \geq 0 \) about \( y \)-axis.

**Fig. 1.82**

\( f(x) \) transform to \( y = |f(x)| \)

\( |f(x)| = f(x) \) if \( f(x) \geq 0 \) and \( |f(x)| = -f(x) \) if \( f(x) < 0 \).

It means that the graph of \( f(x) \) and \( |f(x)| \) would coincide if \( f(x) \geq 0 \) and the parts where \( f(x) < 0 \) would get inverted in the upward direction.

Figure 1.82 would make the procedure clear.
Example 1.103 Sketch the curve \( y = \log |x| \).

Sol. As we know, the curve \( y = \log x \).

\[ \therefore y = \log |x| \] could be drawn in two steps:

a. Leave the graph lying right side of \( y \)-axis as it is.

b. Take the image of \( f(x) \) in the \( y \)-axis as plane mirror.

Example 1.104 Sketch the curve \( y = (x-1)(x-2) \).

Sol.

k. Drawing the graph of \( y = [f(x)] \) from the known graph of \( y = f(x) \)

It is clear that if \( n < f(x) < n+1, n \in I \) then \([f(x)] = n\). Thus, we would draw lines parallel to the \( x \)-axis passing through different integral points. Hence, the values of \( x \) can be obtained so that \( f(x) \) lies between two successive integers.

This procedure can be clearly understood from Fig. 1.90.

Concept Application Exercise 1.14

1. Draw the graph of the following functions: (1 to 5)
   a. \( f(x) = \sin |x| \)
   b. \( f(x) = |x-2| - 3 \)
   c. \( f(x) = \tan x \)
   d. \( f(x) = |x^2 - 3x| + 2 \)
   e. \( f(x) = -[x-1]^2 \)
   f. Find the total number of solutions of \( \sin \pi x = |\ln |x|| \).

2. Solve \( \frac{x^2}{x-1} \leq 1 \) using the graphical method.

3. Which of the following pair(s) of function have same graphs?

   a. \( f(x) = \sec x - \tan x \), \( g(x) = \cos x + \sin x \)
   b. \( f(x) = \sec x - \tan x \), \( g(x) = \cos x + \sin x \)
   c. \( f(x) = e^{x^2+3x+3} \), \( g(x) = x^2 + 3x + 3 \)
   d. \( f(x) = \sec x + \csc x \), \( g(x) = 2 \cos^2 x \)

MISCELLANEOUS SOLVED PROBLEMS

1. Let \( f : X \to Y \) be a function defined by \( f(x) = a \sin \left(x + \frac{\pi}{4}\right) + b \cos x + c \). If \( f \) is both one-one and onto, find sets \( X \) and \( Y \).

Sol. \( f(x) = a \sin \left(x + \frac{\pi}{4}\right) + b \cos x + c \)
Let \( f(x) = a \left( \frac{\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}}{4} \right) + b \cos x + c \)

\[ f(x) = \frac{a}{\sqrt{2}} \sin x + \left( \frac{a}{\sqrt{2}} + b \right) \cos x + c \]

Let \( \left( \frac{a}{\sqrt{2}} \right) = r \cos \alpha \) and \( \left( \frac{a}{\sqrt{2}} + b \right) = r \sin \alpha \)

\[ f(x) = r \left[ \cos \alpha \sin x + \sin \alpha \cos x \right] + c \]

\[ f(x) = r \left[ \sin (x + \alpha) \right] + c \]

where \( r = \sqrt{a^2 + 2ab + b^2} \)

and \( \alpha = \tan^{-1} \left( \frac{a + b\sqrt{2}}{a} \right) \).

For \( f \) to be one-one, we must have \(-\pi/2 \leq x + \alpha \leq \pi/2\). Thus,

domain \( \in \left[ -\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha \right] \) and range \( \in [c-r, c+r] \).

Or \( X = \left[ -\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha \right] \) and \( Y = [c-r, c+r] \).

2. Find the set of all solutions of the equation \( 2^{2^x} - [2^{x-1} - 1] = 2^{x-1} + 1 \).

**Sol.** Here, \( 2^{2^x} - [2^{x-1} - 1] = 2^{x-1} + 1 \).

**Case I:** \( y < 0 \)

\[ 2^y + (2^{x-1} - 1) = 2^{x-1} + 1 \]

\[ \Rightarrow 2^y = 1 \]

Hence, \( y = -1 \), which is true when \( y < 0 \).

**Case II:** \( 0 \leq y < 1 \)

\[ 2^y + (2^{x-1} - 1) = 2^{x-1} + 1 \]

\[ \Rightarrow 2^y = 2 \]

\[ \Rightarrow y = 1, \text{ which shows no solution as } 0 \leq y < 1. \]

**Case III:** \( y \geq 1 \)

\[ 2^y - (2^{x-1} - 1) = 2^{x-1} + 1 \]

\[ \Rightarrow 2^y = 2^{x-1} + 2^{x-1} \]

\[ \Rightarrow 2^y = 2^y \text{, which is an identity, therefore it is true } \forall \ y \geq 1 \]

Hence, from (1), (2) and (3) the solution of the set is \( \{ y : y \geq 1 \} \).

3. Let \( x \in \left( 0, \frac{\pi}{2} \right) \), then find the domain of the function

\[ f(x) = \frac{1}{\sqrt{-\log_{\sin x} \tan x}}. \]

**Sol.** Here \( x \in \left( 0, \frac{\pi}{2} \right) \)

\[ 0 < \sin x < 1 \]

and we know

\[ \log_{a} x < b \Rightarrow x > a^b, \text{ if } 0 < a < 1 \]

\[ x < a^b, \text{ if } a > 1 \]

Thus, \( f(x) = \frac{1}{\sqrt{-\log_{\sin x} \tan x}} \)

\[ \Rightarrow \log_{\sin x} \tan x < 0 \]

[as inequality sign changes on multiplying by -ve]

\[ \tan x > (\sin x)^b \]

[using (1) and (2)]

\[ \tan x < 1 \]

\[ x \in \left( \frac{\pi}{2}, \frac{\pi}{4} \right) \]

[as \( x \in (0, \pi/2) \)].

4. Find whether the given function is even or odd function, where \( f(x) = \left[ \frac{x}{\pi} + \frac{1}{2} \right] \), where \( [ \ ] \) denotes the greatest integer function.

**Sol.**

\[ f(x) = \frac{x}{\pi} + \frac{1}{2} \]

\[ f(-x) = -\frac{x}{\pi} + 0.5 \]

\[ f(-x) = \frac{x}{\pi} + 0.5 \]

Hence, \( f(-x) = -f(x) \) and \( f(-x) = 0 \).
1. If \( f(-x) = -f(x) \). Hence, \( f(x) \) is an odd function if \( x \neq n\pi \) and \( f(x) = 0 \) if \( x = n\pi \) is both even and odd function.

5. Let \( f(x) \) be periodic and \( k \) be a positive real number such that \( f(x + k) + f(x) = 0 \) for all \( x \in \mathbb{R} \). Prove that \( f(x) \) is a periodic function with period \( 2k \).

**Sol.** We have \( f(x + k) = f(x) \), \( \forall x \in \mathbb{R} \).

\[ \Rightarrow \quad f(x + 2k) = f(x) \], \( \forall x \in \mathbb{R} \) (as \( f(x + k) = f(x) \))

\[ \Rightarrow \quad f(x + 2k) = f(x) \], \( \forall x \in \mathbb{R} \)

which clearly shows that \( f(x) \) is periodic with period \( 2k \).

6. If \( f(x) \) be a polynomial function satisfying \( f(x) f \left( \frac{1}{x} \right) = f(x) + f \left( \frac{1}{x} \right) \) and \( f(4) = 65 \). Then find \( f(6) \).

**Sol.** Let \( f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n \).

Then, \( f(x) f \left( \frac{1}{x} \right) = f(x) + f \left( \frac{1}{x} \right) \)

\[ \Rightarrow \quad (a_0 x^n + a_1 x^{n-1} + \cdots + a_n) \left( a_0 x^{-n} + a_1 x^{-n+1} + \cdots + a_n \right) \]

\[ = \quad (a_0 x^n + a_1 x^{n-1} + \cdots + a_n) \left( a_0 x^{-n} + a_1 x^{-n+1} + \cdots + a_n \right) \]

On comparing the coefficient of \( x^n \), we have \( a_0 = 0 \)

\( \Rightarrow \quad a_n = 1 \) \( \frac{a_0}{a_n} \neq 0 \)

On comparing the coefficient of \( x^{-n} \), we have \( a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_n = 0 \)

\( \Rightarrow \quad a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_n = 0 \) \( a_0 = 1 \) \( a_1 = 1 \)

\( \Rightarrow \quad a_n = 0 \) \( a_{n-1} = 0 \)

Similarly, \( a_{n-2} = a_{n-3} = \cdots = a_1 = 0 \)

\( \Rightarrow a_0 = 1 \)

\( \Rightarrow a_1 = 1 \)

\( \Rightarrow a_2 = 0 \)

\( \Rightarrow a_3 = 0 \)

\( \Rightarrow \quad a_n = 65 \) \( f(4) = 65 \)

\( \Rightarrow \quad a_4 = 64 \)

\( \Rightarrow \quad a_5 = 3 \)

\( \Rightarrow \quad a_6 = 0 \)

\( \Rightarrow \quad a_7 = 0 \)

Therefore, \( f(x) = x^3 + 1 \). Hence, \( f(6) = 6^3 + 1 = 217 \).

7. Consider a real-valued function \( f(x) \) satisfying \( 2 f(xy) = f(x) f(y) \forall x, y \in \mathbb{R} \) and \( f(1) = a \) where \( a \neq 1 \). Prove that \( (a - 1) \sum_{i=1}^{n} f(i)^{a^{n+1}} = a^{n+1} - a \).

**Sol.** We have \( 2 f(xy) = f(x) f(y) \).

Replacing \( y \) by 1, we get \( 2 f(x) = f(x) f(1) \Rightarrow f(x) = x^a \)

\[ \Rightarrow \quad \sum_{i=1}^{n} f(i)^{a^{n+1}} = a^{n+1} - a \]

8. If \( (x+y+1) = \left( \sqrt{f(x)} + \sqrt{f(y)} \right)^2 \) and \( f(0) = 1 \), \( \forall x, y \in \mathbb{R} \). Determine \( f(n) \), \( n \in \mathbb{N} \).

**Sol.** Given \( (x+y+1) = \left( \sqrt{f(x)} + \sqrt{f(y)} \right)^2 \)

Putting \( x = y = 0 \),

\( \Rightarrow \quad f(1) = (1+1)^2 = 2^2 \)

Again putting \( x = 0, y = 1 \),

\( \Rightarrow \quad f(2) = (1+2)^2 = 3^2 \)

and for \( x = 1, y = 1 \),

\( \Rightarrow \quad f(3) = (3+2)^2 = 4^2 \)

Hence, \( f(n) = (n+1)^2 \).

9. Check whether the function defined by \( f(x + \lambda) = 1 + \sqrt{2 f(x) - f^2(x)} \) \( \forall x \in \mathbb{R} \) is periodic or not. If yes, then find its period. \( \lambda = 0 \).

**Sol.** For the function to be true, \( 2 f(x) - f^2(x) \geq 0 \)

\[ \Rightarrow \quad f(x) [f(x)-2] \leq 0 \Rightarrow 0 \leq f(x) \leq 2 \]

and from the given function, \( f(x+\lambda) \geq 1 \Rightarrow f(x) \geq 1 \)

From (1) and (2), we have \( 1 \leq f(x) \leq 2 \)

Again, we have \( \{f(x+\lambda)-1\}^2 = f(x) - f^2(x) \)

\[ \Rightarrow \quad \{f(x+\lambda)-1\}^2 = 1 + \{f(x)-1\}^2 \]

Replacing \( x \) by \( x+\lambda \), we get

\[ \{f(x+2\lambda)-1\}^2 = 1 - \{f(x+\lambda)-1\}^2 \]

Subtracting (3) from (4), we get

\[ f(x+2\lambda) = f(x) \]

\[ \Rightarrow \quad f(x+2\lambda) = f(x) \]

\[ \Rightarrow \quad f(x) \text{ is periodic with period } 2\lambda \]

10. If for all real values of \( u \) and \( v \), \( 2 f(u) \cos v = f(u+v) + f(u-v) \), prove that for all real values of \( u \)

\( a. \ f(x) + f(-x) = 2a \cos x \)

\( b. \ f(x) - f(-x) = 0 \)

\( c. \ f(x) + f(x) = 2b \sin x \)

Deduce that \( f(x) = a \cos x + b \sin x \), where \( a, b \) are arbitrary constants.

**Sol.** Given \( 2 f(u) \cos v = f(u+v) + f(u-v) \)

Putting \( u = 0 \) and \( v = \pi \), we get

\( f(x) + f(-x) = 2f(0) \cos x = 2a \cos x \)

or

\( f(0) = f(-x) = 0 \)

or

\( a \) is an arbitrary constant.

Now putting \( u = \pi/2 - x \) and \( v = \pi/2 \), we get

\( f(x) + f(-x) = 0 \)

Again putting \( u = \pi/2 \) and \( v = \pi/2 - x \), we get

\( f(x) + f(x) = 2f(\pi/2) \sin x = 2b \sin x \)

or

\( b \) is an arbitrary constant.

Adding (2) and (4), we get
2f(x) + f(π/2) + f(−x) = 2a cos x + 2b sin x
\implies 2f(x) + 0 = 2a cos x + 2b sin x
\therefore f(x) = a cos x + b sin x.

11. Let \( f(x) = \frac{9^x}{9^x + 3} \). Show \( f(x) + f(1 − x) = 1 \), and hence evaluate
\[ f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \ldots + f\left(\frac{1995}{1996}\right). \]

Sol. \( f(x) = \frac{9^x}{9^x + 3} \) (1)

and \( f(1−x) = \frac{9^{1−x}}{9^{1−x} + 3} \)
\[ \implies f(1−x) = \frac{9}{9^x + 3} = \frac{9}{9 + 3 \cdot 9^{x}} \]
\[ \implies f(1−x) = \frac{3}{3 + 9^x} \] (2)

Adding (1) and (2), we get \( f(x) + f(1−x) = \)
\[ = \frac{9^x}{9^x + 3} + \frac{3}{3 + 9^x} = 1 \]
\[ \implies f(x) + f(1−x) = 1 \] (3)

Now, putting \( x = \frac{1}{1996}, \frac{2}{1996}, \frac{3}{1996}, \ldots, \frac{998}{1996} \) in (3), we get
\[ f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) = \ldots = \]
\[ f\left(\frac{997}{1996}\right) + f\left(\frac{998}{1996}\right) = 1 \]
\[ \text{or } f\left(\frac{998}{1996}\right) = \frac{1}{2} \]

Adding all the above expression, we get
\[ f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + \ldots + f\left(\frac{1995}{1996}\right) \]

\[ = 1 + 1 + \ldots + 997 + \frac{1}{2} = 997 + \frac{1}{2} = 997.5 \]

12. Let \( f(x) \) be defined on \([-2, 2]\) and is given by
\[ f(x) = \begin{cases} -1, & -2 \leq x < 0 \\ x−1, & 0 < x \leq 2 \end{cases} \]

and \( g(x) = f(|x|) + |f(x)| \).

Then find \( g(x) \).

