# MATHEMATICS
## Coordinate Geometry



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**Mathematics: Coordinate Geometry**

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# Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. A combined national-level engineering entrance examination has finally been proposed by the Ministry of Human Resource Development, Government of India. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, NITs, IIITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

**Ghanshyam Tewani**

## COORDINATE GEOMETRY

The coordination of algebra and geometry is called coordinate geometry. Historically, coordinates were introduced to help geometry. And so well did they do this job that the very identity of geometry was changed. The word 'geometry' today generally means coordinate geometry.

In coordinate geometry, all the properties of geometrical figures are studied with the help of algebraic equations. Students should note that the object of coordinate geometry is to use some known facts about a curve in order to obtain its equation and then deduce other properties of the curve from the equation so obtained. For this purpose, we require a coordinate system. There are various types of coordinate systems present in two dimensions e.g., rectangular, oblique, polar, triangular system, etc. Here, we will only discuss rectangular coordinate system in detail.

### Cartesian Coordinates

Let $X'OX$ and $Y'OY$ be two fixed straight lines at right angles. $X'OX$ is called axis of $x$ and $Y'OY$ is called axis of $y$, and $O$ is named as origin. From any point '$P$', a line is drawn parallel to $OY$. The directed line $OM = x$ and $MP = y$. Here, $OM$ is abscissa and $MP$ is ordinate of the point '$P$'. The abscissa $OM$ and the ordinate $MP$ together written as $(x, y)$ are called coordinates of point '$P$'. Here, $(x, y)$ is an ordered pair of real numbers $x$ and $y$, which determine the position of point '$P$'.

Since $X'OX$ is perpendicular to $Y'OY$, this system of representation is called rectangular (or orthogonal) coordinate system (Fig. 1.1(a)).

When the axes coordinates $X'OX$ and $Y'OY$ are not at right angles, they are said to be oblique axes.

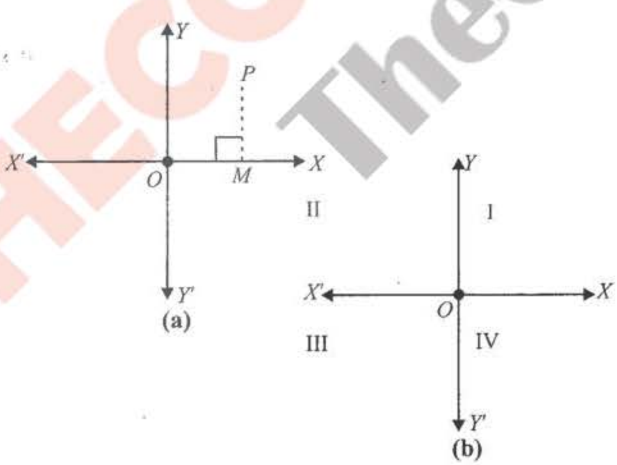



Fig. 1.1

The axes of rectangular coordinate system divide the plane into four infinite regions, called quadrants, each bounded by two half-axes. These are numbered from 1st to 4th and denoted by roman numerals (Fig. 1.1 (b)). The signs of the two coordinates are given below:

| | $x$ | $y$ |
|---|---|---|
| **First quadrant** | + | + |
| **Second quadrant** | − | + |
| **Third quadrant** | − | − |
| **Fourth quadrant** | + | − |

*Lattice point* (with respect is coordinate geometry): Lattice point is defined as a point whose abscissa and ordinate are integers.

## DISTANCE FORMULA

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

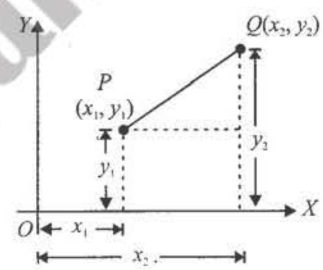



Fig. 1.2

Therefore, distance of $(x_1, y_1)$ from origin $= \sqrt{x_1^2 + y_1^2}$.

**Note:**

- *If distance between two points is given, then use* $\pm$ *sign.*
- *Distance between* $(x_1, 0)$ *and* $(x_2, 0)$ is $|x_1 - x_2|$.
- *Distance between* $(0, y_1)$ *and* $(0, y_2)$ is $|y_1 - y_2|$.
- *Circumcentre $P(x, y)$ is a point which is equidistant from the vertices of triangle $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.*
  *Hence, $AP = BP = CP$, which gives two equations in $x$ and $y$, solving which we get circumcentre.*

**Example 1.1** **In $\Delta ABC$, prove that $AB^2 + AC^2 = 2(AO^2 + BO^2)$, where $O$ is the middle point of $BC$.**

**Sol.**

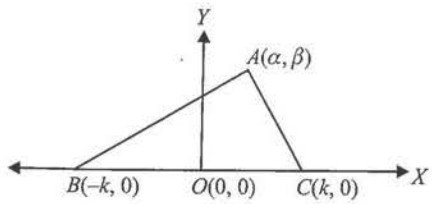



**Fig. 1.3**

We take $O$ as the origin and $OC$ and $OY$ as the $x$-and $y$-axes, respectively.

Let $BC = 2k$, then $B \equiv (-k, 0)$, $C \equiv (k, 0)$.

Let $A \equiv (\alpha, \beta)$

Now L.H.S.

$$= AB^2 + AC^2$$
$$= (\alpha + k)^2 + (\beta - 0)^2 + (\alpha - k)^2 + (\beta - 0)^2$$
$$= \alpha^2 + k^2 + 2\alpha k + \beta^2 + \alpha^2 + k^2 - 2\alpha k + \beta^2$$
$$= 2\alpha^2 + 2\beta^2 + 2k^2$$
$$= 2(\alpha^2 + \beta^2 + k^2)$$

and R.H.S

$$= 2(AO^2 + BO^2)$$
$$= 2[(\alpha - 0)^2 + (\beta - 0)^2 + (-k - 0)^2 + (0 - 0)^2]$$
$$= 2(\alpha^2 + \beta^2 + k^2)$$

$\therefore$ L.H.S. = R.H.S.

**Example 1.2** **Find the coordinates of the circumcentre of the triangle whose vertices are $A(5, -1)$, $B(-1, 5)$, and $C(6, 6)$. Find its radius also.**

**Sol.**

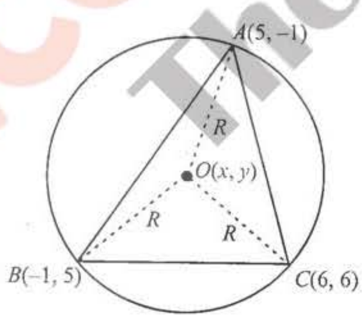



**Fig. 1.4**

Let circumcentre be $O(x, y)$, then

$$(OA)^2 = (OB)^2 = (OC)^2 = (\text{radius})^2 = R^2 \quad \text{(i)}$$

$$\Rightarrow (x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$$
$$= (x - 6)^2 + (y - 6)^2$$

Taking first two relations, we get

$$(x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$$

$$\Rightarrow \quad x = y \quad \text{(ii)}$$

Taking last two relations, we get

$$(x + 1)^2 + (y - 5)^2 = (x - 6)^2 + (y - 6)^2$$

$$\Rightarrow (x + 1)^2 + (x - 5)^2 = (x - 6)^2 + (x - 6)^2 \quad \text{[from (ii)]}$$

$$\Rightarrow \quad 2x^2 - 8x + 26 = 2x^2 - 24x + 72$$

$$x = 23/8$$

$$\Rightarrow \quad \text{Circumcentre} = (23/8, 23/8)$$

$$\Rightarrow \quad R^2 = (x - 5)^2 + (y + 1)^2 = (OA)^2$$

$$\Rightarrow \quad = [(23/8) - 5]^2 + [(23/8) + 1]^2$$
$$= (-17)^2/(64) + (31)^2/(64)$$
$$= 1250/64$$

$$\Rightarrow \quad \text{Radius} = 25\sqrt{2}/8 \text{ units}$$

**Example 1.3** **Two points $O(0, 0)$ and $A(3, \sqrt{3})$ with another point $P$ form an equilateral triangle. Find the coordinates of $P$.**

**Sol.** Let the coordinates of $P$ be $(h, k)$.

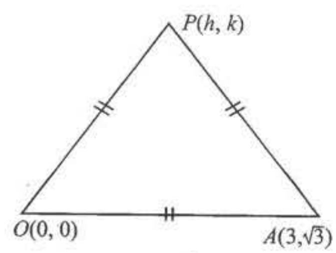



**Fig. 1.5**

$$\therefore \quad OA = OP = AP \text{ or } OA^2 = OP^2 = AP^2$$

$$\therefore \quad OA^2 = OP^2$$

$$\Rightarrow \quad 12 = h^2 + k^2 \quad \text{(i)}$$

and $$OP^2 = AP^2$$

$$\Rightarrow \quad h^2 + k^2 = (h - 3)^2 + (k - \sqrt{3})^2$$

$$\Rightarrow \quad 3h + \sqrt{3}k = 6 \text{ or } h = 2 - k/\sqrt{3} \quad \text{(ii)}$$

Using Eq. (ii) in Eq. (i), we get

$$(2 - k/\sqrt{3})^2 + k^2 = 12 \text{ or } k^2 - \sqrt{3}k - 6 = 0$$

or $(k - 2\sqrt{3})(k + \sqrt{3}) = 0$

$$\therefore \quad k = 2\sqrt{3} \text{ or } -\sqrt{3}$$

From Eq. (ii), we get that

when $\quad k = 2\sqrt{3}, h = 0,$

when $\quad k = -\sqrt{3}, h = 3$

Hence, the coordinates of $P$ are

$(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

**Example 1.4** **If $O$ is the origin and if coordinates of any two points $Q_1$ and $Q_2$ are $(x_1, y_1)$ and $(x_2, y_2)$, respectively, prove that $OQ_1 \cdot OQ_2 \cos \angle Q_1 OQ_2 = x_1x_2 + y_1y_2$.**

**Sol.**

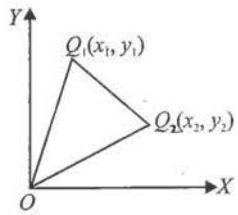



**Fig. 1.6**

In $\Delta Q_2OQ_1$,

$$Q_1Q_2^2 = OQ_1^2 + OQ_2^2 - 2OQ_1OQ_2 \cos \angle Q_1OQ_2$$

$$\Rightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2OQ_1\,OQ_2 \cos\angle Q_1OQ_2$$

(using cosine Rule)

$$\Rightarrow \quad x_1x_2 + y_1y_2 = OQ_1\,OQ_2 \cos Q_1OQ_2$$

**Example 1.5** **Let $A = (3, 4)$ and $B$ is a variable point on the lines $|x| = 6$. If $AB \le 4$, then the number of position of $B$ with integral coordinates is**

**a.** 5 **b.** 4

**c.** 6 **d.** 10

**Sol.** $B = (\pm 6, y)$. So, $AB \le 4$

$$\Rightarrow \quad (3 \mp 6)^2 + (y-4)^2 \le 16$$

$$\therefore \quad 9 + (y-4)^2 \le 16,$$

$$(\because 81 + (y-4)^2 \le 16 \text{ is absurd})$$

$$\Rightarrow \quad y^2 - 8y + 9 \le 0$$

$$\Rightarrow \quad 4 - \sqrt{7} \le y \le 4 + \sqrt{7}$$

But $y$ is an integer.

$$\Rightarrow \quad y = 2, 3, 4, 5, 6$$

## AREA OF A TRIANGLE

The area of a triangle, whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ is

$$\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

**Proof:**

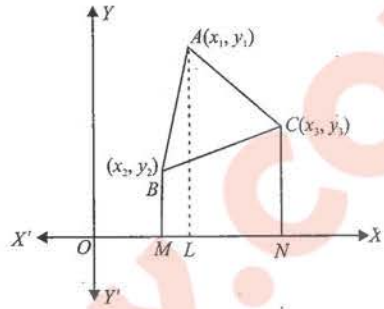



**Fig. 1.7**

Let $ABC$ be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Draw $AL$, $BM$, and $CN$ as perpendiculars from $A$, $B$, and $C$ on the $x$-axis. Clearly, $ABML$, $ALNC$ and $BMNC$ are all trapeziums.

We have,

Area of $\Delta$ $ABC$ = Area of trapezium $ABML$ + Area of trapezium $ALNC$ – Area of trapezium $BMNC$

$$\Rightarrow \text{Area of } \Delta ABC = \frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

**Note:**

- *Area of a triangle can also be found by easy method, i.e., Stair method.*

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}|\{(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\}|$$

- *If three points $A$, $B$, and $C$ are collinear, then area of triangle $ABC$ is zero.*
- ***Sign of area:*** *If the points $A$, $B$, $C$ are plotted in two-dimensional plane and taken on the anticlockwise sense, then the area calculated of the triangle $ABC$ will be positive, while if the points are taken in clockwise sense, then the area calculated will be negative. But, if the points are taken arbitrarily, then the area calculated may be positive or negative, the numerical value being the same in both cases. In case, the area calculated is negative, we will consider it as positive.*

## Area of Polygon

The area of polygon whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, …, $(x_n, y_n)$ is

$$= \frac{1}{2}\left|\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)\}\right|$$

### *Stair Method*

Repeat first coordinates one time in last for down arrow use + ve sign and for up arrow use – ve sign.

$$\text{Area of polygon} = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|\{x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)\}\right|$$

**Note:**

*Points should be taken in cyclic order in coordinate plane.*

**Example 1.6** **Find the area of a triangle whose vertices are $A(3, 2)$, $B(11, 8)$, and $C(8, 12)$.**

**Sol.** Let $A = (x_1, y_1) = (3, 2)$, $B(x_2, y_2) = (11, 8)$, and $C = (x_3, y_3) = (8, 12)$.

Then, area of $\Delta ABC$

$$= \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right|$$

$$= \frac{1}{2}\left|\{3(8 - 12) + 11(12 - 2) + 8(2 - 8)\}\right|$$

$$= \frac{1}{2}\left|\{-12 + 110 - 48\}\right|$$

$$= 25 \text{ sq. units}$$

**Example 1.7** **Prove that the area of the triangle whose vertices are $(t, t - 2)$, $(t + 2, t + 2)$, and $(t + 3, t)$ is independent of $t$.**

**Sol.** Let $A = (x_1, y_1) = (t, t - 2)$, $B = (x_2, y_2) = (t + 2, t + 2)$, and $C = (x_3, y_3) = (t + 3, t)$ be the vertices of the given triangle.

Then, area of $\Delta ABC$

$$= \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right|$$

$$= \frac{1}{2}\left|\{t(t + 2 - t) + (t + 2)(t - t + 2) + (t + 3)(t - 2 - t - 2)\}\right|$$

$$= \frac{1}{2}\left|\{2t + 2t + 4 - 4t - 12\}\right| = |-4| = 4 \text{ sq. units.}$$

Clearly, area of $\Delta ABC$ is independent of $t$.

**Example 1.8** **Find the area of the quadrilateral $ABCD$ whose vertices are respectively $A(1, 1)$, $B(7, -3)$, $C(12, 2)$, and $D(7, 21)$.**

**Sol.** Here, given points are in cyclic order, then

$$\text{Area} = \frac{1}{2}\left\|\begin{matrix} 1 & 1 \\ 7 & -3 \\ 12 & 2 \\ 7 & 21 \\ 1 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|(-3 - 7) + (14 + 36) + (252 - 14) + (7 - 21)\right|$$

$$= 132 \text{ sq. units}$$

**Example 1.9** **For what value of $k$ are the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ are collinear?**

**Sol.** Let three given points be $A = (x_1, y_1) = (k, 2 - 2k)$, $B = (x_2, y_2) = (-k + 1, 2k)$, and $C = (x_3, y_3) = (-4 - k, 6 - 2k)$.

If the given points are collinear, then $\Delta = 0$

$$\Rightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

$$\Rightarrow k(2k - 6 + 2k) + (-k + 1)(6 - 2k - 2 + 2k) + (-4 - k)(2 - 2k - 2k) = 0$$

$$\Rightarrow k(4k - 6) - 4(k - 1) + (4 + k)(4k - 2) = 0$$

$$\Rightarrow 4k^2 - 6k - 4k + 4 + 4k^2 + 14k - 8 = 0$$

$$\Rightarrow 8k^2 + 4k - 4 = 0$$

$$\Rightarrow 2k^2 + k - 1 = 0$$

$$\Rightarrow (2k - 1)(k + 1) = 0$$

$$\Rightarrow k = 1/2 \text{ or } -1$$

Hence, the given points are collinear for $k = 1/2$ or $-1$.

**Example 1.10** **If the vertices of a triangle have rational coordinates, then prove that the triangle cannot be equilateral.**

**Sol.** Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of a triangle $ABC$, where $x_i, y_i$, $i = 1, 2, 3$ are rational. Then, the area of $\Delta ABC$ is given by

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \text{a rational number} \qquad [\because x_i, y_i \text{ are rational}]$$

If possible, let the triangle $ABC$ be an equilateral triangle, then its area is given by

$$\Delta = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 \qquad (\because AB = BC = CA)$$

$= \frac{\sqrt{3}}{4}$ (a rational number) [$\because$ vertices are rational $\therefore AB^2$ is a rational]

= an irrational number

This is a contradication to the fact that the area is a rational number. Hence, the triangle cannot be equilateral.

**Example 1.11** **If the coordinates of two points $A$ and $B$ are (3, 4) and (5, – 2), respectively. Find the coordinates of any point $P$ if $PA = PB$ and area of $\Delta PAB = 10$ sq. units.**

**Sol.** Let the coordinates of $P$ be $(x, y)$. Then, $PA = PB$.

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$$

$$\Rightarrow \quad x - 3y - 1 = 0 \qquad \text{(i)}$$

Now, area of $\Delta PAB = 10$ sq. units

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$$

$$\Rightarrow \quad 6x + 2y - 26 = \pm 20$$

$$\Rightarrow \quad 6x + 2y - 46 = 0$$

or $\quad 6x + 2y - 6 = 0$

$$\Rightarrow \quad 3x + y - 23 = 0 \text{ or } 3x + y - 3 = 0 \qquad \text{(ii)}$$

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get $x = 7$, $y = 2$

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get $x = 1$, $y = 0$

Thus, the coordinates of $P$ are (7, 2) or (1, 0).

**Example 1.12** **Given that $P(3, 1)$, $Q(6, 5)$, and $R(x, y)$ are three points such that the angle $PRQ$ is a right angle and the area of $\Delta RQP = 7$, then find the number of such points $R$.**

**Sol.** Obviously, $R$ lies on the circle with $P$ and $Q$ as end points of diameter.

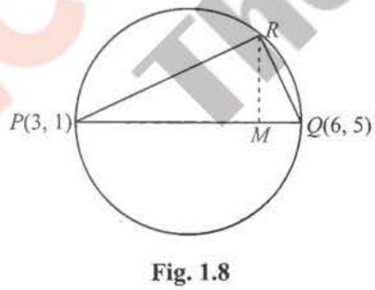



**Fig. 1.8**

Distance between points $P(3, 1)$ and $Q(6, 5)$ is 5 units. Hence, radius is 2.5 units

Area of triangle $PQR = \frac{1}{2} RM \times PQ = 7$ (given)

$$\Rightarrow \quad RM = \frac{14}{5} = 2.8$$

Now the value of $RM$ cannot be greater than the radius. Hence. no such triangle is possible.

## SECTION FORMULA

### Formula for Internal Division

Coordinates of the point that divides the line segment joining the points $(x_1, y_1)$ and $(x_2, y_2)$ internally in the ratio $m : n$ are given by

$$x = \frac{mx_2 + nx_1}{m+n},$$

$$y = \frac{my_2 + ny_1}{m+n}$$

**Proof:**

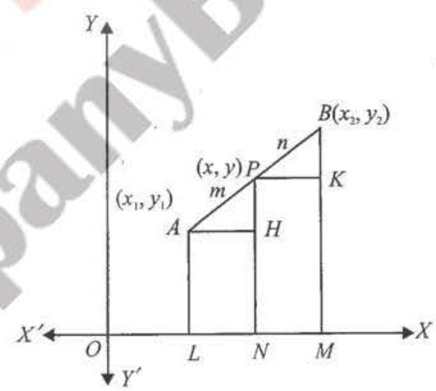



**Fig. 1.9**

From the figure,

Clearly, $\Delta AHP$ and $\Delta PKB$ are similar.

$$\Rightarrow \quad \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Now,

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x}$$

$$\Rightarrow \quad mx_2 - mx = nx - nx_1$$

$$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m+n}$$

Similarly $\quad \frac{m}{n} = \frac{y - y_1}{y_2 - y}$

$$\Rightarrow \qquad my_2 - my = ny - ny_1$$

$$\Rightarrow \qquad y = \frac{my_2 + ny_1}{m+n}$$

Thus, the coordinates of $P$ are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

## Formula for External Division

Coordinates of the point that divides the line segment joining the points $(x_1, y_1)$ and $(x_2, y_2)$ externally in the ratio $m$: n are given by

$$x = \frac{mx_2 - nx_1}{m-n}, y = \frac{my_2 - ny_1}{m-n}$$

**Proof:**

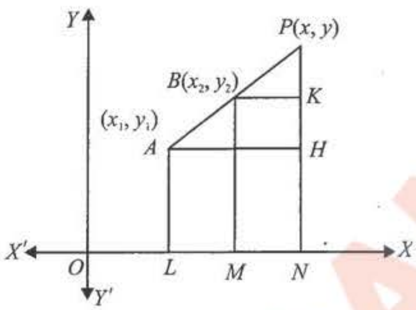



**Fig. 1.10**

From the figure,

Clearly, triangles $PAH$ and $PBK$ are similar. Therefore,

$$\frac{AP}{PB} = \frac{AH}{BK} = \frac{PH}{PK}$$

$$\Rightarrow \qquad \frac{m}{n} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$$

Now,

$$\frac{m}{n} = \frac{x - x_1}{x - x_2}$$

$$\Rightarrow \qquad mx - mx_2 = nx - nx_1$$

$$\Rightarrow \qquad x = \frac{mx_2 - nx_1}{m-n}$$

and

$$\frac{m}{n} = \frac{y - y_1}{y - y_2}$$

$$\Rightarrow \qquad my - my_2 = ny - ny_1$$

$$\Rightarrow \qquad y = \frac{my_2 - ny_1}{m-n}$$

Thus, the coordinates of $P$ are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$.