Sol. We have
\[ f(x) = \begin{cases} -1, & -2 \leq x < 0 \\ x−1, & 0 < x \leq 2 \end{cases} \] (1)

⇒ \[ f(|x|) = \begin{cases} -1, & -2 \leq x \leq 0 \\ |x|−1, & 0 \leq x \leq 2 \end{cases} \]
\[ \text{again, } f(x) = \begin{cases} -1, & -2 \leq x < 0 \\ x−1, & 0 < x \leq 2 \end{cases} \] (2)

\[ \therefore g(x) = f(|x|) + |f(x)| \text{ can be expressed as} \]
\[ g(x) = \begin{cases} (-x−1)+1, & -2 \leq x \leq 0 \\ (x−1)+(1−x), & 0 \leq x \leq 1 \\ (x−1)+(x−1), & 1 \leq x \leq 2 \end{cases} \]

[using (1) and (2)]
\[ \Rightarrow g(x) = \begin{cases} -x, & -2 \leq x \leq 0 \\ 0, & 0 < x \leq 1 \\ 2(x−1), & 1 < x \leq 2 \end{cases} \]

13. For what integral value of \( n \), is the period of the function \( \cos (nx) \sin \left(\frac{5x}{n}\right) \)?
Sol. Let \( f(x) = \cos \left( \frac{5x}{n} \right) \sin \left( \frac{5x}{n} \right) \)

\( f(x) \) is periodic
\[ f(x + \lambda) = f(x) \text{ where } \lambda = \text{ period.} \]
\[ \Rightarrow \cos \left( nx + n\lambda \right) \sin \left( \frac{5x}{n} \right) = \cos nx \sin \left( \frac{5x}{n} \right) \]

at \( x = 0, \cos n\lambda \sin \left( \frac{5\lambda}{n} \right) = 0 \)
If \( \cos n\lambda = 0 \)
\[ \Rightarrow n\lambda = r\pi + \frac{\pi}{2}, r \in \mathbb{I} \]
\[ \Rightarrow n(3\pi) = r\pi + \frac{\pi}{2} \]
\[ \Rightarrow 3n - r = \frac{1}{2} \text{ (Impossible).} \]
Again, let \( \sin \left( \frac{5\lambda}{n} \right) = 0 \)
\[ \Rightarrow \frac{5\lambda}{n} = p\pi \quad (p \in \mathbb{I}) \]
\[ \Rightarrow \frac{5(3\pi)}{n} = p\pi \]
\[ \Rightarrow n = \frac{15}{p} \]
For \( p = \pm 1, \pm 3, \pm 5, \pm 15 \)
\[ n = \frac{15}{p} \]

15. If \( f(x) = 1 + |x - 2|, 0 \leq x \leq 4 \) and \( g(x) = 2 - |x|, -1 \leq x \leq 3 \). Then find \((f \circ g)(x)\) and \((g \circ f)(x)\).

Sol. We have
\[ f(x) = \begin{cases} 1 - x, & 0 \leq x \leq 2 \\ x - 3, & 2 < x \leq 4 \end{cases} \]
and \( g(x) = \begin{cases} 2 + x, & -1 \leq x < 0 \\ 2 - x, & 0 < x \leq 3 \end{cases} \)
\[ \therefore (f \circ g)(x) = \begin{cases} 1 - (g(x)), & 0 \leq g(x) \leq 2 \\ g(x) - 3, & 2 < g(x) \leq 4 \end{cases} \]

Let \( f : R \to R \) and \( g : R \to R \) be functions defined
by \( f(x) = \begin{cases} 2x, & x < 0 \\ 2x^2 - 1, & x \geq 0 \end{cases} \) and \( g(x) = \begin{cases} x + 2, & x < 0 \\ 2x, & x \geq 0 \end{cases} \).

Find (a) \( f + g \), (b) \( f \circ g \).

Sol. \((f + g) : R \to R \) and \((f \circ g) : R \to R \) are functions defined by \((f + g)(x) = f(x) + g(x) \) and \((f \circ g)(x) = f(f(g(x))) \). To find \((f + g)(x) \) and \((f \circ g)(x) \), we rewrite \( f(x) \) and \( g(x) \) as:

\[ f(x) = \begin{cases} 2x, & x < 0 \\ 2x^2 - 1, & x \geq 0 \end{cases} \]
\[ g(x) = \begin{cases} x + 2, & x < 0 \\ 2x, & x \geq 0 \end{cases} \]

Hence, a. \((f + g)(x) = \begin{cases} 3x + 2, & x < 0 \\ 4x, & 0 \leq x < 1 \\ 2x^2 + 2x - 1, & x \geq 1 \end{cases} \)

b. \((f \circ g)(x) = \begin{cases} 2x^2 + 4x, & x < 0 \\ 4x^2, & 0 \leq x < 1 \\ 4x^3 - 2x, & x \geq 1 \end{cases} \)
EXERCISES

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1. Write explicit functions of $y$ defined by the following equations and also find domains of definitions of the given implicit functions:
   a. $x + |y| = 2y$
   b. $e^y - e^{-y} = 2x$
   c. $10^x + 10^{-x} = 10$
   d. $x^2 - \sin^2 y = \frac{\pi}{2}$

2. Let $g(x) = \sqrt{x - 2k}, \forall 2k \leq x < 2(k + 1)$, where $k \in \mathbb{Z}$, check whether $g(x)$ is periodic or not.

3. Let $f(x) = x^3 - 2x$, $x \in R$ and $g(x) = f(x) - 1 + f(-x)$. Show that $g(x) \geq 0 \forall x \in R$.

4. If $f$ and $g$ are two distinct linear functions defined on $R$ such that they map $[-1, 1]$ onto $[0, 2]$ and $h : R \to \{0, 1\}$, then find the domain of $h(x)$.

5. Let $f(x) = x^2 - 4x + 3, x < 3$ and $g(x) = x - 4, x \geq 4$.

6. Let $f(x) = \log_2 \log_3 \log_5 (\sin x + a^2)$. Find the set of values of $a$ for which domain of $f(x)$ is $R$.

7. A certain polynomial $P(x), x \in R$ when divided by $x - a, x - b, x - c$ leaves remainders $a, b, c$, respectively. Then find the remainder when $P(x)$ is divided by $(x - a)(x - b)(x - c)$. (a, b, c are distinct).

8. Let $R = \{(x, y) : x, y \in R, x^2 + y^2 \leq 25\}$ and $R' = \{(x, y) : x, y \in R, y \geq \frac{4}{9}\}$, then find the domain and range of $R \cap R'$.

9. If $f$ is a polynomial function satisfying $f(x^2 + y^2) = f(x) + f(y) + f(x^2 + y^2)$, $\forall x, y \in R$ and if $f(2) = 3$, then find the value of $f(2)$.

10. If $f(a - x) = f(a + x)$ and $f(b - x) = f(b + x)$ for all real $x$, where $a, b \in R$ and $x = b > a$ are constants, then prove that $f(x)$ is a periodic function.

11. If $p, q$ are positive integers, $f$ is a function defined for positive numbers and attains only positive values, such that $f(x^p) = x^q f(x)$, then prove that $p^2 = q$

12. If $f: R \to [0, \infty)$ is a function such that $f(x - 1) + f(x + 1) = f(x)$, then prove that $f(x)$ is periodic and find its period.

13. If $a, b$ are two fixed positive integers such that $f(a + x) = 3$ and $f(b + x) = 3$ for all real $x$, then prove that $f(x)$ is periodic and find its period.

14. Let $f(x, y)$ be a periodic function, satisfying the condition $f(x, y) = f(x + 2y, y - 2x) \forall x, y \in R$ and let $g(x)$ be a function defined as $g(x) = f(x^2, 0)$. Prove that $g(x)$ is periodic function and find its period.

15. Let $f : R \to R, f(x) = \frac{x - a}{(x - b)(x - c)}, b \neq c$. If $f$ is onto, then prove that $a \in (b, c)$.

16. Show that there exists no polynomial $f(x)$ with integral coefficients which satisfy $f(a) = b, f(b) = c, f(c) = a$, where $a, b, c$ are distinct integers.

17. Consider the function $f(x) = \left\lfloor x - \frac{1}{2} \right\rfloor$ if $x \notin I$ where $I$ is the set of integers. Then find $g(x) = \max \{x^2, f(x), |x|; -2 \leq x \leq 2\}$.

18. Determine all functions $f : R \to R$ such that $f(x + y) = f(x) + f(y)$ $\forall x, y \in R$.

19. Let $f(x) = (2 \cos x - 1)(2 \cos 2x - 1)(2 \cos 2x - 1) \cdots (2 \cos 2^{n-1}x - 1)$, where $n \geq 1$. Then prove that $f(x) = 1 \forall x \in R$.

20. If $f(x) = \frac{x^2 - 1}{x^2 + a^2}$, where $a > 0$, then find the value of

Objective Type Solutions on page 1.60

Each question has four choices a, b, c, and d, out of which only one is correct.

1. The function $f : N \to N (N$ is the set of natural numbers) defined by $f(n) = 2n + 3$ is
   a. surjective only
   b. injective only
   c. bijective
   d. None of these

2. The function $f(x) = \sin (\log(x + \sqrt{1 + x^2}))$ is
   a. even function
   b. odd function
   c. neither even nor odd
   d. periodic function

3. If $x$ is real, then the value of the expression $\frac{x^2 + 14x + 9}{x^2 + 2x + 3}$ lies between
   a. 5 and 4
   b. 5 and 4
   c. -5 and 4
   d. None of these

4. The function $f : R \to R$ is defined by $f(x) = \cos^2 x + \sin^4 x$ for $x \in R$, then the range of $f(x)$ is
   a. $\left[\frac{3}{4}, 1\right]$
   b. $\left[\frac{3}{4}, 1\right]$
   c. $\left[\frac{3}{4}, 1\right]$
   d. $\left[\frac{3}{4}, 1\right]$

5. The domain of the function $f(x) = \log_{x^2 - 1}$ is
   a. $(-3, -1) \cup (1, \infty)$
   b. $[-3, -1] \cup [1, \infty)$
   c. $[3, 4]$
   d. $[3, 4]$
6. The domain of the function \( f(x) = \left[ \log_{10} \left( \frac{5x - x^3}{4} \right) \right]^{1/2} \) is
   a. \( -\infty < x < \infty \)  
   b. \( 1 \leq x \leq 4 \)  
   c. \( 4 \leq x \leq 16 \)  
   d. \( -1 \leq x \leq 1 \)  

7. The domain of the function \( f(x) = \sin^{-1}(3 - x) \) is
   a. \( [2, 4] \)  
   b. \( (2, 3) \cup (3, 4) \)  
   c. \( [2, \infty) \)  
   d. \( (-\infty, -3) \cup [2, \infty) \)  

8. The domain of \( f(x) = \log | x - 2 | \) is
   a. \( (0, \infty) \cup (1, \infty) \)  
   b. \( (0, 1) \cup (1, \infty) \)  
   c. \( (0, 1) \)  
   d. \( (-\infty, 1), (1, \infty) \)  

9. The domain of \( f(x) = \log_{3}(x + 3) \) is
   a. \( R \)  
   b. \( (-\infty, -2) \)  
   c. \( (-\infty, -3) \) \( \cap \) \( (-1, 2) \)  
   d. None of these  

10. Let \( f: \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \rightarrow [0, 4] \) be a function defined as \( f(x) = \sqrt{3} \sin x - \cos x + 2 \). Then \( f^{-1}(x) \) is given by
    a. \( \sin^{-1} \left( \frac{x - 2}{2} \right) - \frac{\pi}{6} \)  
    b. \( \sin^{-1} \left( \frac{x - 2}{2} \right) + \frac{\pi}{6} \)  
    c. \( 2\pi + \cos^{-1} \left( \frac{x - 2}{2} \right) \)  
    d. None of these  

11. If \( F(n + 1) = \frac{2F(n) + 1}{2} \), \( n = 1, 2, \ldots \) and \( F(1) = 2 \), then \( F(101) \) equals
    a. 52  
    b. 49  
    c. 48  
    d. 51  

12. The domain of the function \( f(x) = \sqrt{10C_{n-1} - 3\times 10C_{n-2}} \) contains the points
    a. 9, 10, 11  
    b. 9, 10, 12  
    c. all natural numbers  
    d. None of these  

13. The domain of the function \( f(x) = \sqrt{x - \cos(\ln x)} \) is
    a. \( (e^{2\pi}, e^{4\pi + 1/(\ln 2)}) \)  
    b. \( (e^{2\pi + 1/(\ln 2)}, e^{4\pi + 5\pi^2 / 2}) \)  
    c. \( (e^{2\pi + 1/(\ln 2)}, e^{4\pi + 5\pi^2 / 2}) \)  
    d. None of these  

14. If \( f \) is a function such that \( f(0) = 2, f(1) = 3 \) and \( f(x + 2) = 2f(x) - f(x + 1) \) for every real \( x \), then \( f(5) \) is
    a. 7  
    b. 13  
    c. 1  
    d. 5  

15. The range of \( f(x) = \sin^{-1} \left( \frac{x^2 + 1}{x^2 + 2} \right) \) is
    a. \( [0, \pi/2] \)  
    b. \( (0, \pi/6) \)  
    c. \( [\pi/6, \pi/2] \)  
    d. None of these  

16. The function \( f(x) = \frac{\sec^{-1} x}{\sqrt{x - [x]}}, \) where \([x]\) denotes the greatest integer less than or equal to \( x \), is defined for all \( x \in \) \( \mathbb{R} \)
    a. \( R \)  
    b. \( \{(-1, 1) \cup (n \in \mathbb{Z})\} \)  
    c. \( R \)  
    d. \( R \)  

17. The domain of \( f(x) = \cos^{-1} \left( \frac{-2 - x}{4} \right) + [\log(3 - x)]^{-1} \) is
    a. \( [-2, 6] \)  
    b. \( [-6, 2] \cup (2, 3) \)  
    c. \( [-2, 2] \cup (2, 3) \)  
    d. None of these  

18. The domain of the function \( f(x) = \sqrt{\log \frac{1}{1 - \sin x}} \) is
    a. \( R \)  
    b. \( \{\pi \in \mathbb{R} \} \)  
    c. \( \{n \pi | n \in \mathbb{Z}\} \)  
    d. \( (-\infty, \infty) \)  

19. The domain of the function \( f(x) = \log_{2} \left( -\log_{10} \left( \frac{1}{x^{1/4}} \right)^{-1} \right) \) is
    a. \( (0, 1) \)  
    b. \( (0, 1/4) \)  
    c. \( [1, \infty) \)  
    d. \( (1, \infty) \)  

20. The range of \( f(x) = \sin^{-1} (\sqrt{x^2 + x + 1}) \) is
    a. \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \)  
    b. \( \left( 0, \frac{\pi}{2} \right) \)  
    c. \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \)  
    d. None of these  

21. If \( f(x) = \max \left\{ \frac{1}{64}, \frac{1}{2}, \frac{1}{4} \right\} \forall x \in [0, \infty), \) then
    a. \( f(x) = \left\{ \begin{array}{ll} \frac{1}{64}, & 0 \leq x \leq \frac{1}{4} \\frac{1}{2}, & \frac{1}{4} < x \leq \frac{1}{2} \\frac{1}{4}, & \frac{1}{2} < x \leq \frac{3}{4} \\frac{1}{2}, & \frac{3}{4} < x \leq 1 \\frac{1}{4}, & x > 1 \end{array} \right. \)  
    b. \( f(x) = \left\{ \begin{array}{ll} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\frac{1}{4}, & \frac{1}{8} < x \leq \frac{1}{4} \\frac{1}{4}, & \frac{1}{4} < x \leq \frac{3}{8} \\frac{1}{4}, & \frac{3}{8} < x \leq 1 \\frac{1}{4}, & x > 1 \end{array} \right. \)  
    c. \( f(x) = \left\{ \begin{array}{ll} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\frac{1}{4}, & \frac{1}{8} < x \leq \frac{1}{4} \\frac{1}{4}, & \frac{1}{4} < x \leq \frac{3}{8} \\frac{1}{4}, & \frac{3}{8} < x \leq 1 \\frac{1}{4}, & x > 1 \end{array} \right. \)  
    d. None of these  

22. If the period of \( \frac{\cos(\sin(nx))}{\tan(x/n)} \), \( n \in \mathbb{N} \), is 6\( \pi \), then \( n \) is equal to
    a. 3  
    b. 2  
    c. 6  
    d. None of these  

23. The number of real solutions of the equation \( \log_{0.5} |x| = 2 \left| \frac{x}{n} \right| \) is
    a. 1  
    b. 2  
    c. 0  
    d. None of these  

24. The period of the function \( \sin \left( \frac{x}{2} + \cos \left( \frac{x}{5} \right) \right) \) is
    a. \( 2\pi \)  
    b. \( 10\pi \)  
    c. \( 8\pi \)  
    d. \( 5\pi \)  

25. If \( f(x) = \sqrt[n]{x^m} \), \( n \in \mathbb{N} \), is an even function, then \( m \) is
    a. even integer  
    b. odd integer  
    c. any integer  
    d. \( f(x) \) is even but not possible
26. If \( f \) is periodic, \( g \) is polynomial function and \( f(g(x)) \) is periodic and \( g(2) = 3, g(4) = 7 \) then \( g(6) \) is
   a. 13  
   b. 15  
   c. 11  
   d. None of these