**Note:**

- *If the ratio, in which a given line segment is divided, is to be determined, then sometimes, for convenience (instead of taking the ratio m:n), we take the ratio $\lambda$: 1 and apply the formula for internal division* $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$.

  *If the value of $\lambda$ turns out to be positive, it is an internal division, otherwise it is an external division.*
- *The midpoint of* $(x_1, y_1)$ *and* $(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- *To prove that A, B, C, D are vertices of*

| | |
|---|---|
| *Parallelogram* | *Show that diagonals AC and BD bisect each other* |
| *Rhombus* | *Show that diagonals AC and BD bisect each other and adjacent sides AB and BC are equal* |
| *Square* | *Show that diagonals AC and BD are equal and bisect each other and adjacent sides AB and BC are equal* |
| *Rectangle* | *Show that diagonals AC and BD are equal and bisect each other* |

**Example 1.13** **Find the coordinates of the point which divides the line segments joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.**

**Sol.** Let $P(x, y)$ be the required point.

**i.** For internal division, we have


**Fig. 1.11**

$$x = \frac{3(-4) + 2(6)}{3+2}$$

and
$$y = \frac{3(5) + 2(3)}{3+2}$$

$$\Rightarrow \qquad x = 0 \text{ and } y = 21/5$$

So the coordinates of $P$ are (0, 21/5).

**ii.** For external division, we have

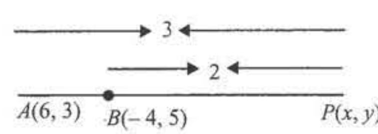



**Fig. 1.12**

$$x = \frac{3(-4) - 2(6)}{3 - 2}$$

and $$y = \frac{3(5) - 2(3)}{3 - 2}$$

$\Rightarrow$ $$x = -24 \text{ and } y = 9$$

So the coordinates of $P$ are $(-24, 9)$.

**Example 1.14** **In what ratio does the $x$-axis divide the line segment joining the points $(2, -3)$ and $(5, 6)$?**

**Sol.** Let the required ratio be $\lambda$:1.

Then, the coordinates of the point of division are

$[(5\lambda + 2)/(\lambda - 1), (6\lambda - 3)/(\lambda + 1)]$

But, it is a point on $x$-axis on which $y$-coordinates of every point is zero.

$\Rightarrow$ $(6\lambda - 3)/(\lambda + 1) = 0$

$\Rightarrow$ $$\lambda = \frac{1}{2}$$

Thus, the required ratio is (1/2): 1 or 1 : 2

**Example 1.15** **Given that $A(1, 1)$ and $B(2, -3)$ are two points and $D$ is a point on $AB$ produced such that $AD = 3AB$. Find the coordinates of $D$.**

**Sol.** We have, $AD = 3AB$. Therefore, $BD = 2AB$.

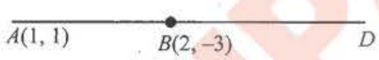



**Fig. 1.13**

Thus, $D$ divides $AB$ externally in the ratio $AD : BD = 3 : 2$

Hence, the coordinates of $D$ are

$$\left(\frac{3(2) - 2(1)}{3 - 2}, \frac{3(-3) - 2(1)}{3 - 2}\right) = (4, -11)$$

**Example 1.16** **Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points $(1, 3)$ and $(2, 7)$.**

**Sol.** Suppose the line $3x + y - 9 = 0$ divides the line segment joining $A(1, 3)$ and $B(2, 7)$ in the ratio $k : 1$ at point $C$. Then, the coordinates of $C$ are

$$\left(\frac{2k + 1}{k + 1}, \frac{7k + 3}{k + 1}\right)$$

But, $C$ lies on $3x + y - 9 = 0$. Therefore,

$$3\left(\frac{2k + 1}{k + 1}\right) + \frac{7k + 1}{k + 1} - 9 = 0$$

$$\Rightarrow 6k + 3 + 7k + 3 - 9k - 9 = 0$$

$\Rightarrow$ $$k = \frac{3}{4}$$

So, the required ratio is 3 : 4 internally

**Example 1.17** **Prove that the points $(-2, -1)$, $(1, 0)$, $(4, 3)$, and $(1, 2)$ are the vertices of a parallelogram. Is it a rectangle?**

**Sol.** Let the given points be $A$, $B$, $C$, and $D$, respectively.

Then, the coordinates of the midpoint of $AC$ are

$$\left(\frac{-2 + 4}{2}, \frac{-1 + 3}{2}\right) = (1, 1)$$

Coordinates of the midpoint of $BD$ are

$$\left(\frac{1 + 1}{2}, \frac{0 + 2}{2}\right) = (1, 1)$$

Thus, $AC$ and $BD$ have the same midpoint.

Hence, $ABCD$ is a parallelogram.

Now, we shall see whether $ABCD$ is a rectangle or not.

We have

$$AC = \sqrt{(4 - (-2))^2 + (3 - (-1))^2}$$

$$= 2\sqrt{13},$$

and

$$BD = \sqrt{(1 - 1)^2 + (0 - 2)^2} = 2$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a rectangle.

**Example 1.18** **If $P$ divides $OA$ internally in the ratio $\lambda_1 : \lambda_2$ and $Q$ divides $OA$ externally in the ratio $\lambda_1 : \lambda_2$, then prove that $OA$ is the harmonic mean of $OP$ and $OQ$.**

**Sol.**


**Fig. 1.14**

We have, $$\frac{1}{OP} = \frac{\lambda_1 + \lambda_2}{\lambda_1 OA}$$

and $$\frac{1}{OP} = \frac{\lambda_1 - \lambda_2}{\lambda_1 OA}$$

$\Rightarrow$ $$\frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OA}$$

$\Rightarrow OP$, $OA$ and $OQ$ are in H.P.

**Example 1.19** **Given that $A_1, A_2, A_3, \ldots, A_n$ are $n$ points in a plane whose coordinates are $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$, respectively. $A_1A_2$ is bisected at the point $P_1$, $P_1A_3$ is divided in the ratio 1 : 2 at $P_2$, $P_2A_4$ is divided in the ratio 1 : 3 at $P_3$, $P_3A_5$ is divided in the ratio 1 : 4 at $P_4$, and so on until all $n$ points are exhausted. Find the final point so obtained.**

**Sol.** The coordinates of $P_1$ (midpoint of $A_1A_2$) are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$P_2$ divides $P_1A_3$ in 1 : 2; therefore, coordinates of $P_2$ are

$$\left(\frac{2\left(\frac{x_1+x_2}{2}\right)+x_3}{2+1}, \frac{2\left(\frac{y_1+y_2}{2}\right)+y_3}{2+1}\right)$$

i.e., $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Now $P_3$ divides $P_1A_4$ in 1 : 3, therefore,

$$P_3 = \left(\frac{3\,\frac{(x_1+x_2+x_3)}{3}+x_4}{3+1}, \frac{3\,\frac{(y_1+y_2+y_3)}{3}+y_4}{3+1}\right)$$
$$= \left[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4)\right]$$

Proceeding in this manner, we can show that the coordinates of the final point are

$$[(x_1+x_2+\cdots+x_n)/n, (y_1+y_2+\cdots+y_n)/n]$$

## COORDINATES OF THE CENTROID, INCENTRE, AND EX-CENTRES OF A TRIANGLE

### Centroid of a Triangle

The point of concurrency of the medians of a triangle is called the centroid of the triangle. The coordinates of the centroid of the triangle with vertices as $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

**Proof:**

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of $\Delta\, ABC$ whose medians are $AD$, $BE$, and $CF$, respectively. So $D$, $E$, and $F$ are respectively the midpoint of $BC$, $CA$, and $AB$.

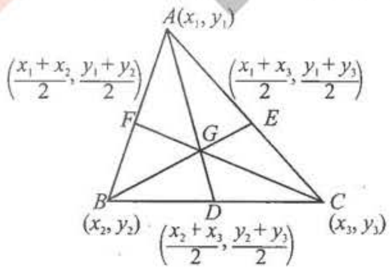



**Fig. 1.15**

Coordinates of $D$ are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

Coordinates of a points $G$ dividing $AD$ in the ratio 2:1 are

$$\left(\frac{1(x_1)+2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1(y_1)+2\left(\frac{y_2+y_3}{2}\right)}{1+2}\right)$$
$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

### Incentre of a Triangle

The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle. The coordinates of the incentre of a triangle with vertices as $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a = $BC$, $b = AC$ and $C = AB$

**Proof:** Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle $ABC$, and let $a$, $b$, $c$ be the lengths of the sides $BC$, $CA$, $AB$, respectively.

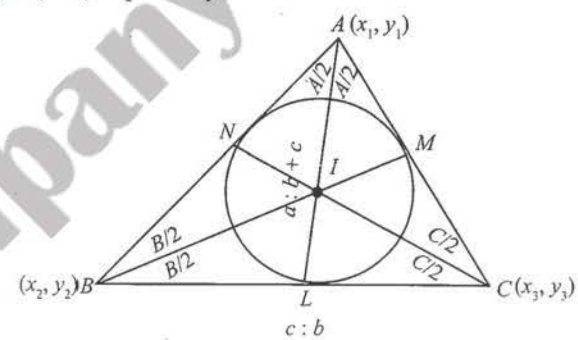



**Fig. 1.16**

The point at which the bisectors of the angles of a triangle intersect is called the incentre of the triangle.

Let $AL$, $BM$, and $CN$ be respectively the internal bisectors of the angles $A$, $B$ and $C$.

As $AL$ bisects $\angle BAC$ internally, we have

$$\frac{BL}{LC} = \frac{BA}{AC} = \frac{c}{b} \qquad \text{(i)}$$

$$\Rightarrow \frac{LC}{BL} = \frac{b}{c}$$

$$\Rightarrow \frac{LC}{BL} + 1 = \frac{b}{c} + 1$$

$$\Rightarrow \frac{LC+BL}{BL} = \frac{b+c}{c}$$

$$\Rightarrow \frac{BC}{BL} = \frac{b+c}{c}$$

$$\Rightarrow \frac{a}{BL} = \frac{b+c}{c}$$

$$\Rightarrow BL = \frac{ac}{b+c} \qquad \text{(ii)}$$

Since $BI$ is the bisector of $\angle B$, so it divides $AIL$ in the ratio $AI : IL$

$$\therefore \quad \frac{AI}{IL} = \frac{AB}{BL} = \frac{c}{(ac)/(b+c)} = \frac{b+c}{a}$$

[Using (ii)]

$$\Rightarrow \quad AI : IL = (b+c) : a \quad \text{(iii)}$$

From (i), $L$ divides $BC$ in the ratio $c : b$

$\Rightarrow$ Coordinates of $L$ are

$$\left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$$

From (iii), $I$ divides $AL$ in the ratio $(b + c) : a$. So the coordinates of $I$ are

$$\left(\frac{ax_1 + (b+c)\left(\frac{bx_2 + cx_3}{b+c}\right)}{a+b+c}, \frac{ay_1 + (b+c)\left(\frac{by_2 + cy_3}{b+c}\right)}{a+b+c}\right)$$

$$\text{or} \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

## Ex-centre of a Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle $ABC$, and let $a$, $b$, $c$ be the lengths of the sides $BC$, $CA$, $AB$, respectively. The circle which touches the side $BC$ and the other two sides $AB$ and $AC$ produced is called the escribed circle opposite to the angle $A$. The bisectors of the external angle $B$ and $C$ meet at a point $I_1$ which is the centre of the escribed circle opposite to the angle $A$.

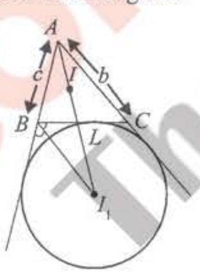



**Fig. 1.17**

$$\frac{BL}{LC} = \frac{c}{b}, \text{ also } \frac{AI_1}{I_1L} = -\frac{(b+c)}{a}$$

The coordinates of $I_1$ are given by

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly, coordinates of $I_2$ and $I_3$ (centres of escribed circles opposite to the angles $B$ and $C$, respectively) are given by

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

## Circumcentre of a Triangle

Circumcentre $P(x, y)$ is a point which is equidistant from the vertices of triangle $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

Hence, $AP = BP = CP$, which gives two equations in $x$ and $y$, solving which we get circumcentre.

Also circumcentre is given by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

## Orthocentre

The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle.

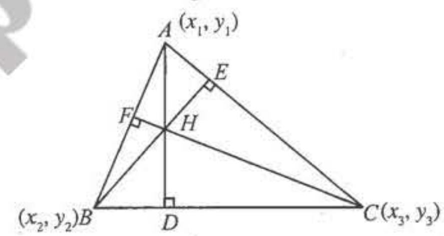



**Fig. 1.18**

In Fig. 1.18, point $H$ is an orthocentre of $\Delta ABC$, and it is given by

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

**Note:**

- *Circumcentre O, Centroid G, and Orthocentre H of an acute $\Delta ABC$ are collinear. G divides OH in the ratio* $1:2$, *i.e.,* $OG : GH = 1 : 2$
- *In an isosceles triangle, centroid, orthocentre, incentre, and circumcentre lie on the same line. In an equilateral triangle, all these four points coincide.*

**Example 1.20** **If a vertex of a triangle is (1, 1), and the middle points of two sides passing through it are (–2, 3) and (5, 2), then find the centroid and the incentre of the triangle.**

**Sol.**

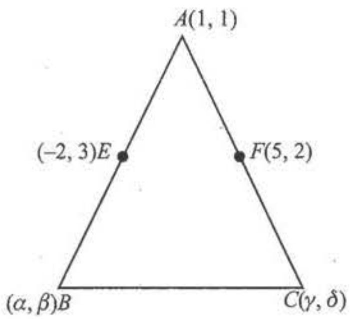



**Fig. 1.19**

Let $E$ and $F$ be the midpoints of $AB$ and $AC$.

Let the coordinates of $B$ and $C$ be $(\alpha, \beta)$ and $(\gamma, \delta)$, respectively, then

$$-2 = \frac{1+\alpha}{2}, 3 = \frac{1+\beta}{2},$$

$$5 = \frac{1+\gamma}{2}, 2 = \frac{1+\delta}{2}$$

$$\therefore \qquad \alpha = -5, \beta = 5, \gamma = 9, \delta = 3$$

Therefore, coordinates of $B$ and $C$ are $(-5, 5)$ and $(9, 3)$, respectively.

Then, centroid is

$$\left(\frac{1-5+9}{3}, \frac{1+5+3}{3}\right), \text{ i.e., } \left(\frac{5}{3}, 3\right)$$

and $\quad a = BC = \sqrt{(-5-9)^2 + (5-3)^2} = 10\sqrt{2}$

$$b = CA = \sqrt{(9-1)^2 + (3-1)^2} = 2\sqrt{17}$$

and $\quad c = AB = \sqrt{(1+5)^2\,(1-5)^2} = 2\sqrt{13}$

Then incentre is

$$\left(\frac{10\sqrt{2}(1) + 2\sqrt{17}(-5) + 2\sqrt{13}(9)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}},\right.$$

$$\left.\frac{10\sqrt{2}(1) + 2\sqrt{17}(5) + 2\sqrt{13}(3)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}\right)$$

$$\text{i.e., } \left(\frac{5\sqrt{2} - 5\sqrt{17} + 9\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}, \frac{5\sqrt{2} + 5\sqrt{17} + 3\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}\right)$$

**Example 1.21** **Find the orthocentre of the triangle whose vertices are (0, 0), (3, 0), and (0, 4).**

**Sol.** This is a right-angled (at origin) triangle, therefore orthocentre = (0, 0).

**Example 1.22** **If the vertices *P*, *Q*, *R* of a triangle *PQR* are rational points, which of the following points of the triangle *PQR* is (are) not necessarily rational?**

**a. centroid**    **b. incentre**

**c. circumcentre**    **d. orthocentre**

**(A rational point is a point both of whose coordinates are rational numbers)**

**Sol.** If $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_2, y_3)$, where $x_1, y_1$ etc., are rational numbers, then $\Sigma x_1$, $\Sigma y_1$ are also rational.

So, the coordinates of the centroid $(\Sigma x_1/3\ \Sigma y_1/3)$ will be rational.

As $AB = c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ may or may not be rational and it may be an irrational number of the form $\sqrt{p}$. Hence, the coordinates of the incentre $(\Sigma ax_1/\Sigma a, \Sigma ay_1/\Sigma a)$ may or may not be rational. If $(\alpha, \beta)$ is the circumcentre or orthocentre, $\alpha$ and $\beta$ are found by solving two linear equations in $\alpha, \beta$ with rational coefficients. So $\alpha, \beta$ must be rational numbers.

**Example 1.23** **If the circumcentre of an acute angled triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$, then find the orthocentre.**

**Sol.** We know from geometry that the circumcentre ($O$) centroid ($G$) and orthocentre ($H$) of a triangle lie on the line joining the circumcentre (0, 0) and the centroid $((a+1)^2/2, (a-1)^2/2)$.

Also $\qquad \dfrac{HG}{GO} = \dfrac{2}{1} \Rightarrow$ H has coordinate

$$\left(\frac{3(a+1)^2}{2}, \frac{3(a-1)^2}{2}\right)$$

**Example 1.24** **If a vertex, the circumcentre, and the centroid of a triangle are (0, 0), (3, 4), and (6, 8), respectively, then the triangle must be**

**a. a right-angled triangle**

**b. an equilateral triangle**

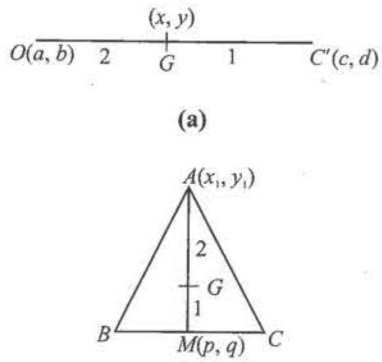
**c. an isosceles triangle**

**d. a right-angled isosceles triangle**

**Sol.** Clearly, (0, 0), (3, 4), and (6, 8) are collinear. So, the circumcentre $M$ and the centroid $G$ are on the median which is also the perpendicular bisector of the side. So, the $\Delta$ must be isosceles.

**Example 1.25** **Orthocentre and circumcentre of a $\Delta ABC$ are $(a, b)$ and $(c, d)$, respectively. If the coordinates of the vertex $A$ are $(x_1, y_1)$, then find the coordinates of the middle point of $BC$.**

**Sol.**




**(a)**

**(b)**

**Fig. 1.20**

$$x = \frac{a+2c}{3};\ y = \frac{b+2d}{3}$$

Now

$$x = \frac{x_1+2p}{3};\ y = \frac{y_1+2q}{3}$$

$$\therefore \quad p = \frac{a+2c-x_1}{2};\ q = \frac{b+2d-y_1}{2}$$

## Concept Application Exercise 1.1

1. If the points $(0, 0)$, $(2, 2\sqrt{3})$, and $(p, q)$ are the vertices of an equilateral triangle, then $(p, q)$ is
   **a.** $(0, -4)$ **b.** $(4, 4)$ **c.** $(4, 0)$ **d.** $(5, 0)$
2. The distance between the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is
   **a.** $a\cos\frac{\alpha-\beta}{2}$ **b.** $2a\cos\frac{\alpha-\beta}{2}$
   **c.** $2a\sin\frac{\alpha-\beta}{2}$ **d.** $a\sin\frac{\alpha-\beta}{2}$
3. Find the length of altitude through $A$ of the triangle $ABC$, where $A \equiv (-3, 0)$, $B \equiv (4, -1)$, $C \equiv (5, 2)$.
4. If the point $(x, -1)$, $(3, y)$, $(-2, 3)$, and $(-3, -2)$ taken in order are the vertices of a parallelogram, then find the values of $x$ and $y$.
5. If the midpoints of the sides of a triangle are $(2, 1)$, $(-1, -3)$, and $(4, 5)$. Then find the coordinates of its vertices.
6. The three points $(-2, 2)$, $(8, -2)$, and $(-4, -3)$ are the vertices of
   a. an isosceles triangle
   b. an equilateral triangle
   **c.** a right angled triangle
   **d.** none of these
7. The points $(a, b)$, $(c, d)$, and $\left(\frac{kc+la}{k+l}, \frac{kd+lb}{k+l}\right)$ are
   **a.** Vertices of an equilateral triangle
   **b.** Vertices of an isosceles triangle
   **c.** Vertices of a right-angled triangle
   **d.** Collinear
8. The points $(-a, -b)$, $(a, b)$, $(a^2, ab)$ are
   **a.** Vertices of an equilateral triangle
   **b.** Vertices of a right-angled triangle
   **c.** Vertices of an isosceles triangle
   **d.** Collinear
9. Circumcentre of the triangle formed by the line $y = x$, $y = 2x$, and $y = 3x + 4$ is
   **a.** $(6, 8)$ **b.** $(6, -8)$
   **c.** $(3, 4)$ **d.** $(-3, -4)$
10. Find the area of the pentagon whose vertices are $A(1, 1)$, $B(7, 21)$, $C(7, -3)$, $D(12, 2)$, and $E(0, -3)$.
11. If the middle points of the sides of a triangle are $(-2, 3)$, $(4, -3)$, and $(4, 5)$, then find the centroid of the triangle.
12. The line joining $A(b\cos\alpha, b\sin\alpha)$ and $B(a\cos\beta, a\sin\beta)$ is produced to the point $M(x, y)$ so that $AM$ and $BM$ are in the ratio $b : a$. Then prove that $x + y\tan(\alpha+\beta/2) = 0$.
13. A point moves such that the area of the triangle formed by it with the points $(1, 5)$ and $(3, -7)$ is 21 sq. units. Then, find the locus of the point.
14. If $(1, 4)$ is the centroid of a triangle and the coordinates of its any two vertices are $(4, -8)$ and $(-9, 7)$, find the area of the triangle.
15. A triangle with vertices $(4, 0)$, $(-1, -1)$, $(3, 5)$ is
    **a.** isosceles and right-angled
    **b.** isosceles but not right-angled
    **c.** right-angled but not isosceles
    **d.** neither right-angled nor isosceles

## LOCUS AND EQUATION TO A LOCUS

### Locus

The curve described by a point which moves under given condition or conditions is called its locus. For example, suppose $C$ is a point in the plane of the paper and $P$ is a variable point in the plane of the paper such that its distance from $C$ is always equal to $a$ (say). It is clear that all the positions of the moving point $P$ lie on the circumference of a circle whose centre is $C$ and whose radius is $a$. The circumference of this circle is, therefore, the "Locus" of point $P$ when it moves under the condition that its distance from the point $C$ is always equal to constant $a$.