27. The period of function \( 2^x + \sin \pi x + 3^{[2x]} + \cos 2\pi x \) (where \([x]\) denotes the fractional part of \( x \)) is
   a. 2  
   b. 1  
   c. 3  
   d. None of these

28. The equation \( |x - 2| + a = 4 \) can have four distinct real solutions for \( x \) if \( a \) belongs to the interval
   a. \((\infty, -4)\)  
   b. \((\infty, 0)\)  
   c. \([4, \infty)\)  
   d. None of these

29. Given the function \( f(x) = \frac{a^x + a^{-x}}{2} \) (where \( a > 2 \)). Then
   a. \( f(x) + f(y) = f(x + y) \)  
   b. \( f(x) f(y) = f(x + y) \)  
   c. \( \frac{f(x)}{f(y)} = f(x/y) \)  
   d. None of these

30. If \( \log_a(x^2 - 6x + 11) \leq 1 \), then exhaustive range of values of \( x \) is
   a. \((\infty, 2) \cup (4, \infty)\)  
   b. \((2, 4)\)  
   c. \((\infty, 1) \cup (1, 3) \cup (4, \infty)\)  
   d. None of these

31. The domain of the function \( f(x) = \sqrt{x^2 - [x]^2} \), where \([x]\) is the greatest integer less than or equal to \( x \), is
   a. \(R\)  
   b. \([0, \infty)\)  
   c. \((\infty, 0)\)  
   d. None of these

32. The range of the function \( f(x) = |x - 1| + |x - 2| \), \(-1 \leq x \leq 3\), is
   a. \([1, 3]\)  
   b. \([1, 5]\)  
   c. \([3, 5]\)  
   d. None of these

33. Which of the following functions is inverse to itself?
   a. \( f(x) = \frac{1-x}{1+x} \)  
   b. \( f(x) = \sqrt{x^2} \)  
   c. \( f(x) = 2^{x-1} \)  
   d. None of these

34. A function \( F(x) \) satisfies the functional equation \( x^2 F(x) + F(1-x) = 2x - x^2 \) for all real \( x \). \( F(x) \) must be
   a. \( x^2 \)  
   b. \( 1-x^2 \)  
   c. \( 1+x^2 \)  
   d. \( x^2 + x + 1 \)

35. If \( f(x) = \begin{cases} x^2 \sin \frac{\pi x}{2}, & |x| < 1 \\ x & |x| \geq 1 \end{cases} \) then \( f(x) \) is
   a. an even function  
   b. an odd function  
   c. a periodic function  
   d. None of these

36. Function \( f : (-\infty, -1) \rightarrow (0, e^5) \) defined by \( f(x) = e^{x^3 - 3x + 2} \) is
   a. many-one and onto  
   b. many-one and into  
   c. one-one and onto  
   d. one-one and into

37. If \( f(x) = \frac{1}{x}, g(x) = \frac{1}{x^2} \) and \( h(x) = x^2 \)
   a. \( f \circ g(x) = x^2 \) \( x \neq 0 \), \( h(g(x)) = \frac{1}{x^2} \)

38. If \([x]\) and \([x]\) represent the integral and fractional parts of \( x \), respectively, then the value of \( \sum_{r=1}^{2000} [x + r] \) is
   a. \( x \)  
   b. \([x]\)  
   c. \([x]\)  
   d. \( x + 2001 \)

39. If \( f(x) \) is a polynomial satisfying \( f(x) f(1/x) = f(x) + f(1/x) \) and \( f(3) = 28 \), then \( f(4) \) is equal to
   a. 63  
   b. 65  
   c. 17  
   d. None of these

40. The values of \( b \) and \( c \) for which the identity \( f(x + 1) - f(x) = 8x + 3 \) is satisfied, where \( f(x) = bx^2 + cx + d \), are
   a. \( b = 2, c = 1 \)  
   b. \( b = 4, c = -1 \)  
   c. \( b = -1, c = 4 \)  
   d. \( b = 1, c = 1 \)

41. Let \( f: R \rightarrow R, g: R \rightarrow R \) be two given functions such that \( f \) is injective and \( g \) is surjective, then which of the following is injective?
   a. \( g \circ f \)  
   b. \( f \circ g \)  
   c. \( g \circ g \)  
   d. None of these

42. If \( f: N \rightarrow N \) where \( f(x) = x - (-1)^x \) then \( f \) is
   a. one-one and onto  
   b. many-one and into  
   c. one-one and onto  
   d. many-one and onto

43. If \( g(x) = x^2 + x - 2 + \frac{1}{2} \) then \( f(x) = 2x^2 - 5x + 2 \), which is not a possible \( f(x) \)?
   a. \( 2x - 3 \)  
   b. \(-2x + 2 \)  
   c. \( x - 3 \)  
   d. None of these

44. If \( f: R \rightarrow R \) is an invertible function such that \( f(x) \) and \( f^{-1}(x) \) are symmetric about the line \( y = x \), then
   a. \( f(x) \) is odd  
   b. \( f(x) \) and \( f^{-1}(x) \) may not be symmetric about the line \( y = x \)  
   c. \( f(x) \) may not be odd  
   d. None of these

45. Let \( f: N \rightarrow N \) defined by \( f(x) = x^2 + x + 1 \), \( x \in N \), then \( f \) is
   a. One-one onto  
   b. Many-one onto  
   c. One-one but not onto  
   d. None of these

46. Let \( f: X \rightarrow Y \) where \( f(x) = \sin x + \cos x + 2 \sqrt{2} \) is invertible. Then which \( X \rightarrow Y \) is not possible?
   a. \( \left[ \frac{\pi}{4}, \frac{5\pi}{4} \right] \rightarrow [\sqrt{2}, 3\sqrt{2}] \)  
   b. \( \left[ \frac{3\pi}{4}, \frac{\pi}{4} \right] \rightarrow [\sqrt{2}, 3\sqrt{2}] \)  
   c. \( \left[ \frac{3\pi}{4}, \frac{3\pi}{4} \right] \rightarrow [\sqrt{2}, 3\sqrt{2}] \)  
   d. None of these
54. The range of the function \( f(x) = e^{x} - e^{-x} \) is
   a. \((-\infty, \infty)\)
   b. \([0, 1)\)
   c. \((-1, 0)\)
   d. \(\{\infty\}\)

55. If \( f: R \rightarrow R \) is a function satisfying the property \( f(2x + 3) + f(2x + 7) = 2, \forall x \in R \), then the fundamental period of \( f(x) \) is
   a. 1
   b. 8
   c. 12
   d. None of these

56. Let \( f: R \rightarrow \left[0, \frac{\pi}{2}\right] \) defined by \( f(x) = \tan^{-1}(x^2 + x + a), \) then the set of values of \( a \) for which \( f \) is onto is
   a. \([0, \infty)\)
   b. \([2, 1]\)
   c. \(\left[\frac{1}{4}, \infty\right)\)
   d. None of these

57. The domain of the function \( f(x) = \frac{1}{\sqrt{\{\sin x\} + \{\sin(x + \pi)\}}} \)
   where \( \{\cdot\} \) denotes the fractional part, is
   a. \([0, \pi]\)
   b. \((2n + 1)\pi/2, n \in Z\)
   c. \((0, \pi)\)
   d. None of these

58. \( f(x) = \frac{\cos x}{\pi} \), where \( x \) is not an integer multiple of \( \pi \)
   and \( \left[\cdot\right] \) denotes the greatest integer function is
   a. An odd function
   b. Even function
   c. Neither odd nor even
   d. None of these

59. Let \( f(x) = (a+\frac{1}{a})x^2 + (a+\frac{1}{a})x - \tan x \)
   is an even function for all \( x \in R \), then the sum of all possible values of \( a \) is (where \( \left[\cdot\right] \) denotes greatest integer function and fractional part functions, respectively)
   a. \(\frac{17}{6}\)
   b. \(-\frac{53}{6}\)
   c. \(\frac{31}{3}\)
   d. \(\frac{35}{3}\)

60. Let \( f: [-10, 10] \rightarrow R \), where \( f(x) = \sin x + \lfloor x^2/a \rfloor \) be an odd function. Then the set of values of parameter \( a \) is/are
   a. \((-10, 10) \cup \{0\}\)
   b. \(\{0\}\)
   c. \([100, \infty)\)
   d. \((100, \infty)\)

61. The function \( f \) satisfies the functional equation
   \[3f(x) + 2f\left(\frac{x + 59}{x - 1}\right) = 10x + 30 \text{ for all } x \neq 1.\]
   The value of \( f(7) \) is
   a. 8
   b. 7
   c. 2
   d. None of these

62. The period of the function \( f(x) = \lfloor 6x + 7 \rfloor + \cos \pi x - 6x \),
   where \( \lfloor \cdot \rfloor \) denotes the greatest integer function, is
   a. 3
   b. 2\pi
   c. 2
   d. None of these

63. If the graph of the function \( f(x) = \frac{a^x - 1}{x^a(a+1)} \) is symmetrical about \( y \)-axis, then \( n \) equals
   a. 2
   b. \(\frac{2}{3}\)
   c. \(\frac{1}{4}\)
   d. \(\frac{1}{3}\)

64. If \( f(x) \) is an even function and satisfies the relation \( x^2 f(x) = g(x) \) where \( g(x) \) is an odd function, then \( f(5) \) equals
   a. 0
   b. \(\frac{50}{75}\)
   c. \(\frac{49}{75}\)
   d. None of these

65. If \( f(x+y) = f(x), f(y) \) for all real \( x, y \) and \( f(0) \neq 0 \), then the function \( g(x) = \frac{f(x)}{1 + [f(x)]^2} \) is
   a. Even function
   b. Odd function
   c. Odd if \( f(x) > 0 \)
   d. Neither even nor odd
66. Possible values of \( a \) such that the equation \( x^2 + 2ax + a = 0 \) has two distinct real roots are given by
   
   a. \([0, 1]\)  
   b. \([-\infty, 0)\)  
   c. \([0, \infty)\)  
   d. \(\left(\frac{3}{4}, \infty\right)\)

67. Let \( g(x) = f(x) - 1 \). If \( f(x) + f(1-x) = 2 \) \( \forall x \in R \), then \( g(x) \) is symmetrical about
   
   a. Origin  
   b. The line \( x = \frac{1}{2} \)  
   c. The point \( (1, 0) \)  
   d. The point \( \left(\frac{1}{2}, 0\right)\)

68. Domain (\( D \)) and range (\( R \)) of \( f(x) = \sin^{-1}(\cos^{-1}(x)) \) where \([\cdot]\) denotes the greatest integer function is
   
   a. \( D = x \in [1, 2), R = \{0\} \)  
   b. \( D = x \in [0, 1], R = \{-1, 0, 1\} \)  
   c. \( D = x \in [-1, 1], R = \left\{\sin^{-1}\left(\frac{\pi}{2}\right), \sin^{-1}(\pi)\right\} \)  
   d. \( D = x \in [-1, 1], R = \emptyset \)

69. If \( f(x + 1) + f(x - 1) = 2f(x) \) and \( f(0) = 0 \), then \( f(n) \) is
   
   a. \( n! \)  
   b. \( f(1)^n \)  
   c. 0  
   d. None of these

70. The range of the function \( f \) defined by \( f(x) = \frac{1}{\sin^{-1}\{x\}} \) (where \([\cdot]\) and \(\{\cdot\}\) respectively denote the greatest integer and the fractional part functions) is
   
   a. \( I, \) the set of integers  
   b. \( N, \) the set of natural numbers  
   c. \( W, \) the set of whole numbers  
   d. \( \{1, 2, 3, 4, \ldots\} \)

71. If \( [\cos^{-1} x] + [\cot^{-1} x] = 0 \), where \([\cdot]\) denotes the greatest integer function, then the complete set of values of \( x \) is
   
   a. \( [\cos 1, 1] \)  
   b. \( [\cos 1, \cot 1] \)  
   c. \( [\cot 1, 1] \)  
   d. \( [0, \cot 1] \)

72. If \( f(x) \) and \( g(x) \) are periodic functions with period 7 and 11, respectively. Then the period of \( F(x) = f(x) g\left(\frac{x}{5}\right) - g(x) \) is
   
   a. 177  
   b. 222  
   c. 433  
   d. 1155

73. The period of the function
   
   \[ f(x) = c \sin^2 x + \sin^\left(x + \frac{x}{3}\right) \cos^\left(x + \frac{x}{3}\right) \]
   
   is (where \( c \) is constant)
   
   a. 1  
   b. \( \frac{\pi}{2} \)  
   c. \( \pi \)  
   d. Cannot be determined

74. If \( f(x + f(y)) = f(x) + y \) \( \forall x, y \in R \) and \( f(0) = 1 \), then the value of \( f(7) \) is
   
   a. 1  
   b. 7  
   c. 6  
   d. 8

75. Let \( f(x) = \sqrt{x} \{x\} \) (where \( \{\cdot\} \) denotes the fractional part of \( x \)) and \( X, Y \) are its domain and range, respectively, then
   
   a. \( x \in \left(\frac{1}{2}, \infty\right) \) and \( Y \in \left[\frac{1}{2}, \infty\right) \)  
   b. \( x \in \left(-\infty, -\frac{1}{2}\right) \cup (0, \infty) \) and \( Y \in \left[\frac{1}{2}, \infty\right) \)  
   c. \( X \in \left(-\infty, -\frac{1}{2}\right) \cup (0, \infty) \) and \( Y \in [0, \infty) \)  
   d. None of these

76. Let \( f \) be a function satisfying \( f(xy) = \frac{f(x)}{y} \) for all positive real numbers \( x \) and \( y \) if \( f(30) = 20 \), then the value of \( f(40) \) is
   
   a. 15  
   b. 20  
   c. 40  
   d. 60

77. The domain of the function \( f(x) = \sqrt{x + 4 + 4x^2} \) is
   
   a. \( \left[-3, 4\right] \cup [1, 2] \)  
   b. \( \left(-3, -1\right) \cup [2, \infty) \)  
   c. \( \left(-\infty, -3\right) \cup (-1, -1) \cup (2, \infty) \)  
   d. None of these

78. The range of \( f(x) = [1 + \sin x] + \left[2 + \sin \frac{x}{2}\right] + \left[3 + \sin \frac{x}{3}\right] + \cdots + \left[n + \sin \frac{x}{n}\right] \) \( \forall x \in [0, \pi] \), where \([\cdot]\) denotes the greatest integer function, is
   
   a. \( \left\{\frac{n^2 + n - 2}{2}, n(n + 1)\right\} \)  
   b. \( \left\{\frac{n(n + 1)}{2}\right\} \)  
   c. \( \left\{\frac{n^2 + n - 2}{2}, n(n + 1), n^2 + n + 2\right\} \)  
   d. \( \left\{n(n + 1), n^2 + n + 2\right\} \)

79. The total number of solutions of \( [x]^2 = x + 2 \{x\} \), where \([\cdot]\) and \(\{\cdot\}\) denote the greatest integer function and fractional part, respectively, is equal to
   
   a. 2  
   b. 4  
   c. 6  
   d. None of these

80. The domain of \( f(x) = \sqrt{2(x)^2 - 3\{x\} + 1} \), where \(\{\cdot\}\) denotes the fractional part in \([0, 1] \), is
   
   a. \( [-1, 1] \)  
   b. \( \left[\frac{1}{2}, 1\right] \)  
   c. \( \left[0, \frac{1}{2}\right] \)  
   d. None of these
81. The range of \( \sin^{-1} \left[ x^2 + \frac{1}{2} \right] + \cos^{-1} \left[ x^2 - \frac{1}{2} \right] \), where \( [\cdot] \) denotes the greatest integer function, is
   a. \( \left[ \frac{\pi}{2}, \pi \right] \)  
   b. \( \{ \pi \} \)  
   c. \( \left[ \frac{\pi}{2} \right] \)  
   d. None of these

82. If the period of \( \frac{\cos(\sin(n\pi))}{\tan \left( \frac{x}{n} \right)} \), \( n \in \mathbb{N} \) is 6\( \pi \) then \( n = \)
   a. 3  
   b. 2  
   c. 6  
   d. 1

83. The domain of \( f(x) = \ln \left( ax^2 + (a+b)x + (b+c) \right) \), where \( a > 0 \), \( b^2 - 4ac = 0 \), is (where \( [\cdot] \) represents greatest integer function).
   a. \( (-1, \infty) \sim \left( -\frac{b}{2a} \right) \)  
   b. \( (1, \infty) \sim \left( -\frac{b}{2a} \right) \)  
   c. \( (-1, 1) \sim \left( -\frac{b}{2a} \right) \)  
   d. None of these

84. The period of \( f(x) = \left[ x \right] + \left[ 2x \right] + \left[ 3x \right] + \left[ 4x \right] + \cdots + \left[ nx \right] \) \( n(n+1) \) \( x \), where \( n \in \mathbb{N} \), is (where \( [\cdot] \) represents greatest integer function)
   a. \( n \)  
   b. 1  
   c. \( \frac{1}{n} \)  
   d. None of these

85. If \( f \left( x + \frac{1}{2} \right) + f \left( x - \frac{1}{2} \right) = f(x) \) for all \( x \in \mathbb{R} \), then the period of \( f(x) \) is
   a. 1  
   b. 2  
   c. 3  
   d. 4

86. If \( f: \mathbb{R} \rightarrow \mathbb{R} \), \( f(x) + 3xf \left( \frac{1}{x} \right) = 2(x+1) \), then \( f(99) \) is equal to
   a. 40  
   b. 30  
   c. 50  
   d. 60

87. If \( f: X \rightarrow Y \), where \( X \) and \( Y \) are sets containing natural numbers, \( f(x) = \frac{x+5}{x+2} \) then the number of elements in the domain and range of \( f(x) \) are respectively
   a. 1 and 1  
   b. 2 and 1  
   c. 2 and 2  
   d. 1 and 2

88. If \( f(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ x & \text{for } x < 0 \end{cases} \) then \( f(f(x)) \) is given by
   a. \( x^2 \) for \( x \geq 0 \), \( x \) for \( x < 0 \)  
   b. \( x^2 \) for \( x \geq 0 \), \( x^2 \) for \( x < 0 \)  
   c. \( x^2 \) for \( x \geq 0 \), \( -x^2 \) for \( x < 0 \)  
   d. \( x^2 \) for \( x \geq 0 \), \( x \) for \( x < 0 \)

89. If the graph of \( y = f(x) \) is symmetrical about lines \( x = 1 \) and \( x = 2 \), then which of the following is true?
   a. \( f(x+1) = f(x) \)  
   b. \( f(x+3) = f(x) \)  
   c. \( f(x+2) = f(x) \)  
   d. None of these

90. Let \( f(x) = x^2 + 2x + 1 + 2|x - 1| \). If \( f(x) = k \) has exactly one real solution, then the value of \( k \) is
   a. 3  
   b. 0  
   c. 1  
   d. 2

91. The domain of \( f(x) = \cos^{-1} \left( 1 + x^2 \right) \) \( + 2 - x^2 \) is
   a. \( \left( 0, \frac{\pi}{2} \right) \)  
   b. \( \{ 0, 1 + \pi \} \)  
   c. \( \left( 1, \frac{\pi}{2} \right) \)  
   d. \( \{ 1, 1 + \pi \} \)

92. If \( f(x) = \cos^{-1} \left( \frac{1+x^2}{2x} \right) + \sqrt{2-x^2} \) is
   a. \( \left( 0, \frac{\pi}{2} \right) \)  
   b. \( \{ 0, 1 + \pi \} \)  
   c. \( \left( 1, \frac{\pi}{2} \right) \)  
   d. \( \{ 1, 1 + \pi \} \)

93. If \( f(x) = \begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases} \) then \( f(f(x)) \) is
   a. \( x \forall x \in \mathbb{R} \)  
   b. \( \left( x, x \text{ is irrational} \right) \)  
   c. \( \left( 1-x, x \text{ is irrational} \right) \)  
   d. None of these

94. The range of \( f(x) = \left[ \sin x \right] + \left[ \cos x \right] \), where \( [\cdot] \) denotes the greatest integer function, is
   a. \( \{ 0 \} \)  
   b. \( \{ 0, 1 \} \)  
   c. \( \{ 1 \} \)  
   d. None of these

95. If \( f(x) = \log_e \left( \frac{x^2+e}{x^2+1} \right) \), then the range of \( f(x) \) is
   a. \( (0, 1) \)  
   b. \( [0, 1] \)  
   c. \( [0, 1] \)  
   d. \( (0, 1) \)
96. The domain of the function \( f(x) = \frac{1}{\sqrt{4x - [x^2 - 10x + 9]}} \) is
   \[ \text{a. } (7 - \sqrt{40}, 7 + \sqrt{40}) \quad \text{b. } (0, 7 + \sqrt{40}) \]
   \[ \text{c. } (7 - \sqrt{40}, \infty) \quad \text{d. None of these} \]

97. If the function \( f : [1, \infty) \to [1, \infty) \) is defined by
   \[ f(x) = 2^{x-1}, \quad \text{then } f^{-1}(x) = \begin{cases} \frac{1}{x} \quad &\text{if } x \neq 1 \\ \frac{1}{2} \quad &\text{if } x = 1 \end{cases} \]
   \[ \text{a. } \frac{1}{2} (1 + \sqrt{1 + 4 \log_2 x}) \quad \text{b. } \frac{1}{2} (1 + \sqrt{1 + 4 \log_2 x}) \]
   \[ \text{c. } \frac{1}{2} (1 - \sqrt{1 + 4 \log_2 x}) \quad \text{d. Not defined} \]

98. The number of roots of the equation \( x \sin x = 1 \), \( x \in [-2\pi, 0) \cup (0, 2\pi] \), is
   \[ \text{a. 2} \quad \text{b. 3} \quad \text{c. 4} \quad \text{d. 0} \]

99. The number of solutions of \( 2 \cos x = \left| \sin x \right| \), \( 0 \leq x \leq 4\pi \), is
   \[ \text{a. 0} \quad \text{b. 2} \quad \text{c. 4} \quad \text{d. Infinite} \]

100. If \( \alpha(x + 1) + \beta f \left( \frac{1}{x + 1} \right) = x, x \neq -1, a \neq b \), then \( f(2) \) is equal to
      \[ \text{a. } \frac{2a + b}{2(a^2 - b^2)} \quad \text{b. } \frac{a}{a^2 - b^2} \]
      \[ \text{c. } \frac{a + 2b}{a^2 - b^2} \quad \text{d. None of these} \]

101. The number of solutions of \( \tan x - m x = 0 \), \( m > 1 \) in
      \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) is
      \[ \text{a. 1} \quad \text{b. 2} \quad \text{c. 3} \quad \text{d. } m \]

102. The range of \( f(x) = [\sin x + \cos x + \lfloor \tan x + \lfloor \sec x \rfloor \rfloor], x \in (0, \pi/4) \), where \( \lfloor x \rfloor \) denotes the greatest integer function \( \leq x \), is
      \[ \text{a. } \{0, 1\} \quad \text{b. } \{-1, 0, 1\} \quad \text{c. } \{1\} \quad \text{d. None of these} \]

103. If \( f(3x + 2) + f(3x + 29) = 0 \), \( x \in R \), then the period of \( f(x) \) is
      \[ \text{a. 7} \quad \text{b. 8} \quad \text{c. 10} \quad \text{d. None of these} \]

104. Let \( f(x) = \begin{cases} \sin x + \cos x & \text{if } 0 < x < \frac{\pi}{2} \\ \tan^2 x + \sec x & \text{if } \frac{\pi}{2} < x < \pi \end{cases} \)
    then its odd extension is
    \[ \begin{cases} -\tan^2 x - \sec x & -\pi < x < -\frac{\pi}{2} \\ -a_x & x = -\frac{\pi}{2} \\ -\sin x + \cos x & -\frac{\pi}{2} < x < 0 \end{cases} \]

105. If \( f \) and \( g \) are one-one function, then
      \[ \text{a. } f + g \text{ is one-one} \quad \text{b. } fg \text{ is one-one} \]
      \[ \text{c. } f \circ g \text{ is one-one} \quad \text{d. None of these} \]

106. The domain of \( f(x) = (0, 1) \), then domain of \( f(e^x) + f(\ln|x|) \) is
      \[ \text{a. } (-1, e) \quad \text{b. } (1, e) \quad \text{c. } (-e, 1) \quad \text{d. None of these} \]

107. The domain of \( f(x) = \frac{1}{\sqrt{\cos x} + \cos x} \) is
      \[ \text{a. } [-2\pi, 2\pi], n \in Z \quad \text{b. } (2n\pi, 2n+\pi], n \in Z \]
      \[ \text{c. } \left( \frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2} \right], n \in Z \quad \text{d. } \left( \frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2} \right], n \in Z \]

108. If \( f(2x + 3y, 2x - 7y) = 20x \), then \( f(x, y) \) equals
      \[ \text{a. } 7x - 3y \quad \text{b. } 7x + 3y \quad \text{c. } 3x - 7y \quad \text{d. } x - ky \]

109. Let \( X = \{a_1, a_2, \ldots, a_8\} \) and \( Y = \{b_1, b_2, b_3\} \). The number of functions \( f \) from \( x \) to \( y \) such that it is onto and there are exactly three elements \( x \) in \( X \) such that \( f(x) = b_1 \) is
      \[ \text{a. 75} \quad \text{b. } 90 \quad \text{c. } 100 \quad \text{d. } 120 \]

110. Let \( f : R \to R \) and \( g : R \to R \) be two one-one and onto functions such that they are the mirror images of each other about the line \( y = a \). If \( h(x) = f(x) + g(x) \), then \( h(x) \) is
      \[ \text{a. One-one and onto} \quad \text{b. Only one-one and not onto} \]
      \[ \text{c. Only onto but not one-one} \quad \text{d. Neither one-one nor onto} \]

111. If \( f(x) = \left\lfloor \frac{2x}{\pi} \right\rfloor \), \( g(x) = \left| \sin x \right| - \left| \cos x \right| \) and \( \phi(x) = f(x)g(x) \) (where \( \lfloor x \rfloor \) denotes the greatest integer function) then the respective fundamental periods of \( f(x) \), \( g(x) \) and \( \phi(x) \) are
112. Let \( f(n) = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \), then

\[ f(1) + f(2) + f(3) + \ldots + f(n) \]

is equal to

a. \( n(f(n)) - 1 \)

b. \((n + 1)f(n) - n\)

c. \((n + 1)f(n) + n\)

d. \(n(f(n)) + n\)

113. Let \( f(x) = \frac{e^{tx}}{x} \) and \( g(x) = \frac{e^{tx}}{x} \), \( x \in R \) where \( \{ \} \) and \( [ ] \) denote the fractional and integral part functions, respectively. Also \( h(x) = \log(f(x)) + \log(g(x)) \) then for real \( x, h(x) \) is

a. An odd function.

b. An even function.

c. Neither an odd nor an even function.

d. Both odd as well as even function.

114. Let \( f_1(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & x > 1 \\ 0, & \text{otherwise} \end{cases} \)

and \( f_2(x) = f_1(-x) \) for all \( x \)

\( f_3(x) = f_2(x) \) for all \( x \)

\( f_4(x) = f_2(x) \) for all \( x \)

Which of the following is necessarily true?

a. \( f_1(x) = f_2(x) \) for all \( x \)

b. \( f_3(x) = f_2(x) \) for all \( x \)

c. \( f_2(x) = f_3(x) \) for all \( x \)

d. \( f_3(x) + f_4(x) = 0 \) for all \( x \)

115. The number of solutions of the equation \( \{y + [y]\} = \frac{1}{2} \cos x \)

where \( y = \frac{1}{3} \) \([\sin x + \sin x + \sin x]\) (where \([\cdot]\) denotes the greatest integer function) is

a. 4

b. 2

c. 3

d. 53

116. The sum of roots of the equation \( \cos^{-1}(\cos x) = [x] \) \([x] \) denotes the greatest integer function is

a. \( 2\pi + 3 \)

b. \( \pi + 3 \)

c. \( 2\pi - 3 \)

d. None of these

117. The range of \( \sqrt{(1 - \cos x)(1 - \cos x)(1 - \cos x)} \ldots \infty \)

is

a. \( [0, 1] \)

b. \( [0, 1/2] \)

c. \([0, 2] \)

d. None of these

118. Let \( h(x) = [kx + 5] \), the domain of \( f(x) \) is \([-5, 7] \), the domain of \( f(h(x)) \) is \([-6, 1] \) and the range of \( h(x) \) is the same as the domain of \( f(x) \), then the value of \( k \) is

a. 1

b. 2

c. 3

d. 4

119. The range of \( f(x) = (x + 1)(x + 2)(x + 3)(x + 4) + 5 \) for \( x \in [-6, 6] \) is

a. \([4, 5045]\)

b. \([0, 5045]\)

c. \([-20, 5045]\)

d. None of these

120. The exhaustive domain of \( f(x) = \sqrt{x^2 - x^2 + x^4 - x + 1} \)

is

a. \([0, 1]\)

b. \([1, \infty]\)

c. \((-\infty, 1]\)

d. None of these

121. The range of \( f(x) = \sec^{-1}(\log_3 \tan x + \log_3 \tan x) \) is

a. \( \left[ \frac{2\pi}{3}, \pi \right] \)

b. \( \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \)

c. \( \left[ 2\pi, \pi \right] \)

d. None of these

122. The range of the function \( f(x) = \frac{7-x}{x-3} \) is

a. \( \{1, 2, 3\} \)

b. \( \{1, 2, 3, 4, 5, 6\} \)

c. \( \{1, 2, 3, 4, 5\} \)

d. \( \{1, 2, 3, 4\} \)

123. A real-valued function \( f(x) \) satisfies the functional equation \( f(x-y) = f(x)f(y) - f(x-2f(y)) \), where \( a \) is a given constant and \( f(0) = 1 \). \( f(2a) - x \) is equal to

a. \( f(x) \)

b. \(-f(x) \)

c. \( f(-x) \)

d. \( f(a) + f(a-x) \)

Multiple Correct Answers Type

Solutions on page 1.71
c. range of \( f(x) \) is \([-2, \infty)\)
d. range of \( f(x) \) is \([2, \infty)\)

6. Let \( f(x) + f(y) = f(xy^2) + f(y^3) \) (if \( f(x) \) is not identically zero). Then
a. \( f(4x^2 - 3x) + 3f(x) = 0 \)
b. \( f(4x^3) - 3x = 0 \)
c. \( f(2x^2 - x^3) + 2f(x) = 0 \)
d. \( f(2x^3 - x^2) = 2f(x) \)

7. Consider the real-valued function satisfying \( 2f(\sin x) + f(\cos x) = x \). Then
a. domain of \( f(x) \) is \( R \)
b. domain of \( f(x) \) is \([-1, 1]\)
c. range of \( f(x) \) is \([-\frac{2\pi}{3}, \frac{\pi}{3}]\)
d. range of \( f(x) \) is \( R \)

8. If \( f(x) \) satisfies the relation \( f(x + y) = f(x) + f(y) \) for all \( x, y \in R \) and \( f(1) = 5 \), then
a. \( f(x) \) is an odd function
b. \( f(x) \) is an even function

c. \( \sum_{r=1}^{m} f(r) = 5 \cdot m + 1 \)
d. \( \sum_{r=1}^{m} f(r) = \frac{5m(m + 2)}{3} \)

9. Let \( f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases} \)
and \( g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases} \)
then, which of the following is/are true?

a. \( (f + g)(3.5) = 0 \)
b. \( f(3) = 3 \)
c. \( (f \circ g)(2) = 1 \)
d. \( (f - g)(1) = 0 \)

10. \( f(x) = x^2 - 2ax + a(a + 1) \), find \( f_x \) \( \rightarrow (a, \infty) \), \( f_x \) \( \rightarrow (a, \infty) \), if one of the solutions of the equation \( f(x) = f^{-1}(x) \) is 5049, then the other may be
a. 5051
b. 5048
c. 5052
d. 5050

11. Which of the following function is/are periodic

a. \( f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases} \)
b. \( f(x) = \begin{cases} x - \lfloor x \rfloor, & 2n \leq x < 2n + 1 \\ 0, & 2n + 1 \leq x < 2n + 2 \end{cases} \)
where \( \lfloor x \rfloor \) denotes the greatest integer function, \( n \in Z \)
c. \( f(x) = \lfloor 2x \rfloor \), where \( \lfloor x \rfloor \) denotes the greatest integer function

d. \( f(x) = x - [x + 3] + \tan \left( \frac{\pi x}{2} \right) \), where \( \lfloor x \rfloor \) denotes the greatest integer function, and \( a \) is a rational number