Let $A$ and $B$ be two fixed points in the plane of the paper, and $P$ be a variable point in the plane of the paper which moves in such a way that its distance from $A$ and $B$ is always same. Thus, the "locus" of $P$ is the perpendicular bisector of $AB$ when it moves under the condition that its distance from A and $B$ is always equal.

## Equation to Locus of a Point

The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.

**Steps to find locus of a points**

**Step I:** Assume the coordinates of the point say $(h, k)$ whose locus is to be found.

**Step II:** Write the given condition in mathematical form involving $h, k$.

**Step III:** Eliminate the variable(s), if any.

**Step IV:** Replace $h$ by $x$ and $k$ by $y$ in the result obtained in step III.

The equation so obtained is the locus of the point which moves under some stated condition(s).

**Example 1.26** **The sum of the squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.**

**Sol.** Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$, $B(-a, 0)$ be the given points.

Then, we have

$$PA^2 + PB^2 = 2c^2 \text{ (given)}$$

$$\Rightarrow (h-a)^2 + (k-0)^2 + (h+a)^2 + (k-0)^2 = 2c^2$$

$$2h^2 + 2k^2 + 2a^2 = 2c^2$$

$$\Rightarrow \quad h^2 + k^2 = c^2 - a^2$$

Hence, equation to locus $(h, k)$ is $x^2 + y^2 = c^2 - a^2$.

**Example 1.27** **Find the locus of a point, so that the join of $(-5, 1)$ and $(3, 2)$ subtends a right angle at the moving point.**

**Sol.** Let $P(h, k)$ be a moving point and let $A(-5, 1)$ and $B(3, 2)$ be given points.

By the given condition, we have

$$\angle APB = 90°$$

$\Rightarrow \Delta APB$ is a right-angled triangle

$$\Rightarrow \quad AB^2 = AP^2 + PB^2$$

$$\Rightarrow (3+5)^2 + (2-1)^2 = (h+5)^2 + (k-1)^2 + (h-3)^2 + (k-2)^2$$

$$\Rightarrow \quad 65 = 2(h^2 + k^2 + 2h - 3k) + 39$$

$$\Rightarrow \quad h^2 + k^2 + 2h - 3k - 13 = 0$$

Hence, locus of $(h, k)$ is $x^2 + y^2 + 2x - 3y - 13 = 0$.

**Example 1.28** **A rod of length $l$ slides with its ends on two perpendicular lines. Find the locus of its midpoint.**

**Sol.** Let the two perpendicular lines be the coordinates axes.

Let $AB$ be a rod of length $l$.

Let the coordinates of $A$ and $B$ be $(a, 0)$ and $(0, b)$, respectively.

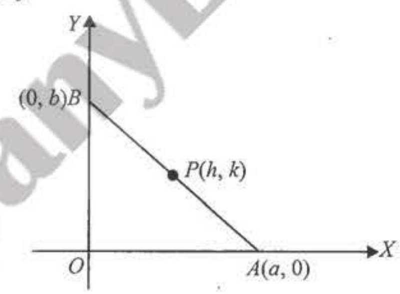



**Fig. 1.21**

As the rod slides, the values of $a$ and $b$ change.

So $a$ and $b$ are two variables.

Let $P(h, k)$ be the midpoint of the rod $AB$ in one of the infinite positions it attains.

Then, $$h = \frac{a+0}{2} \text{ and } k = \frac{0+b}{2}$$

$$\Rightarrow \quad h = \frac{a}{2} \text{ and } k = \frac{b}{2} \quad \text{(i)}$$

From $\Delta OAB$, we have

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad a^2 + b^2 = l^2$$

$$\Rightarrow \quad (2h)^2 + (2k)^2 = l^2 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 4h^2 + 4k^2 = l^2$$

Hence, the locus of $(h, k)$ is $4x^2 + 4y^2 = l^2$

**Example 1.29** ***AB* is a variable line sliding between the coordinate axes in such a way that $A$ lies on $x$-axis and $B$ lies on $y$-axis. If $P$ is a variable point on $AB$ such that $PA = b$, $PB = a$, and $AB = a + b$, find the equation of the locus of $P$.**

**Sol.** Let $P(h, k)$ be a variable point on $AB$ such that $\angle OAB = \theta$.

Here $\theta$ is a variable.

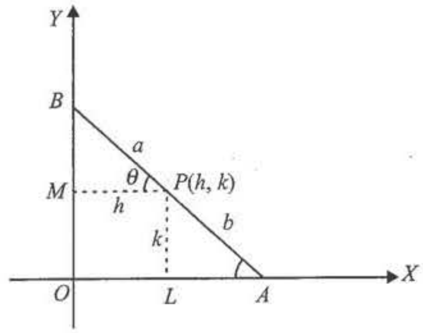



**Fig. 1.22**

From triangles $ALP$ and $PMB$, we have

$$\sin\theta = \frac{k}{b} \qquad \text{(i)}$$

$$\cos\theta = \frac{h}{a} \qquad \text{(ii)}$$

Here $\theta$ is a variable. So, we have to eliminate $\theta$.

Squaring (i) and (ii) and adding, we get

$$\frac{k^2}{b^2} + \frac{h^2}{a^2} = 1$$

Hence, the locus of $(h, k)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

**Example 1.30** **Two points $P$ and $Q$ are given, $R$ is a variable point on one side of the line $PQ$ such that $\angle RPQ - \angle RQP$ is a positive constant $2\alpha$. Find the locus of the point $R$.**

**Sol.** Let the $x$-axis along $QP$ and the middle point of $PQ$ be origin and let $R \equiv (x_1, y_1)$.

Let $OP = OQ = a$ and $\angle RPM = \theta$ and $\angle RQM = \phi$

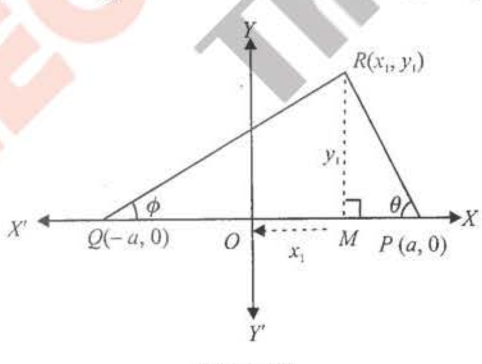



**Fig. 1.23**

In $\Delta RMP$,

$$\tan\theta = \frac{RM}{MP} = \frac{y_1}{a - x_1} \qquad \text{(i)}$$

In $\Delta RQM$,

$$\tan\phi = \frac{RM}{QM} = \frac{y_1}{a + x_1} \qquad \text{(ii)}$$

But given $\angle RPQ - \angle RQP = 2\alpha$ (constant)

$$\Rightarrow \qquad \theta - \phi = 2\alpha$$

$$\Rightarrow \qquad \tan(\theta - \phi) = \tan 2\alpha$$

$$\Rightarrow \qquad \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan 2\alpha$$

$$\Rightarrow \qquad \frac{\dfrac{y_1}{a - x_1} - \dfrac{y_1}{a + x_1}}{1 + \dfrac{y_1}{a - x_1}\dfrac{y_1}{a + x_1}} = \tan 2\alpha$$

$$\Rightarrow \qquad \frac{2x_1y_1}{a^2 - x_1^2 + y_1^2} = \tan 2\alpha$$

$\Rightarrow a^2 - x_1^2 + y_1^2 = 2x_1y_1 \cot 2\alpha$ or $x_1^2 - y_1^2 + 2x_1y_1 \cot 2\alpha = a^2$

Hence, locus of the point $R(x_1, y_1)$ is $x^2 - y^2 + 2xy \cot 2\alpha = a^2$.

**Example 1.31** **If the coordinates of a variable point $P$ is $(a\cos\theta, b\sin\theta)$, where $\theta$ is a variable quantity, then find the locus of $P$.**

**Sol.** Let $P \equiv (x, y)$. According to the question

$$x = a\cos\theta \qquad \text{(i)}$$

$$y = b\sin\theta \qquad \text{(ii)}$$

Squaring and adding (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

**Example 1.32** **Find the locus of a point such that the sum of its distance from the points (0, 2) and (0, – 2) is 6.**

**Sol.** Let $P(h, k)$ be any point on the locus and let $A(0, 2)$ and $B(0, -2)$ be the given points.

By the given condition, we get

$$PA + PB = 6$$

$$\Rightarrow \sqrt{(h-0)^2+(k-2)^2}+\sqrt{(h-0)^2+(k+2)^2}=6$$

$$\Rightarrow \quad \sqrt{h^2+(k-2)^2}=6-\sqrt{(h-0)^2+(k+2)^2}$$

$$\Rightarrow \quad h^2+(k-2)^2=36-12\sqrt{h^2+(k+2)^2}+h^2+(k+2)^2$$

$$\Rightarrow \quad -8k-36=-12\sqrt{h^2+(k+2)^2}$$

$$\Rightarrow \quad (2k+9)=3\sqrt{h^2+(k+2)^2}$$

$$\Rightarrow \quad (2k+9)^2=9(h^2+(k+2)^2)$$

$$\Rightarrow \quad 4k^2+36k+81=9h^2+9k^2+36k+36$$

$$\Rightarrow \quad 9h^2+5k^2=45$$

Hence, locus of $(h, k)$ is $9x^2+5y^2=45$.