12. If \( f: R^+ \rightarrow R^+ \) is a polynomial function satisfying the functional equation \( f(f(x)) = 6x - f(x) \), then \( f(17) \) is equal to
a. 17
b. -51
c. 34
d. -34

13. Let \( f: R \rightarrow R \) be a function defined by \( f(x + 1) = \frac{f(x) - 5}{f(x) - 3} \)
\( \forall x \in R \). Then which of the following statement(s) is/are true
d. \( f(2006) = f(2018) \)

14. Let \( f(x) = \sec^{-1}[1 + \cos^2 x] \) where \( [.] \) denotes the greatest integer function. Then
a. the domain of \( f(x) \) is \( R \)
b. the domain of \( f(x) \) is \([1, 2]\)
c. the domain of \( f(x) \) is \([1, 2]\)
d. the range of \( f(x) \) is \([1, 2]\)

15. Which of the following pairs of functions is/are identical?
a. \( f(x) = \tan^{-1}(x) \) and \( g(x) = \cot^{-1}(x) \)
b. \( f(x) = \arcsin(x) \) and \( g(x) = \arccos(\sin(x)) \)
c. \( f(x) = \cos^2 x \) and \( g(x) = \cos^2 x \)
d. \( f(x) = e^{\sin(x)} \) and \( g(x) = \sec^2(x) \)

16. \( f: R \rightarrow [0, \infty) \) and \( f(x) = \ln([\sin 2x + |\cos 2x|]) \) (where \( [.] \) is the greatest integer function).
a. \( f(x) \) has range \( Z \)
b. \( f(x) \) is periodic with fundamental period \( \pi/4 \)
c. \( f(x) \) is invertible in \([0, \pi/4]\)
d. \( f(x) \) is into function

17. Which of the following is/are not a function (\([.] \) and \( \{x\} \) denotes the greatest integer and fractional part functions respectively)

a. \( \frac{1}{\ln[1 - |x|]} \)
b. \( \frac{\ln(x)}{\sqrt{1 - x^2}} \)
c. \( \ln(x) \{x\} \)
d. \( \ln(x - 1) \)

18. If the following functions are defined from \([-1, 1]\) to \([-1, 1]\), select those which are not objective

a. \( \sin(\sin^{-1} x) \)
b. \( \frac{2\pi}{\pi} \sin^{-1} (\sin x) \)
c. \( (\operatorname{sgn}(x)) \ln(e^x) \)
d. \( x^2 \cdot (\operatorname{sgn}(x)) \)

19. If \( f: R \rightarrow N \cup \{0\} \), where \( f(\text{area of triangle joining points } P(5, 0), Q(8, 4) \) and \( R(x, y) \) such that the angle \( \text{PRO is } \pi \text{-right} \) number of triangle. Then, which of the following is true?
a. \( f(5) = 4 \)
b. \( f(7) = 0 \)
c. \( f(6.25) = 2 \)
d. \( f(x) \) is into function

20. If \( f(x) \) is a polynomial of degree \( n \) such that \( f(0) = 0, f(1) = \frac{1}{2}, \ldots, f(n) = \frac{n}{n + 1} \), then the value of \( f(n + 1) \) is
Consider the function \( f(x) = \sin(kx) + \{x\} \), where \( \{x\} \) represents the fractional part function.

**Statement 1:** \( f(x) \) is periodic for \( k = m\pi \) where \( m \) is a rational number.

**Statement 2:** The sum of two periodic functions is always periodic.

**Statement 1:** Function \( f(x) = x^2 + \tan^{-1}x \) is a non-periodic function.

**Statement 2:** The sum of two non-periodic functions is always non-periodic.

**Statement 1:** Given \( x \in [1, \sqrt{3}] \), then the range of \( f(x) = \tan^{-1}x \) is \([\pi/4, \pi/3] \).

**Statement 2:** If \( x \in [a, b] \), then the range of \( f(x) \) is \([\arctan(a), \arctan(b)]\).

**Statement 1:** If \( f: N \rightarrow R, f(x) = \sin x \) is a one-one function.

**Statement 2:** The period of \( \sin x \) is \(2\pi \) and \( 2\pi \) is an irrational number.

**Statement 1:** A continuous surjective function \( f: R \rightarrow R, f(x) \) can never be a periodic function.

**Statement 2:** For a surjective function \( f: R \rightarrow R, f(x) \) to be periodic, it should necessarily be a discontinuous function.

**Statement 1:** The solution of equation \(|x^2 - 5x + 4| = |2x - 3|\) is \(x \in (-\infty, 1) \cup \left[ \frac{3}{2}, 4 \right] \).

**Statement 2:** If \( x + y = |x| + |y| \), then \( x, y \geq 0 \).

Consider \( f \) and \( g \) be real-valued functions such that \( f(x + y) + f(x - y) = 2f(x) \cdot g(x) \forall x, y \in R \).

**Statement 1:** If \( f(x) \) is not identically zero and \( |f(x)| \leq 1 \forall x \in R \), then \( |g(x)| \leq 1 \forall x \in R \).

**Statement 2:** For any two real numbers \( x \) and \( y \), \(|x + y| = |x| + |y| \).

**Statement 1:** \( f(x) = \cos(x^2 - \tan x) \) is a non-periodic function.

**Statement 2:** \( x^2 - \tan x \) is a non-periodic function.

**Statement 1:** The period of function \( f(x) = \sin \{x\} \) is \(1\), where \( \{x\} \) represents fractional part function.

**Statement 2:** \( g(x) = \{x\} \) has period \(1\).

**Statement 1:** If \( f : R \rightarrow R, y = f(x) \) is periodic and continuous function, then \( y = f(x) \) cannot be onto.

**Statement 2:** A continuous periodic function is bounded.

Consider the functions \( f(x) = \log x \) and \( g(x) = x^2 + 3 \).

**Statement 1:** \( f(g(x)) \) is a one-one function.

**Statement 2:** \( g(x) \) is a one-one function.

Consider the functions \( f: R \rightarrow R, f(x) = x^3 \) and \( g: R \rightarrow R, g(x) = 3x + 4 \).

**Statement 1:** \( f(g(x)) \) is an onto function.

**Statement 2:** \( g(x) \) is an onto function.

**Statement 1:** \( f(x) = \sin x \) and \( g(x) = \cos x \) are identical functions.

**Statement 2:** Both the functions have the same domain and range.

**Statement 1:** The period of \( f(x) = \sin x \) is \(2\pi \) \(\Rightarrow\) the period of \( g(x) = \sin x \) is \(\pi \).

**Statement 2:** The period of \( f(x) = \cos x \) is \(2\pi \) \(\Rightarrow\) the period of \( \arcsin x \) is \(\pi \).
18. **Statement 1:** \( f(x) = \sqrt{ax^2 + bx + c} \) has a range \([0, \infty)\) if \( b^2 - 4ac > 0 \).

**Statement 2:** \( ax^2 + bx + c = 0 \) has real roots if \( b^2 - 4ac = 0 \).

19. **Statement 1:** If \( f(x) = \cos x \) and \( g(x) = x^2 \), then \( f(g(x)) \) is an even function.

**Statement 2:** If \( f(g(x)) \) is an even function, then both \( f(x) \) and \( g(x) \) must be even function.

20. **Statement 1:** The graph of \( y = \sec^2 x \) is symmetrical about \( y \)-axis.

**Statement 2:** The graph of \( y = \tan x \) is symmetrical about origin.

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**Linked Comprehension Type Solutions on page 1.76**

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which only one is correct.

**For Problems 1–3**

Consider the functions

\[
  f(x) = \begin{cases} x + 1, & x \leq 1 \\ 2x + 1, & 1 < x \leq 2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2, & -1 \leq x < 2 \\ 2x + 2, & 2 \leq x \leq 3 \end{cases}.
\]

1. **The domain of the function** \( f(g(x)) \) **is**
   a. \([0, \sqrt{2}]\)
   b. \([-1, 2]\)
   c. \([-\sqrt{2}, \sqrt{2}]\)
   d. None of these

2. **The range of the function** \( g(x) \) **is**
   a. \([1, 5]\)
   b. \([2, 3]\)
   c. \([1, \sqrt{2}] \cup [3, 5]\)
   d. None of these

3. **The number of roots of the equation** \( f(g(x)) = 2 \) **is**
   a. 1
   b. 2
   c. 4
   d. None of these

---

**For Problems 4–6**

Consider the function \( f(x) \) satisfying the identity \( f(x) + f\left(\frac{x-1}{x}\right) = 1 + x, \forall x \in R \setminus \{0, 1\} \) and \( g(x) = 2f(x) - x + 1 \).

4. **The domain of** \( y = \sqrt{g(x)} \) **is**
   a. \((-\infty, -\frac{1}{\sqrt{2}}] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)\)
   b. \((-\infty, 0] \cup (0, 1) \cup \left[\frac{1}{\sqrt{2}}, \infty\right)\)
   c. \(\left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left[\frac{1}{\sqrt{2}}, 1\right]\)
   d. None of these

5. **The range of** \( y = g(x) \) **is**
   a. \((-\infty, 5]\)
   b. \([1, \infty)\)
   c. \((-\infty, 1] \cup [5, \infty)\)
   d. None of these

6. **The number of roots of the equation** \( g(x) = 1 \) **is**
   a. 2
   b. 1
   c. 3
   d. 0

---

**For Problems 7–9**

Let \( f: N \to \mathbb{R} \) be a function satisfying the following conditions,

\[
 f(1) = 1/2 \quad \text{and} \quad f(1) + f(2) + f(3) + \ldots + n f(n) = n(n + 1),
\]

\( f(n) \) for \( n \geq 2 \).

7. **The value of** \( f(1003) \) **is** \( \frac{1}{K} \) **where** \( K \) **equals**
   a. 1003
   b. 2003
   c. 2005
   d. 2006

8. **The value of** \( f(999) \) **is** \( \frac{1}{K} \) **where** \( K \) **equals**
   a. 999
   b. 1000
   c. 1998
   d. 2000

9. \( f(1), f(2), f(3), f(4), \ldots \) represents a series of
   a. an A.P.
   b. a G.P.
   c. a H.P.
   d. An arithmetico-geometric

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**For Problems 10–12**

If \( (f(x))^2 \times f\left(\frac{1-x}{1+x}\right) = 64x \), \( \forall x \in D_f \), then

10. **\( f(x) \) is equal to**
    a. \( 4x^2 \left(\frac{1+x^{1/3}}{1-x^{1/3}}\right) \)
    b. \( x^{\sqrt{3}} \left(\frac{1-x}{1+x}\right)^{\sqrt{3}} \)
    c. \( \frac{1}{x^{\sqrt{3}} \left(\frac{1-x}{1+x}\right)^{\sqrt{3}}} \)
    d. \( x \left(\frac{1+x}{1-x}\right)^{\sqrt{3}} \)

11. **The domain of** \( f(x) \) **is**
    a. \([0, \infty)\)
    b. \(R - \{1\}\)
    c. \((-\infty, \infty)\)
    d. None of these

12. **The value of** \( f(9/7) \) **is**
    a. \(8(\frac{9}{7})^{20}\)
    b. \(-8(\frac{9}{7})^{3}\)
    c. \(4(\frac{9}{7})^{3}\)
    d. None of these

---

**For Problems 13–15**

\[
 f(x) = \begin{cases} x - 1, & -1 \leq x \leq 0 \\ x^2, & 0 \leq x \leq 1 \end{cases} \quad \text{and} \quad g(x) = \sin x.
\]

Consider the functions \( h_1(x) = f([g(x)]) \) and \( h_2(x) = f(g(x)) \).

13. **Which of the following is not true about** \( h_1(x) \) ?
    a. It is periodic function with period \( \pi \)
    b. Range is \([0, 1]\)
    c. Domain is \( R \)
    d. None of these

14. **Which of the following is not true about** \( h_2(x) \) ?
    a. Domain is \( R \)
    b. It periodic function with period \( 2\pi \)
    c. Range is \([0, 1]\)
    d. None of these

15. **For** \( h_1(x) \) **and** \( h_2(x) \) **are identical function, then which of the following is not true?**
    a. Domain of \( h_1(x) \) and \( h_2(x), x \in [2n\pi, (2n + 1)\pi], n \in \mathbb{Z} \)
    b. Range of \( h_1(x) \) and \( h_2(x) \) is \([0, 1]\)
    c. Period of \( h_1(x) \) and \( h_2(x) \) is \( \pi \)
    d. None of these
For Problems 16–18
If \( a_0 = x, a_{n+1} = f(a_n) \), where \( n = 0, 1, 2, \ldots \), then answer the following questions.

16. If \( f(x) = \sqrt[n]{a-x^n}, x > 0, m \geq 2, m \in \mathbb{N} \). Then
   a. \( a_n = x, n = 2k + 1 \), where \( k \) is integer
   b. \( a_n = f(x) \) if \( n = 2k \), where \( k \) is integer
   c. Inverse of \( a_n \) exists for any value of \( n \) and \( m \)
   d. None of these

17. If \( f(x) = \frac{1}{1-x} \), then which of the following is not true?
   a. \( a_n = \frac{1}{1-x} \) if \( n = 3k + 1 \)
   b. \( a_n = \frac{x-1}{x} \) if \( n = 3k + 2 \)
   c. \( a_n = x \) if \( n = 3k \)
   d. None of these

18. If \( f: R \to R \) be given by \( f(x) = 3 + 4x \) and \( a_n = A + Bx \), then which of the following is not true?
   a. \( A + B + 1 = 2n \)
   b. \( |A - B| = 1 \)
   c. \( \lim_{n \to B} = -1 \)
   d. None of these

For Problems 19–21
Let \( f(x) = f(f(x)) - 2f_1(x) \),
where \( f_1(x) = \begin{cases} \min \{ x^2, |x| \}, & |x| \leq 1 \\ \max \{ x^2, |x| \}, & |x| > 1 \end{cases} \)
and \( f_2(x) = \begin{cases} \min \{ x^2, |x| \}, & |x| \leq 1 \\ \max \{ x^2, |x| \}, & |x| > 1 \end{cases} \)
and \( g(x) = \begin{cases} \min \{ f(t) : -3 \leq t \leq x, -3 \leq t < 0 \} \\ \max \{ f(t) : 0 \leq t \leq x, 0 \leq t \leq 3 \} \end{cases} \)

19. For \( -3 \leq x \leq -1 \), the range of \( g(x) \) is
   a. \([-1, 3]\)
   b. \([-1, -15]\)
   c. \([-1, 9]\)
   d. None of these

20. For \( x \in (-1, 0) \), \( f(x) + g(x) \) is
   a. \( x^2 - 2x + 1 \)
   b. \( x^2 + 2x - 1 \)
   c. \( x^2 + 2x + 1 \)
   d. None of these

21. The graph of \( y = g(x) \) in its domain is broken at
   a. 1 point
   b. 2 points
   c. 3 points
   d. None of these

For Problems 22–24
Let
\[
\begin{align*}
    f(x) &= \begin{cases} 
        2x + a, & x \geq -1 \\
        bx^2 + 3, & x < -1
    \end{cases} \\
    g(x) &= \begin{cases} 
        x + 4, & 0 \leq x \leq 4 \\
        3x - 2, & -2 < x < 0
    \end{cases}
\end{align*}
\]
and \( g(f(x)) \) is not defined if
a. \( a \in (10, \infty), b \in (5, \infty) \)

b. \( a \in (4, 10), b \in (5, \infty) \)

c. \( a \in (10, \infty), b \in (0, 1) \)

d. \( a \in (4, 10), b \in (1, 5) \)

23. If the domain of \( g(f(x)) \) is \([-1, 4]\), then
   a. \( a = 1, b > 5 \)
   b. \( a = 2, b > 7 \)
   c. \( a = 2, b > 10 \)
   d. \( a = 0, b \in R \)

For Problems 25–27
Let \( f: R \to R \) is a function satisfying \( f(2 - x) = f(2 + x) \) and \( f(20 - x) = f(x) \), \( \forall x \in R \). For this function \( f \), answer the following.

25. If \( f(0) = 5 \), then the minimum possible number of values of \( x \) satisfying \( f(x) = 5 \), for \( x \in [0, 170] \), is
   a. 21
   b. 12
   c. 11
   d. 22

26. The graph of \( y = f(x) \) is not symmetrical about
   a. symmetrical about \( x = 2 \)
   b. symmetrical about \( x = 10 \)
   c. symmetrical about \( x = 8 \)
   d. None of these

27. If \( f(2) \neq f(6) \), then the
   a. fundamental period of \( f(x) \) is 1
   b. fundamental period of \( f(x) \) may be 1
   c. period of \( f(x) \) cannot be 1
   d. fundamental period of \( f(x) \) is 8

For Problems 28–30
Consider two functions \( f(x) = \begin{cases} [x], & -2 \leq x \leq -1 \\
         [x] + 1, & -1 < x < 2 \\
          \sin x, & 0 \leq x \leq \pi \end{cases} \), where \([\cdot]\) denotes the greatest integer function.

28. The exhaustive domain of \( g(f(x)) \) is
   a. \([0, 2]\)
   b. \([-2, 0]\)
   c. \([-2, 2]\)
   d. \([-1, 2]\)

29. The range of \( g(f(x)) \) is
   a. \([\sin 3, \sin 1]\)
   b. \([\sin 3, 1] \cup \{-2, -1, 0\}\)
   c. \([\sin 1, 1] \cup \{-2, -1\}\)
   d. \([\sin 1, 1]\)

30. The number of integral points in the range of \( g(f(x)) \) is
   a. 2
   b. 4
   c. 3
   d. 5

Matrix-Match Type

Each question contains statements given in two columns which have to be matched.

Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match is a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4x4 matrix should be as follows:

```
<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```
1. The function \( f(x) \) is defined on the interval \([0, 1]\).
Then match the following columns

<table>
<thead>
<tr>
<th>Column I: Function</th>
<th>Column II: Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( f(\tan x) )</td>
<td>p. ( \left[ 2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2} \right], n \in \mathbb{Z} )</td>
</tr>
<tr>
<td>b. ( f(\sin x) )</td>
<td>q. ( \left[ 2n\pi, 2n\pi + \frac{\pi}{6} \right] \cup \left[ 2n\pi + \frac{5\pi}{6}, (2n+1)\pi \right], n \in \mathbb{Z} )</td>
</tr>
<tr>
<td>c. ( f(\cos x) )</td>
<td>r. ( [2n\pi, (2n+1)\pi], n \in \mathbb{Z} )</td>
</tr>
<tr>
<td>d. ( f(2\sin x) )</td>
<td>s. ( \left[ n\pi, n\pi + \frac{\pi}{4} \right], n \in \mathbb{Z} )</td>
</tr>
</tbody>
</table>

2. Match the columns:

<table>
<thead>
<tr>
<th>Column I: Function</th>
<th>Column II: Type of function</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( f(x) = \left( (\text{sgn} x) \cdot x \right)^n, x \neq 0, ) ( n ) is an odd integer</td>
<td>p. odd function</td>
</tr>
<tr>
<td>b. ( f(x) = \frac{x^2 + 1}{e^x - 1} )</td>
<td>q. even function</td>
</tr>
<tr>
<td>c. ( f(x) = \begin{cases} 0, &amp; \text{if } x \text{ is rational} \ 1, &amp; \text{if } x \text{ is irrational} \end{cases} )</td>
<td>r. neither odd nor even function</td>
</tr>
<tr>
<td>d. ( f(x) = \max { \tan x, \cot x } )</td>
<td>s. periodic</td>
</tr>
</tbody>
</table>

3. Match the columns:

<table>
<thead>
<tr>
<th>Column I: Functions</th>
<th>Column II: Values of ( x ) for which both the functions in any option of the column I are identical</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( f(x) = \tan^{-1} \left( \frac{2x}{1-x^2} \right), g(x) = 2\tan^{-1}x )</td>
<td>p. ( x \in {-1, 1} )</td>
</tr>
<tr>
<td>b. ( f(x) = \sin^{-1}(\sin x) ) and ( g(x) = \sin(\sin^{-1}x) )</td>
<td>q. ( x \in [-1, 1] )</td>
</tr>
<tr>
<td>c. ( f(x) = \log_2 5 ) and ( g(x) = \log_5 )</td>
<td>r. ( x \in (-1, 1) )</td>
</tr>
<tr>
<td>d. ( f(x) = \sec^{-1}x + \cos^{-1}x, g(x) = \sin^{-1}x + \cos^{-1}x )</td>
<td>s. ( x \in (0, 1) )</td>
</tr>
</tbody>
</table>

4. Match the columns:

<table>
<thead>
<tr>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( f: \mathbb{R} \to \left[ \frac{3\pi}{4}, \pi \right] ) and ( f(x) = \cot^{-1}(2x - x^2 - 2) ), then ( f(x) ) is</td>
<td>p. one-one</td>
</tr>
<tr>
<td>b. ( f: \mathbb{R} \to \mathbb{R} ) and ( f(x) = e^{px} \sin q x ) where ( p, q \in \mathbb{R}^* ), then ( f(x) ) is</td>
<td>q. into</td>
</tr>
<tr>
<td>c. ( f: \mathbb{R} \to [4, \infty) ) and ( f(x) = 4 + 3x^2 ), then ( f(x) ) is</td>
<td>r. many-one</td>
</tr>
<tr>
<td>d. ( f: X \to X ) and ( f(f(x)) = x \ \forall x \in X ), then ( f(x) ) is</td>
<td>s. onto</td>
</tr>
</tbody>
</table>

5. Let \( f: \mathbb{R} \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \) be functions such that \( f(g(x)) \) is a one-one function.

<table>
<thead>
<tr>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Then ( g(x) )</td>
<td>p. must be one-one</td>
</tr>
<tr>
<td>b. Then ( f(x) )</td>
<td>q. may not be one-one</td>
</tr>
<tr>
<td>c. If ( g(x) ) is onto then ( f(x) )</td>
<td>r. may be many-one</td>
</tr>
<tr>
<td>d. If ( g(x) ) is into then ( f(x) )</td>
<td>s. must be many-one</td>
</tr>
<tr>
<td>Column I: Function</td>
<td>Column II: Period</td>
</tr>
<tr>
<td>------------------------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>a. $f(x) = \cos((\sin x -</td>
<td>\cos x</td>
</tr>
<tr>
<td>b. $f(x) = \cos(2\pi x)$</td>
<td>q. $\pi/2$</td>
</tr>
<tr>
<td>c. $f(x) = \sin^{-1}(\sin x) + e^{\tan x}$</td>
<td>r. $4\pi$</td>
</tr>
<tr>
<td>d. $f(x) = \sin^3 x \sin 3x$</td>
<td>s. $2\pi$</td>
</tr>
</tbody>
</table>

7. \{\} denotes the fractional part function and [] denotes the greatest integer function:

<table>
<thead>
<tr>
<th>Column I: (Function)</th>
<th>Column II: (Period)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $f(x) = \cos(2\pi x) + \sin(2\pi x)$</td>
<td>p. $1/3$</td>
</tr>
<tr>
<td>b. $f(x) = \sin^3 x + \cos^3 x$</td>
<td>q. $1/4$</td>
</tr>
<tr>
<td>c. $f(x) = \sin(\pi x) + \cos(\pi x)$</td>
<td>r. $1/2$</td>
</tr>
<tr>
<td>d. $f(x) = x - [x]$</td>
<td>s. $1$</td>
</tr>
</tbody>
</table>

8. Column I: (Function) | Column II: (Range)
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $f(x) = \log_{3}(5 + 4x - x^2)$</td>
<td>p. function not defined</td>
</tr>
<tr>
<td>b. $f(x) = \log_{3}(x^2 - 4x - 5)$</td>
<td>q. $[0, \infty)$</td>
</tr>
<tr>
<td>c. $f(x) = 3x + 5$</td>
<td>r. $(-\infty, 2)$</td>
</tr>
<tr>
<td>d. $f(x) = \log_{2}(4 - x)$</td>
<td>s. $R$</td>
</tr>
</tbody>
</table>

9. Column I: Equation | Column II: Number of roots
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $x^2 \tan x - 4 = 0, x \in [0, 2\pi]$</td>
<td>p. 5</td>
</tr>
<tr>
<td>b. $2 \cos^2 x + \sin x, x \in [0, 2\pi]$</td>
<td>q. 2</td>
</tr>
<tr>
<td>c. If $f(x)$ is a polynomial of degree 5 with real coefficients such that $f(1) = 0$ has 8 real roots, then the number of roots of $f(x) = 0$</td>
<td>r. 3</td>
</tr>
<tr>
<td>d. $7x^4(5 -</td>
<td>x</td>
</tr>
</tbody>
</table>

**Integer Type Solutions on page 1.81**

1. Let $f$ be a real-valued invertible function such that $f(\frac{2x-3}{x-2}) = 5x - 2, x \neq 2$. Then the value of $f^{-1}(13)$ is $x^2 + x - 12$.

2. Number of values of $x$ for which $|\sqrt{x^2 - x + 4} - 2| > -3$ is $x^2 + x - 12$.

3. Let $f(x) = 3x^2 - 7x + c$, where $c$ is a variable coefficient and $x > \frac{7}{6}$. Then the value of $c$ such that $f(\frac{7}{6})$ touches $f^{-1}(x)$ is (where $\lfloor \cdot \rfloor$ represents greatest integer function).

4. Number of integral values of $x$ for which $\frac{\pi}{2\tan x - 4}(x - 4)(x - 10) < 0$ is $x! - (x - 1)!$.

5. Let $f: R^+ \rightarrow R$ be a function which satisfies $f(x) \cdot f(y) = f(xy) + 2\left(\frac{1}{x} + \frac{1}{y}\right)$ for $x, y > 0$, then possible value of $f(1/2)$ is $f(2)$.

6. A continuous function $f(x)$ on $R \rightarrow R$ satisfies the relation $f(x)^2 + f(x + y) + 5xy = f(3x + y) + 2x^2 - 1$ for all $x, y \in R$, then the value of $f(4)$ is $f(x)^2$.

7. Let $a > 2$ be a constant. If there are just 18 positive integers satisfying the inequality $(x - a)(x - 2a)(x - a^2) < 0$, then the value of $a$ is $x$.

8. Number of integers in the domain of the function, satisfying $f(x) + f(x^{-1}) = \frac{x^3 + 1}{x}$ is $x$.

9. $f: R \rightarrow R, f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17$ for all $x \in R$, then the value of $f(5)$ is $x$.

10. If $f(x)$ is an odd function and $f(1) = 3$, and $f(x + 2) = f(x)$, then the value of $f(3)$ is $x$.

11. Let $f: R \rightarrow R$ be a continuous onto function satisfying $f(x) + f(-x) = 0, \forall x \in R$.

12. Number of integral values of $x$ for which the function $\sqrt{\sin x + \cos x + \sqrt{7x - x^2 - 6}}$ is defined is $x$. 
13. Suppose that \( f \) is an even, periodic function with period 2, and that \( f(x) = x \) for all \( x \) in the interval \([0, 1]\). The value of \( \lceil 10 f(3.14) \rceil \) is (where \( \lceil \cdot \rceil \) represents the greatest integer function).

14. If \( f(x) = x^2 + \sqrt{x^2 - 1} \), then the maximum value of \( f(x)^2 \) is __________.

15. The function \( f(x) = \frac{x + 1}{x^3 + 1} \) can be written as the sum of an even function \( g(x) \) and an odd function \( h(x) \). Then the value of \( g(0) \) is __________.

16. If \( T \) is the period of the function \( f(x) = [8x + 7] + \tan 2\pi x + \cot 2\pi x - 8x \) (where \([ \cdot ]\) denotes the greatest integer function), then the value of \( 1/T \) is __________.

17. If \( a, b \) and \( c \) are non-zero rational numbers, then the sum of all the possible values of \( \frac{|a| + |b| + |c|}{a+b+c} \) is __________.

18. An even polynomial function \( f(x) \) satisfies a relation \( f(2x - 1) + f(\frac{2x+1}{2}) = f(x) \forall x \in \mathbb{R} \). Find the value of \( f(0) \).

19. If \( f(x) = \sin^2 x + \sin^2 \left( x + \frac{\pi}{3} \right) + \cos x \cos \left( x + \frac{\pi}{3} \right) \) and \( g \left( \frac{5}{4} \right) = 1 \), then \( (gof)(x) \) is __________.

20. Let \( E = \{1, 2, 3, 4\} \) and \( F = \{1, 2\} \). If \( N \) is the number of onto functions from \( E \) to \( F \), then the value of \( N/2 \) is __________.

21. The function \( f \) is continuous and has the property \( f(f(x)) = 1-x \), then the value of \( f(\frac{1}{4}) + f(\frac{3}{4}) \) is __________.

22. Number of integral values of \( x \) satisfying the inequality \( \frac{3}{4} \leq \frac{6x + 10 - x^2}{64} \) is __________.

23. A function \( f \) from integers to integers is defined as \( f(x) = \begin{cases} n+3, & n \text{ odd} \\ n/2, & n \text{ even} \end{cases} \). Suppose \( k \in \mathbb{Z} \) and \( f(f(k)) = 27 \), then the sum of digits of \( k \) is __________.

24. If \( \theta \) is the fundamental period of the function \( f(x) = \sin^{99} x + \sin^{99} \left( x + \frac{2\pi}{3} \right) + \sin^{99} \left( x + \frac{4\pi}{3} \right) \), then complex number \( z = e^{i\theta} \) lies in the quadrant number __________.

25. If \( x = \frac{4}{9} \), satisfy the equation \( \log_2(x^2 - 2x + 2) > \log_2(x^2 - 2x + 2) + 2x + 3 \), then sum of all possible distinct values of \( [x] \) is (where \([ \cdot ]\) represents the greatest integer function). __________.

26. If \( 4^x - 2^{x+2} + 5 = \lceil x \rceil \), then \( x, y, b \in \mathbb{R} \), then the possible value of \( b \) is __________.

27. Let \( f: N \rightarrow N \), and \( x_1 > x_2 \Rightarrow f(x_2) > f(x_1), \forall x_1, x_2 \in N \) and \( f(n) = 3n, \forall n \in N \), then \( f(2) = \) __________.

28. Number of integral values of \( a \) for which \( f(x) = \log \left( \log \left( \log \left( \log \left( \sin x + a \right) \right) \right) \right) \) is defined for every real value of \( x \) __________ and its range is __________.

29. Let \( f(x) = \sin^2 x - \cos^2 x \) and \( g(x) = 1 + \frac{1}{2} \tan^{-1} x \), then the number of values of \( x \) in interval \([-10\pi, 8\pi]\) satisfying the equation \( f(x) = g(x) \) is __________.

30. Suppose that \( f(x) \) is a function of the form \( f(x) = ax^8 + bx^6 + cx^4 + dx^2 + ex + f(0) = 1 \). If \( f(5) = 2 \), then the value of \( x \) __________ is __________.

**Subjective**

1. Find the domain and range of the function \( f(x) = \frac{x^2}{1 + x^2} \).

2. Is the function one-to-one? (IIT-JEE, 1978)

3. Draw the graph of \( y = x^{1/3} \) for \(-1 \leq x \leq 1\) (IIT-JEE, 1978)

4. If \( f(x) = x^3 - 6x^2 + 12x - 8 \), find \( f(6) \). (IIT-JEE, 1979)

5. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a one-one function with domain \( \{x, y, z\} \) and range \( \{1, 2, 3\} \). It is given that exactly one of the following statements is true and the remaining two are false \( f(x) = 1 \), \( f(x) = 2 \), \( f(x) = 3 \) determine \( f^{-1}(1) \). (IIT-JEE, 1982)

6. Find the natural number \( a \) for which \( \sum_{k=1}^{n} f(a + k) = 16(2^n - 1) \), where the function \( f \) satisfies the relation \( f(x + y) = f(x)f(y) \) for all natural numbers \( x, y \) and further \( f(1) = 2 \). (IIT-JEE, 1992)

7. Let \( \{x\} \) and \( [x] \) denote the fractional and integral part of a real number \( x \), respectively. Solve \( 4 \{x\} = x + [x] \). (IIT-JEE, 1992)

8. A function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( f(x) = ax^2 + bx + c \), find the interval of values of \( a \) for which \( f \) is onto. Is the function one-to-one for \( a = 3 \)? Justify your answer. (IIT-JEE, 1996)

**Objective**

**Fill the blanks**

1. The values of \( f(x) = 3 \sin \left( \sqrt{\frac{\pi^2}{16} - x^2} \right) \) lie in the interval _________. (IIT-JEE, 1983)

2. The domain of the function \( f(x) = \sin^{-1} \left( \log_2 \left( \frac{x^2}{2} \right) \right) \) is given by _________. (IIT-JEE, 1984)

3. Let \( A \) be a set of \( n \) distinct elements. Then the total number of distinct functions from \( A \) to \( A \) is ________ and out of these ________ are onto functions. (IIT-JEE, 1985)

4. If \( f(x) = \sin \log_a \left( \sqrt[3]{4 - x^2} \right) \), then the domain of \( f(x) \) is ________ and its range is _________. (IIT-JEE, 1985)
5. There are exactly two distinct linear functions, \( \frac{x+1}{x+2} \) and \( \frac{x}{x+1} \) which map \([-1, 1]\) onto [0, 2].

6. If \( f \) is an even function defined on the interval \((-5, 5)\), then four real values of \( x \) satisfying the equation \( f(x) = f\left(\frac{x+1}{x+2}\right) \) are \( \frac{1}{2} \), \( -1 \), \( -2 \), and \( 3 \). (IIT-JEE, 1985)

7. If \( f(x) = \sin^2 x + \sin^2 \left(\frac{x + \pi}{3}\right) + \cos x \cos \left(\frac{x + \pi}{3}\right) \) and \( g\left(\frac{x}{4}\right) = 1 \), then \( (gof)(x) = \frac{8(3x^2 - 1)}{1 - 3(2x - 1)^2} \). (IIT-JEE, 1996)

8. The domain of the function \( f(x) = \sin^{-1}\left(\frac{1}{8(3x^2 - 1)}\right) \) is \((0, 2)\). (IIT-JEE, 2011)

9. The function \( f(x) = x^2 + 4x + 30 \) is not onto. (IIT-JEE, 1983)

10. If \( f_1(x) \) and \( f_2(x) \) are defined on the domain \( D_1 \) and \( D_2 \) respectively, then \( f_1(x) + f_2(x) \) is defined on \( D_1 \cup D_2 \). (IIT-JEE, 1988)

11. True or false
   
   1. If \( f(x) = (a - x)^{1/n} \) where \( a > 0 \) and \( n \) is a positive integer, then \( f[f(x)] = x \). (IIT-JEE, 1983)
   
   2. The function \( f(x) = x^2 + 4x + 30 \) is not onto. (IIT-JEE, 1983)
   
   3. If \( f_1(x) \) and \( f_2(x) \) are defined on the domain \( D_1 \) and \( D_2 \) respectively, then \( f_1(x) + f_2(x) \) is defined on \( D_1 \cup D_2 \). (IIT-JEE, 1988)

12. Multiple choice questions with one correct answer
   
   1. Let \( R \) be the set of real numbers. If \( f: R \to R \) is a function defined by \( f(x) = x^2 \), then \( f\) is
      
      a. Injective but not surjective
      
      b. Surjective but not injective
      
      c. Bijective
      
      d. None of these
      
      (IIT-JEE, 1979)

   2. The entire graph of the equation \( y = x^2 + kx - x + 9 \) is strictly above the \( x \)-axis if and only if
      
      a. \( k < 7 \)
      
      b. \(-5 < k < 7 \)
      
      c. \( k > 3 \)
      
      d. None of these
      
      (IIT-JEE, 1979)

   3. Let \( f(x) = |x - 1| \). Then
      
      a. \( f(x^2) = (f(x))^2 \)
      
      b. \( f(x + y) = f(x) + f(y) \)
      
      c. \( f(|x|) = |f(x)| \)
      
      d. None of these
      
      (IIT-JEE, 1983)

   4. If \( x \) satisfies \( |x - 1| + |x - 2| + |x - 3| \geq 6 \), then
      
      a. \( 0 \leq x \leq 4 \)
      
      b. \( x \leq 2 \) or \( x \geq 4 \)
      
      c. \( x \leq 0 \) or \( x \geq 4 \)
      
      d. None of these
      
      (IIT-JEE, 1983)

   5. If \( f(x) = \cos(\log_e x) \), then \( f(x)f(y) - \frac{1}{2} \left[ f\left(\frac{x}{y}\right) + f(xy) \right] \)
      has the value
      
      a. \(-1\)
      
      b. \(1/2\)
      
      c. \(-2\)
      
      d. None of these
      
      (IIT-JEE, 1983)

   6. The domain of definition of the function
      
      \( y = \frac{1}{\log_{10}(1-x)} + \sqrt{x + 2} = \)
      
      a. \((-3, -2)\) excluding \(-2.5\)
      
      b. \([0, 1]\) excluding \(0.5\)
      
      c. \([-2, 1]\) excluding \(0\)
      
      d. None of these
      
      (IIT-JEE, 1983)

   7. Which of the following functions is periodic?
      
      a. \( f(x) = x - [x] \) where \([x]\) denotes the largest integer less than or equal to the real number \(x\)
      
      b. \( f(x) = \sin \frac{1}{x} \) for \( x \neq 0 \), \( f(0) = 0 \)
      
      c. \( f(x) = x \cos x \)
      
      d. None of these
      
      (IIT-JEE, 1983)

   8. If the function \( f: [1, \infty) \to [1, \infty) \) is defined by \( f(2^x - 1) \), then \( f^{-1}(x) \) is
      
      a. \( \left(\frac{1}{2}\right)^x(2^x - 1) \)
      
      b. \( \frac{1}{2} \left(1 + \sqrt{1 + 4 \log_2 x} \right) \)
      
      c. \( \frac{1}{2} \left(1 - \sqrt{1 + 4 \log_2 x} \right) \)
      
      d. Not defined
      
      (IIT-JEE, 1992)

   9. Let \( f(x) = \sin x \) and \( g(x) = \log_2 |x| \). If the ranges of the composition function \( f \circ g \) and \( g \circ f \) are \( R_1 \) and \( R_2 \) respectively, then
      
      a. \( R_1 = \{ u: -1 \leq u < 1 \} \), \( R_2 = \{ v: v < 0 \} \)
      
      b. \( R_1 = \{ u: u < 0 \} \), \( R_2 = \{ v: v < 0 \} \)
      
      c. \( R_1 = \{ u: -1 < u < 1 \} \), \( R_2 = \{ v: v < 0 \} \)
      
      d. \( R_1 = \{ u: -1 < u < 1 \} \), \( R_2 = \{ v: v < 0 \} \)
      
      (IIT-JEE, 1994)

   10. Let \( f(x) = (x+1)^2 - 1, x \geq -1 \). Then the set \( \{ x: f(x) = f^{-1}(x) \} \) is
      
      a. \( \{ 0, 1, -1 \} \)
      
      b. \([0, 1, -1]\)
      
      c. \(\{0, 1\}\)
      
      d. Empty
      
      (IIT-JEE, 1995)

   11. Let \( f(x) \) be defined for all \( x > 0 \) and be continuous. Let \( f(x) \) satisfy \( f\left(\frac{x}{y}\right) = f(x) - f(y) \) for all \( x, y \) and \( f(1) = 1 \). Then
      