### Concept Application Exercise 1.2

1. Find the locus of a point whose distance from $(a, 0)$ is equal to its distance from $y$-axis.
2. The coordinates of the points $A$ and $B$ are $(a, 0)$ and $(-a, 0)$, respectively. If a point $P$ moves so that $PA^2 - PB^2 = 2k^2$, when $k$ is constant, then find the equation to the locus of the point $P$.
3. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a $\Delta ABC$, then as $\alpha$ varies, then find the locus of its centroid.
4. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a triangle $ABC$. If the centroid of the triangle moves on the line $2x+3y=1$, then find the locus of the vertex $C$.
5. $Q$ is a variable point whose locus is $2x+3y+4=0$; corresponding to a particular position of $Q$, $P$ is the point of section of $OQ$, $O$ being the origin, such that $OP : PQ = 3 : 1$. Find the locus of $P$?
6. Find the locus of the middle point of the portion of the line $x\cos\alpha + y\sin\alpha = p$ which is intercepted between the axes, given that $p$ remains constant.
7. Find the locus of the point of intersection of lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$ ($\alpha$ is a variable).

## SHIFTING OF ORIGIN

Let $O$ be the origin and let $X'OX$ and $Y'OY$ be the axis of $x$ and $y$, respectively. Let $O'$ and $P$ be two points in the plane having coordinates $(h, k)$ and $(x, y)$, respectively referred to $X'OX$ and $Y'OY$ as the coordinates axes. Let the origin be transferred to $O'$ and let $X'O'X$ and $Y'O'Y$ be new rectangular axes. Let the coordinates of $P$ referred to new axes as the coordinates axes be $(X, Y)$.

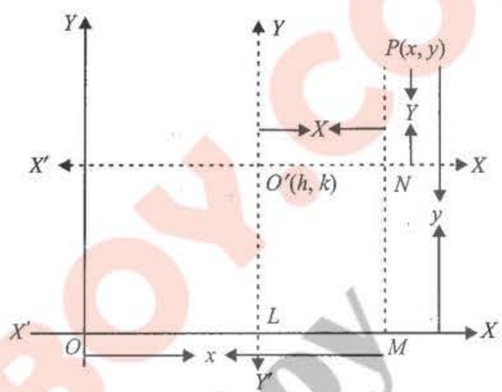



**Fig. 1.24**

Then,

$O'N = X$, $PN = Y$, $OM = x$, $PM = y$, $OL = h$, and $O'L = k$

Now,

$x = OM = OL + LM = OL + O'N = h + X$

and

$y = PM = PN + NM = PN + O'L = Y + k$

$\Rightarrow x = X + h$ and $y = Y + k$

Thus, if $(x, y)$ are coordinates of a point referred to old axes and $(X, Y)$ are the coordinates of the same point referred to new axes, then $x = X + h$ and $y = Y + k$. Therefore, the origin is shifted at a point $(h, k)$, we must substitute $X + h$ and $Y + k$ for $x$ and $y$, respectively.

The transformation formula from new axes to old axes is

$X = x - h$, $Y = y - k$

The coordinates of the old origin referred to the new axes are $(-h, -k)$.

**Example 1.33** **If the origin is shifted to the point $(1, -2)$ without rotation of axes what do the following equations become?**

**i. $2x^2 + y^2 - 4x + 4y = 0$ and**

**ii. $y^2 - 4x + 4y + 8 = 0$.**

**Sol. i.** Substituting $x = X + 1$, $y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$, we get

$2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0$

or $2X^2 + Y^2 = 6$

**ii.** Substituting $x = X + 1$, $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get

$$(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0$$

or $Y^2 = 4X$

**Example 1.34** **At what point the origin be shifted, if the coordinates of a point (4, 5) become (– 3, 9)?**

**Sol.** Let $(h, k)$ be the point to which the origin is shifted. Then,

$$x = 4, y = 5, X = -3, Y = 9$$

$\therefore$ $x = X + h$ and $y = Y + k$

$\Rightarrow$ $4 = -3 + h$ and $5 = 9 + k$

$\Rightarrow$ $h = 7$ and $k = -4$

Hence, the origin must be shifted to $(7, -4)$.

**Example 1.35** **Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in $y$ and the constant term.**

**Sol.** Let the origin be shifted to $(h, k)$. Then,

$$x = X + h \text{ and } y = Y + k$$

Substituting $x = X + h$, $y = Y + k$ in the equation $y^2 + 4y + 8x - 2 = 0$, we get

$$(Y + k)^2 + 4(Y + k) + 8(X + h) - 2 = 0$$

$\Rightarrow$ $Y^2 + (4 + 2k)Y + 8X + (k^2 + 4k + 8h - 2) = 0$

For this equation to be free from the term containing $Y$ and the constant term, we must have

$$4 + 2k = 0 \text{ and } k^2 + 4k + 8h - 2 = 0$$

$\Rightarrow$ $k = -2$ and $h = 3/4$

Hence, the origin is shifted at the point $(3/4, -2)$.

**Example 1.36** **The equation of a curve referred to new axes, axes retaining their directions, and origin (4, 5) is $X^2 + Y^2 = 36$. Find the equation referred to the original axes.**

**Sol.** With the above notation, we have

$$x = X + 4, y = Y + 5$$

$\Rightarrow$ $X = x - 4$, $Y = y - 5$

$\therefore$ The required equation is

$$(x - 4)^2 + (y - 5)^2 = 36$$

$\Rightarrow$ $x^2 + y^2 - 8x - 10y + 5 = 0$ which is equation referred to the original axes.

**Example 1.37** **Find the equation to which the equation**

**$x^2 + 7xy - 2y^2 + 17x - 26y - 60 = 0$**

**is transformed if the origin is shifted to the point (2, – 3), the axes remaining parallel to the original axis.**

**Sol.** Here the new origin is (2, - 3).

Then, $x = X + 2$, $y = Y - 3$.

and the given equation transforms to

$(X + 2)^2 + 7(X + 2)(Y - 3) - 2(Y - 3)^2 + 17(X + 2) - 26(Y - 3) - 60 = 0$

$\Rightarrow$ $X^2 + 7XY - 2Y^2 - 4 = 0$

## ROTATION OF AXIS

### Rotation of Axes without Changing the Origin

Let $O$ be the origin. Let $P \equiv (x, y)$ with respect to axes $OX$ and $OY$ and let $P \equiv (x', y')$ with respect to axes $OX'$ and $OY'$ where $\angle X'OX = \angle YOY' = \theta$

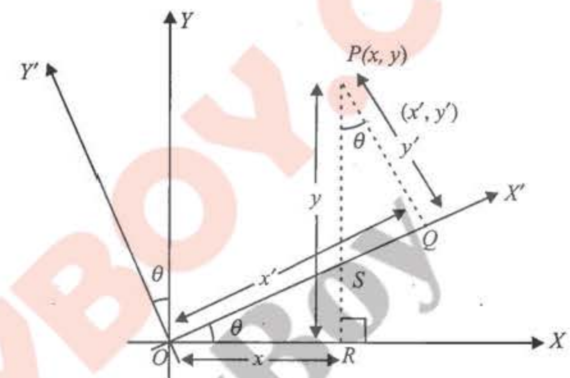



**Fig. 1.25**

In Fig. 1.25, we have

$$SR = x \tan\theta, OS = x \sec\theta,$$
$$PS = y - x \tan\theta$$

Now in triangle $PQS$,

$$\sin\theta = \frac{SQ}{PS} = \frac{x' - x\sec\theta}{y - x\tan\theta}$$

$\Rightarrow$ $x' = y\sin\theta - x\dfrac{\sin^2\theta}{\cos\theta} + \dfrac{x}{\cos\theta}$

$$= y\sin\theta + x\frac{1 - \sin^2\theta}{\cos\theta}$$

$\Rightarrow$ $x' = x\cos\theta + y\sin\theta$

Also $\cos\theta = \dfrac{PQ}{PS} = \dfrac{y'}{y - x\tan\theta}$

$\Rightarrow$ $y' = -x\sin\theta + y\cos\theta$

$\Rightarrow$ $x = x'\cos\theta - y'\sin\theta,$

$y = x'\sin\theta + y'\cos\theta$

| | $x$ | $y$ |
|---|---|---|
| $x'$ | $\cos\theta$ | $\sin\theta$ |
| $y'$ | $-\sin\theta$ | $\cos\theta$ |

**Note:**

*Compare real and imaginary parts of the equation $(x + iy) = (x' + iy')(\cos\theta + i\sin\theta)$ to remember the formula*

**Example 1.38** **The equation of a curve referred to a given system of axes is $3x^2 + 2xy + 3y^2 = 10$. Find its equation if the axes are rotated through an angle 45°, the origin remaining unchanged.**

**Sol.** With the above notation, we have

$$x = x' \cos 45° - y' \sin 45° = \frac{x' - y'}{\sqrt{2}}$$

and $y = x' \sin 45° + y' \cos 45° = \dfrac{x' + y'}{\sqrt{2}}$

Thus, $3x^2 + 2xy + 3y^2 = 10$ transforms to

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + 3\left(\frac{x'+y'}{\sqrt{2}}\right)^2 = 10$$

$\Rightarrow \; 2x'^2 + y'^2 = 5.$

## Removal of the term $xy$, from $f(x, y) = ax^2 + 2hxy + by^2$ without changing the origin

Clearly, $h \neq 0$.

Rotating the axes through an angle $\theta$, we have

$x = x' \cos \theta - y' \sin \theta$

$y = x' \sin \theta + y' \cos \theta.$

$\therefore f(x, y) = ax^2 + 2hxy + by^2$

$= a(x' \cos \theta - y' \sin \theta)^2 + 2h(x' \cos \theta - \theta y' \sin \theta).(x' \sin \theta + y' \cos \theta) + b(x' \sin \theta + y' \cos \theta)^2$

$= (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) x'^2 + 2[(b - a)\cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta)] x'y' + (a \sin^2 \theta - 2h \cos \theta \sin \theta + b \cos^2 \theta) y'^2$

$= F(x', y')$, (*say*).

In $F(x', y')$, we require that the coefficient of the $XY$-term to be zero.

$\therefore 2[(b - a) \cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta)] = 0.$

$\Rightarrow (a - b) \sin 2\theta = 2h \cos 2\theta.$

$\Rightarrow \tan 2\theta = \dfrac{2h}{a-b}$ or $\cot 2\theta = \dfrac{a-b}{2h}$

We use the former or the later equation according as $a \neq b$ or $a = b$. These yield $\theta$, the angle through which the axes are to be rotated (the origin remaining unchanged) in order to remove the $xy$-term from $f(x, y)$.

**Example 1.39** **Given the equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 1$. Through what angle should the axes be rotated so that the term $xy$ is removed from the transformed equation.**

**Sol.** Comparing the given equation, with

$ax^2 + 2hxy + by^2$, we get $a = 4$, $h = \sqrt{3}$, $b = 2$.

Let $\theta$ be the angle through which the axes are to be rotated.

Then $\quad \tan 2\theta = \dfrac{2h}{a-b}$

$\Rightarrow \quad \tan 2\theta = \dfrac{2\sqrt{3}}{4-2} = \sqrt{3} = \tan \dfrac{\pi}{3}$

$\Rightarrow \quad 2\theta = \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$

$\Rightarrow \quad \theta = \dfrac{\pi}{6}, \dfrac{2\pi}{3}$

## Change of Origin and Rotation of Axes

If origin is changed to $O'(\alpha, \beta)$ and axes are rotated about the new origin $O'$ by an angle $\theta$ in the anticlockwise sense such that the new coordiantes of $P(x, y)$ becomes $(x', y')$, then the equations of transformation will be

$$x = \alpha + x' \cos \theta - y' \sin \theta$$

and

$$y = \beta + x' \sin \theta + y' \cos \theta$$

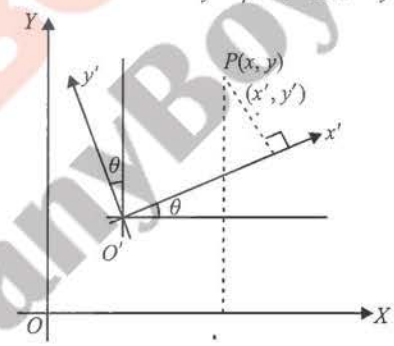



**Fig. 1.26**

**Example 1.40** **What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to rectangular axes through the point $(-2, -3)$, the new axes being inclined at an angle of 45° with the old axes?**

**Sol.** Let $O'$ be $(-2, -3)$. Since the axes are rotated about $O'$ by an angle 45° in anticlockwise direction, let $(x', y')$ be the new coordinates with respect to new axes and $(x, y)$ be the coordinates with respect to old axes. Then, we have

$$x = -2 + x' \cos 45° - y' \sin 45° = -2 + \left(\frac{x'-y'}{\sqrt{2}}\right)$$

$$y = -3 + x' \sin 45° + y' \cos 45° = -3 + \left(\frac{x'+y'}{\sqrt{2}}\right)$$

The new equation will be

$$2\left\{-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right\}^2 + 4\left(-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right)\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right)$$

$$-5\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right)^2 + 20\left(-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right)$$

$$-22\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right) - 14 = 0$$

$$\Rightarrow \quad x'^2 - 14x'y' - 7y'^2 - 2 = 0$$

Hence, new equation of curve is

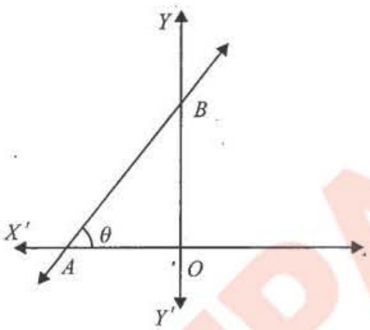
$$x^2 - 14xy - 7y^2 - 2 = 0$$

## STRAIGHT LINE

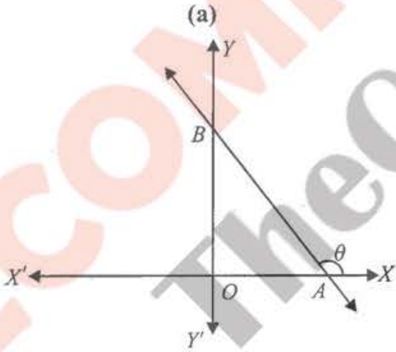
A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

### Slope (Gradient) of a Line

The trigonometrical tangent of an angle that a line makes with the positive direction of the $x$-axis in anticlockwise sense is called the slope or gradient of the line.




(a)




(b)

Fig. 1.27

**Note:**

- *The slope of a line is generally denoted by m. Thus,* $m = \tan\theta$.
- *Since a line parallel to x-axis makes an angle of* $0°$ *with x-axis; therefore, its slope is* $\tan 0° = 0$.
- A line parallel to $y$-axis, i.e., perpendicular to $x$-axis makes an angle of 90° with $x$-axis, so its slope is $\tan \pi/2 = \infty$.
- *The slope of a line equally inclined with the axis is 1 or – 1, as it makes 45° or 135° angle with x-axis. Slope of a line in terms of coordinates of any two points on it is given as shown below:*

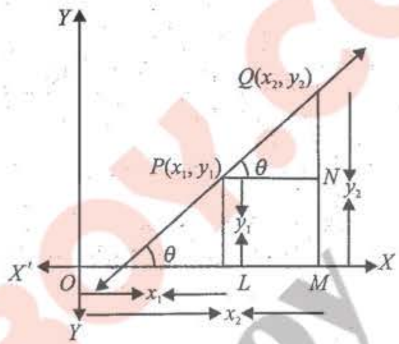



Fig. 1.28

*From the figure, slope is*

$$\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\textit{Difference of ordinates}}{\textit{Difference of abscissae}}$$

### Angle between Two Lines

The angle $\theta$ between the lines having slope $m_1$ and $m_2$ is given by

$$\tan\theta = \pm\ \frac{m_2 - m_1}{1 + m_1 m_2}$$

**Proof:**

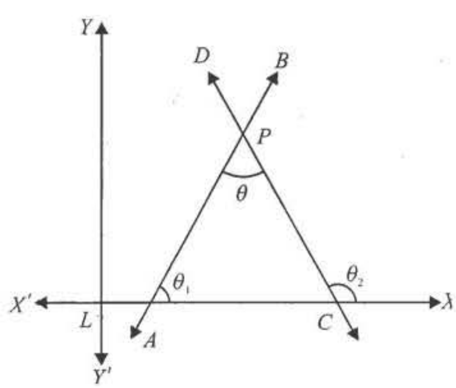



Fig. 1.29

Let $m_1$ and $m_2$ be the slopes of two given lines $AB$ and $CD$ which intersect at a point $P$ and make angles $\theta_1$ and $\theta_2$, respectively with the positive direction of $x$-axis.

Then,

$$m_1 = \tan\theta_1 \text{ and } m_2 = \tan\theta_2$$

Let $\angle APC = \theta$ be the angle between the given lines.

Then,

$$\theta_2 = \theta + \theta_1$$

$$\Rightarrow \quad \theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}.$$

$$\Rightarrow \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{(i)}$$

Since $\angle APD = \pi - \theta$ is also the angle between $AB$ and $CD$.

Therefore,

$$\tan\angle APD = \tan(\pi - \theta) = -\tan\theta$$

$$= -\frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{(ii) [Using (i)]}$$

From (i) and (ii), we find that the angle between two lines of slopes $m_1$ and $m_2$ is given by

$$\tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\pm\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

The acute angle between the lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

**Example 1.41** **If $A(-2, 1)$, $B(2, 3)$, and $C(-2, -4)$ are three points, then find the angle between $BA$ and $BC$.**

**Sol.** Let $m_1$ and $m_2$ be the slopes of $BA$ and $BC$, respectively. Then,

$$m_1 = \frac{3 - 1}{2 - (-2)} = \frac{2}{4} = \frac{1}{2}$$

and $$m_2 = \frac{-4 - 3}{-2 - 2} = \frac{7}{4}$$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right|$$

$$= \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right| = \frac{2}{3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

**Example 1.42** **Determine $x$ so that the line passing through (3, 4) and ($x$, 5) makes 135° angle with the positive direction of $x$-axis**

**Sol.** Since the line passing through (3, 4) and ($x$, 5) makes an angle of 135° with $x$-axis; therefore, its slope is

$$\tan 135° = -1.$$

But, the slope of the line is also equal to

$$\frac{5 - 4}{x - 3}$$

$$\Rightarrow \quad -1 = \frac{5 - 4}{x - 3}$$

$$\Rightarrow \quad -x + 3 = 1$$

$$\Rightarrow \quad x = 2$$

## Condition for Parallelism of Lines

It two lines of slopes $m_1$ and $m_2$ are parallel, then the angle $\theta$ between is 0°.

$$\therefore \quad \tan\theta = \tan 0° = 0$$

$$\Rightarrow \quad \frac{m_2 - m_1}{1 + m_1 m_2} = 0$$

$$\Rightarrow \quad m_2 = m_1$$

Thus, when two lines are parallel, their slopes are equal.

## Condition for Perpendicularity of Two Lines

If two lines of slopes $m_1$ and $m_2$ are perpendicular, then the angle $\theta$ between them is 90°.

$$\therefore \quad \cot\theta = 0$$

$$\Rightarrow \quad \frac{1 + m_1 \cdot m_2}{m_2 - m_1} = 0$$

$$\Rightarrow \quad m_1 m_2 = -1$$

Thus, when lines are perpendicular, the product of their slope is $-1$.

If $m$ is the slope of a line, then the slope of a line perpendicular to it is $-(1/m)$.

**Example 1.43** **Let $A(6, 4)$ and $B(2, 12)$ be two given points. Find the slope of a line perpendicular to $AB$.**

**Sol.** Let $m$ be the slope of $AB$, than

$$m = \frac{12 - 4}{2 - 6} = \frac{8}{-4} = -2$$

So, the slope of a line $\perp$ to $AB$

$$= -\frac{1}{m} = \frac{1}{2}$$

### Intercepts of a Line on the Axes

If a straight line cuts $x$-axis at $A$ and the $y$ – axis at $B$ then $OA$ and $OB$ are known as the intercepts of the line on $x$-axis and $y$-axis respectively.

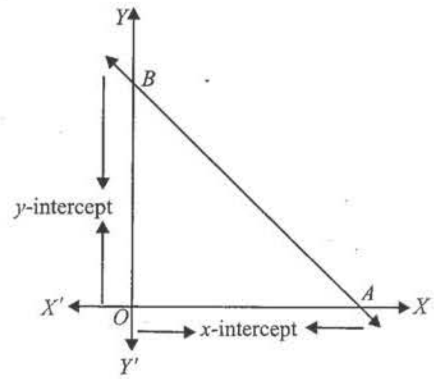



**Fig. 1.30**

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinates axes.

In figure $OA$ = $x$-intercept, $OB$ = $y$-intercept

$OA$ is positive or negative according as $A$ lies on $OX$ or $OX'$ respectively.

Similarly $OB$ is positive or negative according as $B$ lies on $OY$ or $OY'$ respectively.

**Note:**

- *If line has equal intercept on axes, then its slope is* $-1$.

### Equation of a Line Parallel to $x$-Axis

Equation of a line parallel to $x$-axis at a distance $b$ from it.

Then, clearly the ordinates of each point on $AB$ is $b$.

Thus, $AB$ can be considered as the locus of a point at a distance $b$ from $x$-axis.

Thus, if $P(x, y)$ is any point on $AB$, then $y = b$.

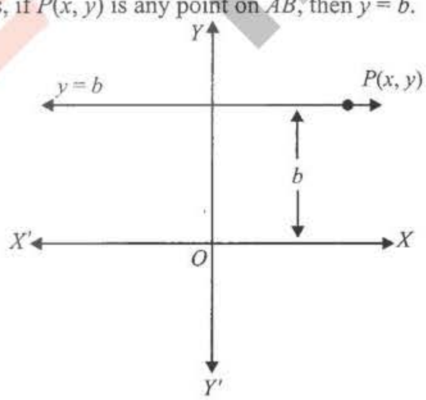



**Fig. 1.31**

Hence, the equation of a line parallel to $x$-axis at a distance $b$ from it is $y = b$.

Since $x$-axis is parallel to itself at a distance 0 from it; therefore, the equation of $x$-axis is $y = 0$.