      a. \( f(x) \) is bounded
      
      b. \( f\left(\frac{1}{x}\right) \to 0 \) as \( x \to 0 \)
      
      c. \( x f(x) \to 1 \) as \( x \to 0 \)
      
      d. \( f(x) = \log x \)
      
      (IIT-JEE, 1995)

   12. The domain of definition of the function \( f(x) \) given by the equation \( 2^x + 2^y = 2 \) is
      
      a. \( 0 \leq x \leq 1 \)
      
      b. \( 0 \leq x \leq 1 \)
      
      c. \( -\infty < x < 0 \)
      
      d. \( -\infty < x < 1 \)
      
      (IIT-JEE, 2000)

   13. Let \( g(x) = 1 + x - [x] \) and \( f(x) = \left\{ \begin{array}{ll} 0, & x = 0 \end{array} \right. \). Then for all \( x \), \( f(g(x)) \) is equal to (where \([\cdot]\) represents greatest integer function)
      
      a. \( x \)
      
      b. \( 1 \)
      
      c. \( f(x) \)
      
      d. \( g(x) \)
      
      (IIT-JEE, 2001)

   14. If \( f: [1, \infty) \to [2, \infty) \) is given by \( f(x) = \frac{x}{x - 1} \), then \( f^{-1}(x) \) equals
      
      a. \(-3, -2) \) excluding \(-2.5\)
      
      b. \([0, 1]\) excluding \(0.5\)
      
      c. \([-2, 1]\) excluding \(0\)
      
      d. None of these
      
      (IIT-JEE, 2001)
23. If \( f(x) = \sin x + \cos x \), \( g(x) = x^2 - 1 \), then \( g(f(x)) \) is invertible in the domain
   a. \( \left[ 0, \frac{\pi}{2} \right] \)
   b. \( \left[ \frac{\pi}{2}, \pi \right] \)
   c. \( \left[ -\frac{\pi}{2}, -1 \right] \)
   d. \( \left[ 0, \pi \right] \)  (IIT-JEE, 2004)

24. If the functions \( f(x) \) and \( g(x) \) are defined on \( R \to R \) such that \( f(x) = \begin{cases} 0 & x \in \text{rational} \\ x & x \in \text{irrational} \end{cases} \) and \( g(x) = \begin{cases} 0 & x \in \text{irrational} \\ x & x \in \text{rational} \end{cases} \), then \( (f-g)(x) \) is
   a. one-one and onto
   b. neither one-one nor onto
   c. one-one but not onto
   d. onto but not one-one  (IIT-JEE, 2005)

25. \( X \) and \( Y \) are two sets and \( f: X \to Y \). If \( f(c)=y, c \subset X, y \subset Y \) and \( f^{-1}(d)=x, d \subset Y, x \subset X \), then the true statement is
   a. \( f^{-1}(b)=b \)
   b. \( f^{-1}(a)=a \)
   c. \( f^{-1}(b)=b, c \subset y \)
   d. \( f^{-1}(a)=a, a \subset x \)  (IIT-JEE, 2005)

Multiple choice questions with one or more than one correct answer

1. If \( y = f(x) = \frac{x+2}{x-1} \) then
   a. \( x = f(y) \)
   b. \( f(1) = 3 \)
   c. \( y \) increases with \( x \) for \( x < 1 \)
   d. \( f \) is a rational function of \( x \)  (IIT-JEE, 1984)

2. Let \( g(x) \) be a function defined on \([-1, 1]\). If the area of the equilateral triangle with two of its vertices at \((0, 0)\) and \((x, g(x))\) is \( \sqrt{3}/4 \) then the function \( g(x) \) is
   a. \( g(x) = \pm \sqrt{1-x^2} \)
   b. \( g(x) = \sqrt{1-x^2} \)
   c. \( g(x) = -\sqrt{1-x^2} \)
   d. \( g(x) = \sqrt{1+x^2} \)  (IIT-JEE, 1989)

3. If \( f(x) = \cos [\pi^2] x + \cos [-\pi^2] x \), where \([x] \) stands for the greatest integer function, then
   a. \( f\left(\frac{\pi}{2}\right) = -1 \)
   b. \( f(\pi) = 1 \)
   c. \( f(-\pi) = 0 \)
   d. \( f\left(\frac{\pi}{4}\right) = 1 \)  (IIT-JEE, 1991)

4. If \( f(x) = 3x - 5 \), then \( f^{-1}(x) \)
   a. is given by \( \frac{1}{3x-5} \)
   b. is given by \( \frac{x+5}{3} \)
   c. does not exist because \( f \) is not one-one
   d. does not exist because \( f \) is not onto  (IIT-JEE, 1998)
9.5. If \( g(f(x)) = |\sin x| \) and \( f(g(x)) = (\sin \sqrt{x})^2 \), then

a. \( f(x) = \sin^2 x \), \( g(x) = \sqrt{x} \)
b. \( f(x) = \sin x \), \( g(x) = |x| \)
c. \( f(x) = x^2 \), \( g(x) = \sin \sqrt{x} \)
d. \( f \) and \( g \) cannot be determined

(IIT-JEE, 1998)

**Match the following type**

This question contains statements given in two columns which have to be matched. Statements \( a, b, c, d \) in Column I have to be matched with statements \( p, q, r, s \) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct match is \( a-p \), \( a-s \), \( b-q \), \( b-r \), \( c-q \) and \( d-s \), then the correctly bubbled \( 4 \times 4 \) matrix should be as follows:

<table>
<thead>
<tr>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. If (-1 &lt; x &lt; 1), then ( f(x) ) satisfies ( p. 0 &lt; f(x) &lt; 1 )</td>
<td>b. If (1 &lt; x &lt; 2), then ( f(x) ) satisfies ( q. f(x) &lt; 0 )</td>
</tr>
<tr>
<td>c. If (3 &lt; x &lt; 5), then ( f(x) ) satisfies ( r. f(x) &gt; 0 )</td>
<td>d. If (x &gt; 5), then ( f(x) ) satisfies ( s. f(x) &lt; 1 )</td>
</tr>
</tbody>
</table>

(IIT-JEE, 2007)

### ANSWERS AND SOLUTIONS

#### Subjective Type

1. \( a + |y| = 2y \)

If \( y \geq 0 \), we have \( x + y = 2y \Rightarrow y = x \)

\( \Rightarrow y = x, x \geq 0 \)

If \( y < 0 \)
\( x - y = 2y \Rightarrow y = \frac{x}{3} \)
\( \Rightarrow y = \frac{x}{3}, x < 0 \)
\( \Rightarrow y = \frac{x}{3}, x \geq 0 \)

\( D_j = R \)

2. \( e^x - e^{-y} = 2x \)

(Multiplying by \( e^y \))

\( e^x - 2e^y - 1 = 0 \)

\( e^y = 2x \pm \sqrt{4x^2 + 4} \)
\( x = \pm \sqrt{x^2 + 1} \)
\( \Rightarrow y = x + \sqrt{x^2 + 1} \) (as \( \sqrt{x^2 + 1} > x \), then \( x - \sqrt{x^2 + 1} < 0 \), which is not possible)

\( \Rightarrow y = \ln(x + \sqrt{x^2 + 1}) \)

\( D_j = R \)

3. \( 10^x + 10^y = 10 \)

\( y = \log_{10}(10 - 10^x) \)

For domain \( 10 - 10^x > 0 \) \( \Rightarrow 10^x < 10 \) \( \Rightarrow x < 1 \)

\( D_j = (-\infty, 1) \)

4. \( x^2 - \sin^{-1} y = \frac{\pi}{2} \)

\( \sin^{-1} y = x^2 - \frac{\pi}{2} \)

\( \Rightarrow y = \sin(x^2 - \frac{\pi}{2}) \)

\( D_j = \mathbb{R} \)

5. \( g(x) = \sqrt{x - 2x^2} \), \( \forall 2k < x < 2(k + 1) \), where \( k \in \), integer

\( \Rightarrow g(x) = \begin{cases} 
\sqrt{x + 2}, & -2 \leq x < 0 \\
\sqrt{x}, & 0 \leq x \leq 2 \\
\sqrt{x - 2}, & 2 \leq x < 4 \\
\sqrt{x - 4}, & 4 \leq x \leq 6 
\end{cases} \)

\( \Rightarrow g \) is periodic with period = 2

\( y = \sqrt{x + 2} \)

\( y = \sqrt{x - 2} \)

\( y = \sqrt{x - 4} \)

**Fig. 1.92**

3. Given \( f(x) = x^2 - 2x = (x - 1)^2 - 1 \)

\( \Rightarrow g(x) = f(f(x) - 1) + f(5 - f(x)) \)
\( = f[(x - 1)^2 - 2] + f[6 - (x - 1)^2] \)
\( = [(x - 1)^2 - 2] - 1 + [6 - (x - 1)^2 - 1]^2 - 1 \)
\( = (x - 1)^4 - 6(x - 1)^2 + 9 - 1 + (x - 1)^4 \)
\( = 2(x - 1)^4 + 75 - 1 \)
4. Let two linear functions be \( f(x) = ax + b \) and \( g(x) = cx + d \). They map \([-1, 1] \rightarrow [0, 2]\) and mapping is onto.

\[
\begin{align*}
\Rightarrow f(-1) = 0 \quad & \text{and} \quad f(1) = 2 \quad \text{and} \quad g(-1) = 0 \quad \text{and} \quad g(1) = 2 \\
\Rightarrow -a + b = 0 \quad & \text{and} \quad a + b = 2 \\
\text{and} \quad -c + d = 0 \quad & \text{and} \quad c + d = 0 \\
\Rightarrow a = b = 1 \quad & \text{and} \quad c = -1, d = 1 \\
\Rightarrow f(x) = x + 1 \quad & \text{and} \quad g(x) = -x + 1 \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow h(x) = \frac{x + 1}{1 - x} \Rightarrow h(h(x)) = \frac{x + 1}{1 - \frac{x + 1}{1 - x}} = \frac{x + 1}{x} \\
\Rightarrow h(h(1/x)) = x \\
\Rightarrow |h(h(x)) + h(h(1/x))| = |x + 1/x| > 2.
\end{align*}
\]

5. \( f(x) = \begin{cases} 
  x^2 - 4x + 3, & x < 3 \\
  x - 4, & x \geq 3
\end{cases} 
\]

\[
\begin{align*}
\Rightarrow f(x) = \begin{cases} 
  x^2 - 4x + 3, & x < 3 \\
  3 \leq x < 4 \\
  x - 4, & x \geq 4
\end{cases}
\end{align*}
\]

\[
\begin{align*}
g(x) = \begin{cases} 
  x - 3, & x < 4 \\
  3 \leq x < 4 \\
  x^2 + 2x + 2, & x \geq 4
\end{cases}
\end{align*}
\]

From (1) and (2), we have

\[
\begin{align*}
f(x) = \begin{cases} 
  x^2 - 4x + 3, & x < 3 \\
  x - 4, & 3 \leq x < 4 \\
  x^2 + 2x + 2, & x \geq 4
\end{cases}
\end{align*}
\]

Clearly, \( f(x) / g(x) \) is not defined at \( x = 3 \), hence the domain is \( R - \{3\} \).

6. Given \( f(x) = \log_3 \log_4 \log_5 \log_6 (\sin x + a^2) \)

\[
\begin{align*}
f(x) \text{ is defined only if} \log_3 \log_4 \log_5 \log_6 (\sin x + a^2) > 0, \forall x \in R \\
\Rightarrow \log_3 \log_4 \log_5 (\sin x + a^2) > 1, \forall x \in R \\
\Rightarrow \log_3 \log_4 \log_5 (\sin x + a^2) > 4, \forall x \in R \\
\Rightarrow (\sin x + a^2) > 64, \forall x \in R \\
\Rightarrow a^2 > 625 - \sin x, \forall x \in R \\
\Rightarrow a^2 \text{ must be greater than maximum value of } 625 - \sin x \\
\Rightarrow a^2 > 626
\end{align*}
\]

\[\Rightarrow a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)\]

7. By remainder theorem, \( P(a) = a, P(b) = b \) and \( P(c) = c \).

Let the required remainder be \( R(x) \), then \( P(x) = (x - a)(x - b)(x - c)Q(x) + R(x) \), where \( R(x) \) is a polynomial of degree at most 2.

We get \( R(a) = a, R(b) = b \) and \( R(c) = c \).

So, the equation \( R(x) - x = 0 \) has three roots \( a, b \) and \( c \).

But its degree is at most 2, So, \( R(x) - x \) must be zero polynomial (or identity). Hence, \( R(x) = x \).

8. \[
\begin{align*}
\text{Fig. 1.93}
\end{align*}
\]

The equation \( x^2 + y^2 = 25 \) represents a circle with centre \((0, 0)\) and radius 5 and the equation \( y = \frac{4}{9}x^2 \) represents a parabola with vertex \((0, 0)\). Hence, \( R \cap R' \) is the set of points indicated in the figure = \{(x, y): -3 \leq x \leq 3, 0 \leq y \leq 5\}.

Thus, the domain \( R \cap R' = \{0, 5\} \).

9. Put \( y = \frac{1}{x} \)

\[
\begin{align*}
\Rightarrow 2 + f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1) \quad (1)
\end{align*}
\]

Now put \( x = 1 \)

\[
\begin{align*}
\Rightarrow 2 + (f(1))^2 = 3f(1) \\
\Rightarrow f(1) = 1 \text{ or } 2
\end{align*}
\]

But \( f(1) \neq 1 \), otherwise from the given relation \( 2 + f(x) = f(x) + f(1) + f(x) \) or \( f(x) = 1 \), which is not possible as given that \( f(2) = 5 \).

Hence, \( f(1) = 2 \).

\[
\begin{align*}
\Rightarrow \text{From (1), we have } f(x) = f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \\
\Rightarrow f(x) = \pm x^2 + 1 \\
\Rightarrow f(2) = \pm 2^2 + 1 = 5 \\
\Rightarrow 2^n = 4 \Rightarrow n = 2 \\
\Rightarrow f(x) = x^2 + 1 \\
\Rightarrow f(f(2)) = f(5) = 26
\end{align*}
\]

10. \( f(x) = f(b + (x - b)) \\
\Rightarrow = f(b - (x - b)) \\
\Rightarrow = f(2b - x) \\
\Rightarrow = f(a + (2b - x - a)) \\
\Rightarrow = f(a - (2b - x - a)) \\
\Rightarrow = f(2a - 2b + x) \\
\]

Hence, \( f(x) \) is periodic with period \( 2a - 2b \).
11. Given \( f(xf(y)) = x^p y^q \)

\[
\Rightarrow x = \left[ \frac{f(xf(y))}{y^q} \right]^{1/p}
\]

Let \( xf(y) = 1 \Rightarrow x = \frac{1}{f(y)^{1/p}} \), then from (1)

\[
f(y) = \left[ \frac{f(1)}{f(y)^{1/p}} \right]^{1/p}
\]

\[
\Rightarrow f(1) = 1
\]

\[
\Rightarrow f(y) = y^{q/p}
\]

Now, \( f(xy^{q/p}) = x^p y^q \). Put \( y^{q/p} = z \), we get

\[
f(xz) = (xz)^p
\]

From (2) and (3) \( x^p = x^{q/p} \) \( \Rightarrow p^2 = q \).

12. \( f(x-1) + f(x+1) = \sqrt{3} f(x) \)

Putting \( x + 2 \) for \( x \) in relation (1) we get

\[
f(x+1) + f(x+3) = \sqrt{3} f(x+2)
\]

From (1) and (2), we get

\[
f(x-1) + 2f(x+1) + f(x+3) = \sqrt{3} \left( f(x) + f(x+2) \right)
\]

\[
\Rightarrow \left[ \frac{f(x-1)}{3} + f(x+1) + \frac{f(x+3)}{3} \right] = f(x+1)
\]

\[
\Rightarrow f(x-1) + f(x+3) = f(x+1)
\]

Putting \( x + 2 \) for \( x \) in relation (3), we get

\[
f(x+1) + f(x+5) = f(x+3)
\]

Adding (3) and (4) in \( f(x-1) = -f(x+5) \)

Now, put \( x+1 \) for \( x \), we get

\[
f(x+6) = f(x)
\]

Put \( x + 6 \) in place of \( x \) in (5), we get

\[
f(x + 12) = f(x)
\]

\[
\therefore \text{the period of } f(x) \text{ is 12.}
\]

13. \( f(a+x) = b + [1 + b - 3b^2 f(x) + 3b^2 f(x)^2 - (f(x))^3]^{1/3} \)

\[
\Rightarrow f(a+x) - b = [1 - \{ f(x) - b \}^3]^{1/3}
\]

\[
\Rightarrow \phi(a+x) = 1 - \{ \phi(x) \}^{1/3}
\]

where \( \phi(x) = f(x) - b \)

\[
\Rightarrow \phi(2a+x) = 1 - \{ \phi(x+a) \}^{1/3} = \phi(x) \text{ from (1)}
\]

\[
\Rightarrow f(x+2a) = f(x) - b
\]

\[
\Rightarrow f(x+2a) = f(x)
\]

\[
\therefore f(x) \text{ is periodic with period } 2a.
\]

14. \( f(x,y) = f(2x+2y, 2y-2x) \)

(Replacing \( x \) by \( 2x+2y \) and \( y \) by \( 2y-2x \))

\[
f(x, y) = f(2(2x+2y) + 2(2y-2x), 2(2y-2x) - 2(2x+2y))
\]

\[
f(x, y) = f(8x, -8y) = f(8(-8x), -8(8y))
\]

\[
= f(-64x, -64y)
\]

\[
= f(-64(-64x), -64y(-64y)) = f(2^{12} x, 2^{12} y)
\]

\[
f(x, 0) = f(2^{12} x, 0)
\]

\[
f(2^{12} x, 0) = f(2^{12} \cdot 2^{12} x, 0)
\]

\[
\Rightarrow g(y) = g(y+12)
\]

Hence, \( g(x) \) is periodic and its period is 12.

15. \( y = \frac{x-a}{(x-b)(x-c)} \Rightarrow xy^2 - [(b+c)y + 1]x + bcy + a = 0 \)

Now \( x \) is real, \( \Rightarrow D \geq 0 \)

\[
\Rightarrow (b+c)y^2 - 2(b+c-y) + 1 \geq 0, \forall y \in R,
\]

(as given that \( f(x) \) is an onto function)

\[
\Rightarrow (b-c)^2 - 2(b+c-y) + 1 \geq 0, \forall y \in R
\]

\[
D \leq 0
\]

\[
\Rightarrow 4(b+c-2a)^2 - 4(b-c)^2 \leq 0
\]

\[
\Rightarrow (b+c-2a)^2 - (b-c)^2 \leq 0
\]

\[
\Rightarrow (b+c-2a-b+c)(b+c-2a+b-c) \leq 0
\]

\[
\Rightarrow (c-a)(b-a) \leq 0
\]

\[
\Rightarrow c \leq a \text{ and } b \leq a \text{ or } c \geq a \text{ and } b \geq a
\]

\[
\Rightarrow c \leq a \leq b \text{ (as } b > c)
\]

\[
\Rightarrow \text{a is } (b, c)
\]

16. Let \( f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, a_i \in I (i = 0, 1, 2, \ldots, n) \)

\[
\text{Now, } f(a) = a_0 + a_1 a + a_2 a^2 + \ldots + a_n a^n = b
\]

\[
f(b) = a_0 + a_1 b + a_2 b^2 + \ldots + a_n b^n = c
\]

\[
f(c) = a_0 + a_1 c + a_2 c^2 + \ldots + a_n c^n = a
\]

\[
\Rightarrow f(a) - f(b) = (a-b)f(a, b) = b-c,
\]

where \( f_i(a, b) \) is an integer

Similarly, \( b-a \) \( f_i(b, c) = c-a \)

and \( c-a \) \( f_i(c, a) = a-b \)

Multiplying all these, we get \( f_1(a, b) f_2(b, c) f_3(c, a) = 1 \)

\[
\Rightarrow f_1(a, b) = f_2(b, c) = f_3(c, a) = 1
\]

\[
\Rightarrow a - b = b - c, c - a = a - b \text{ and } c - a = a - b
\]

\[
\Rightarrow b = c \text{ which is a contradiction.}
\]

Hence, no such polynomial exists.

17. Clearly, from graph \( g(x) = \frac{1}{2} x + \frac{1}{4} \)

\[
\text{for } -1 < x < -1/4
\]

\[
\text{for } 1/4 < x < 1
\]

\[
\text{for } x = \{ 1, 2 \}
\]

\[
\text{for } x^2, \quad 1 \leq x \leq 2
\]

Fig. 1.94
18. Given \( f(x - f(y)) = f(f(y)) + x f(y) + f(x) - 1 \) 

Putting \( x = f(y) = 0 \), then \( f(0) = f(0) + 0 + f(0) - 1 \)

\[ \therefore f(0) = 1 \]  

(2) Again putting \( x = f(y) = \lambda \) in (1) 

Then \( f(0) = f(\lambda) + \lambda^2 + f(\lambda) - 1 \)

\[ \Rightarrow 1 = 2 f(\lambda) + \lambda^2 - 1 \] 

(from 2)

\[ \therefore f(\lambda) = \frac{2 - \lambda^2}{2} = 1 - \frac{\lambda^2}{2} \]

Hence, \( f(x) = 1 - \frac{x^2}{2} \) is the unique function.

19. Since \( f(x) = \frac{(2 \cos x - 1)(2 \cos 2x - 1)(2 \cos 2^2 x - 1) \ldots}{(2 \cos 2^n x - 1)} \)

\[ f(x) = \frac{(2 \cos 2^n x - 1)(2 \cos 2^{n-1} x - 1) \ldots (2 \cos 2^2 x - 1)(2 \cos 2x - 1) \ldots (2 \cos 2x + 1)}{(2 \cos 2^n x + 1) \ldots (2 \cos 2^2 x + 1)(2 \cos 2x + 1)} \]

\[ f(x) = \frac{(2 \cos 2^n x - 1)(2 \cos 2^{n-1} x - 1) \ldots (2 \cos 2^2 x - 1)(2 \cos 2x - 1)}{(2 \cos 2^n x + 1) \ldots (2 \cos 2^2 x + 1)(2 \cos 2x + 1)} \]

Proceeding in similarly way

\[ f(x) = \frac{(2 \cos 2^n x + 1)(2 \cos 2^{n-1} x - 1)}{(2 \cos 2^n x + 1)} \]

\[ = \frac{(2 \cos 2^n x - 1)(2 \cos 2^{n-1} x - 1) \ldots (2 \cos 2^2 x - 1)(2 \cos 2x - 1)}{(2 \cos 2^n x + 1) \ldots (2 \cos 2^2 x + 1)(2 \cos 2x + 1)} \]

\[ = \frac{(2 \cos 2^n x + 1)(2 \cos 2^{n-1} x - 1)}{(2 \cos 2^n x + 1)} \]

\[ = \frac{(2 \cos 2^n x - 1)(2 \cos 2^{n-1} x - 1) \ldots (2 \cos 2^2 x - 1)(2 \cos 2x - 1)}{(2 \cos 2^n x + 1) \ldots (2 \cos 2^2 x + 1)(2 \cos 2x + 1)} \]

\[ = \frac{(2 \cos 2^n x + 1)(2 \cos 2^{n-1} x - 1)}{(2 \cos 2^n x + 1)} \]

\[ = \frac{(2 \cos 2^n x - 1)(2 \cos 2^{n-1} x - 1) \ldots (2 \cos 2^2 x - 1)(2 \cos 2x - 1)}{(2 \cos 2^n x + 1) \ldots (2 \cos 2^2 x + 1)(2 \cos 2x + 1)} \]

1.60 Calculus

20. \( f(x) = \frac{a^x}{a^x + \sqrt{a}} \)

\[ \Rightarrow f(1-x) = \frac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \frac{a}{a + \sqrt{aa^x}} = \frac{\sqrt{a}}{a + \sqrt{a}} \]

\[ \Rightarrow f(x) + f(1-x) = 1 \]

Also, \( f\left(\frac{1}{2}\right) = \frac{1}{2} \)

\[ \Rightarrow \sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right) = 2 + f\left(\frac{n}{2n}\right) + f\left(\frac{n+1}{2n}\right) + \ldots + f\left(\frac{n-1}{2n}\right) + 1 \]

\[ = 2\left[ f\left(\frac{1}{2n}\right) + f\left(\frac{2n-1}{2n}\right) \right] + f\left(\frac{2}{2n}\right) + f\left(\frac{2n-2}{2n}\right) \]

\[ + \ldots + f\left(\frac{n-1}{2n}\right) + f\left(\frac{n+1}{2n}\right) + f\left(\frac{1}{2}\right) \]

\[ = 2\left[ f\left(\frac{1}{2n}\right) + f\left(\frac{2}{2n}\right) + f\left(\frac{3}{2n}\right) + \ldots + f\left(\frac{n-1}{2n}\right) + 1 \right] \]

\[ = 2n-1 \]

Objective Type

1. b. \( f: N \rightarrow N, f(n) = 2n + 3 \).

Here, the range of the function is \( \{5, 6, 7, \ldots\} \) or \( N - \{1, 2, 3, 4\} \)

which is a subset of \( N \) (co-domain).

Hence, function is into.

Also, it is clear that \( f(n) \) is one-one or injective.

Hence, \( f(n) \) is injective only.

2. b. \( f(x) = \sin \left(\log (x + \sqrt{1+x^2})\right) \)

\[ \Rightarrow f(-x) = \sin [\log (-x + \sqrt{1+x^2})] \]

\[ \Rightarrow f(-x) = \sin \left(\sqrt{1+x^2} - x \right) \left(\sqrt{1+x^2} + x \right) \]

\[ \Rightarrow f(-x) = \sin \left(\frac{1}{x + \sqrt{1+x^2}}\right) \]