### Equation of a Line Parallel to $y$-Axis

Let $AB$ be a line parallel to $y$-axis and at a distance $a$ from it. Then the abscissa of every point on $AB$ is $a$. So it can be treated as the locus of a point at a distance $a$ from $y$-axis.

Thus, if $P(x, y)$ is any point in $AB$, then $x = a$.

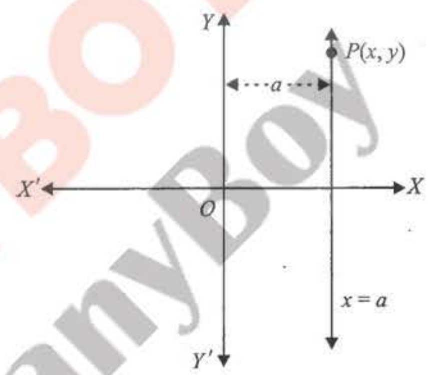



**Fig. 1.32**

## DIFFERENT FORMS OF LINE

### Slope Intercept Form of a Line

The equation of a line with slope $m$ that makes an intercept $c$ on $y$-axis is

$$y = mx + c$$

**Proof:** Let the given line intersects $y$-axis at $Q$ and makes an angle $\theta$ with $x$-axis. Then $m = \tan\theta$. Let $P(x, y)$ be any point on the line as shown in the figure.

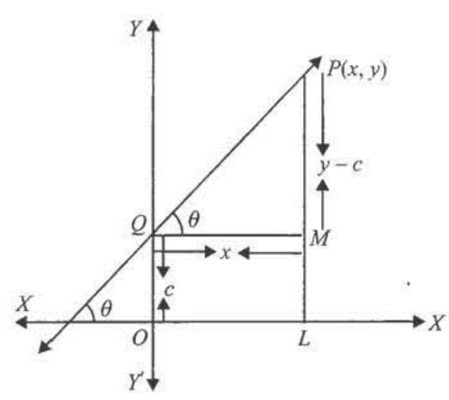



**Fig. 1.33**

From $\Delta PMQ$, we have

$$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$$

$$\Rightarrow \quad m = \frac{y-c}{x}$$

$$\Rightarrow \quad y = mx + c$$

which is the required equation of the line.

## Point–Slope Form of a Line

The equation of a line which passes through the point $(x_1, y_1)$ and has the slope '$m$' is

$$y - y_1 = m(x - x_1)$$

**Proof:** Let $Q(x_1, y_1)$ be the point through which the line passes and let $P(x, y)$ be any point on the line. Then, the slope of the line is

$$\frac{y-y_1}{x-x_1}$$

But $m$ is the slope of the line. Therefore,

$$m = \frac{y-y_1}{x-x_1} \Rightarrow y - y_1 = m(x - x_1)$$

Thus, $y - y_1 = m(x - x_1)$ is the required equation of the line.

**Example 1.44** **Find the equation of a straight line which cuts-off an intercept of 5 units on negative direction of $y$-axis and makes an angle of 120° with the positive direction of $x$-axis.**

**Sol.** Here, $m = \tan 120° = \tan(90 + 30°) = -\cot 30° = -\sqrt{3}$ and $c = -5$. So, the equation of the line is

$$y = -\sqrt{3}\,x - 5 \Rightarrow \sqrt{3}\,x + y + 5 = 0$$

**Example 1.45** **Find the equation of a straight line cutting off an intercept $-1$ from $y$-axis and being equally inclined to the axes.**

**Sol.** Since the required line is equally inclined with coordinates axes; therefore, it makes an angle of either 45° or 135° with the $x$-axes.

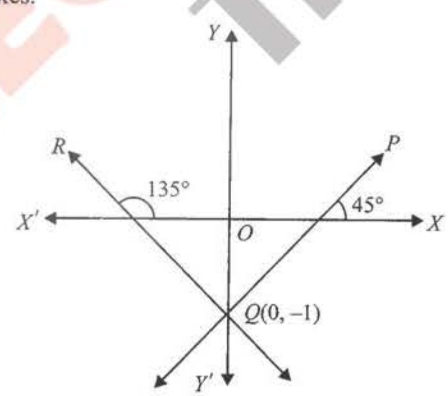



**Fig. 1.34**

So, its slope is either $m = \tan 45°$ or $m = \tan 135°$, i.e., $m = 1$ or $-1$. It is given that $c = -1$. Hence, the equations of the lines are

$$y = x - 1 \text{ and } y = -x - 1$$

**Example 1.46** **Find the equation of a line that has $y$- intercept 4 and is perpendicular to the line joining $(2, -3)$ and $(4, 2)$.**

**Sol.** Let $m$ be the slope of the required line.

Since the required line is perpendicular to the line joining $A(2, -3)$ and $B(4, 2)$. Therefore,

$$m \times \text{slope of } AB = -1$$

$$\Rightarrow \quad m \times \frac{2+3}{4-2} = -1$$

$$\Rightarrow \quad m = -\frac{2}{5}$$

The required line cuts-off an intercept 4 on $y$-axis, so $c = 4$.

Hence, the equation of the required line is

$$y = -\frac{2}{5}x + 4$$

$$\Rightarrow \quad 2x + 5y - 20 = 0$$

## Two-Point Form of a Line

The equation of a line passing through two points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

**Proof:**

Let $m$ be the slope of the passing through $(x_1, y_1)$ and $(x_2, y_2)$, then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m(x - x_1) \text{ (Using point – slope form)}$$

Substituting the value of $m$, we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is the required equation of the line in two-point form.

**Example 1.47** **Find the equation of the perpendicular bisector of the line segment joining the points $A(2, 3)$ and $B(6, -5)$.**

**Sol.** The slope of $AB$ is given by $m$

$$= \frac{-5-3}{6-2} = -2$$

$\Longrightarrow$ The slope of a line $\perp$ to $AB$

$$= -\frac{1}{m} = \frac{1}{2}$$

Let $P$ be the midpoint of $AB$, then the coordinates of $P$ are

$$\left(\frac{2+6}{2}, \frac{3-5}{2}\right), \text{ i.e., } (4, -1)$$

Thus, the required line passes through $P(4, -1)$ and has slope 1/2.

So its equation is

$$y + 1 = \frac{1}{2}(x-4) \; [(\text{Using } y - y_1 = m(x - x_1))]$$

or $\quad x - 2y - 6 = 0$

**Example 1.48** **Find the equations of the medians of the triangle $ABC$ whose vertices are $A(2, 5)$, $B(-4, 9)$, and $C(-2, -1)$.**

**Sol.** Let $D$, $E$, $F$ be the midpoints of $BC$, $CA$ and $AB$, respectively. Then the coordinates of these points are $D(-3, 4)$, $E(0, 2)$, and $F(-1, 7)$, respectively.

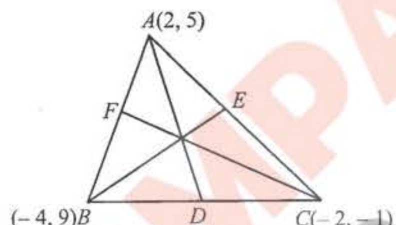



**Fig. 1.35**

The median $AD$ passes through points $A(2, 5)$ and $D(-3, 4)$.

So, the equation of $AD$ is

$$y - 5 = \frac{4-5}{-3-2}(x-2)$$

$\Rightarrow \quad y - 5 = \frac{1}{5}(x-2)$

$\Rightarrow \quad x - 5y + 23 = 0$

The median $BE$ passes through points $B(-4, 9)$ and $E(0, 2)$

So, the equation of median $BE$ is

$$(y - 9) = \left(\frac{2-9}{0+4}\right)(x+4)$$

$\Rightarrow \quad 7x + 4y - 8 = 0$

Similarly, the equation of the median $CF$ is

$$(y + 1) = \frac{7+1}{-1+2}(x+2)$$

$\Rightarrow \quad 8x - y + 15 = 0$

**Example 1.49** **In what ratio does the line joining the points (2, 3) and (4, 1) divide the segment joining the points (1, 2) and (4, 3)?**

**Sol.** The equation of the line joining the points (2, 3) and (4, 1) is

$$y - 3 = \frac{1-3}{4-2}(x-2)$$

$\Rightarrow \quad y - 3 = -x + 2$

$\Rightarrow \quad x + y - 5 = 0 \qquad \text{(i)}$

Suppose the line joining (2, 3) and (4, 1) divides the segment joining (1, 2) and (4, 3) at point $P$ in the ratio $\lambda : 1$.

Then the coordinates of $P$ are

$$\left(\frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+2}{\lambda+1}\right)$$

Clearly, $P$ lies on (i)

$$\Rightarrow \frac{4\lambda+1}{\lambda+1} + \frac{3\lambda+2}{\lambda+1} - 5 = 0$$

$\Rightarrow \quad \lambda = 1$

Hence, the required ratio is $\lambda : 1$, i.e., $1 : 1$.

**Example 1.50** **Find the equations of the altitudes of the triangle whose vertices are $A(7, -1)$, $B(-2, 8)$, and $C(1, 2)$ and hence orthocentre of triangle.**

**Sol.**

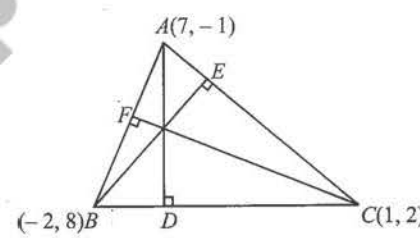



**Fig. 1.36**

Let $AD$, $BE$, and $CF$ be three altitudes of triangle $ABC$.

Let $m_1$, $m_2$, and $m_3$ be the slopes of $AD$, $BE$, and $CF$, respectively.

Then, $AD \perp BC$

$\Rightarrow$ Slope of $AD \times$ Slope of $BC = -1$

$\Rightarrow \quad m_1 \times \left(\frac{2-8}{1+2}\right) = -1$

$\Rightarrow \quad m_1 = \frac{1}{2}$

$BE \perp AC$

$\Rightarrow$ Slope of $BE \times$ Slope of $AC = -1$

$\Rightarrow \quad m_2 \times \left(\frac{-1-2}{7-1}\right) = -1$

$\Rightarrow \quad m_2 = 2$

And, $CF \perp AB$

$\Rightarrow$ Slope of $CF$ × Slope of $AB = -1$

$$\Rightarrow \quad m_3 \times \frac{-1-8}{7+2} = -1$$

$$\Rightarrow \quad m_3 = 1$$

Since $AD$ passes through $A(7, -1)$ and has slope

$$m_1 = 1/2.$$

So, its equation is

$$y + 1 = \frac{1}{2}(x - 7)$$

$$\Rightarrow \quad x - 2y - 9 = 0$$

Similarly, equation of $BE$ is

$$y - 8 = 2(x + 2)$$

$$\Rightarrow \quad 2x - y + 12 = 0$$

Equation of $CF$ is

$$y - 2 = 1\,(x - 1)$$

$$\Rightarrow \quad x - y + 1 = 0$$

## Intercept Form of a Line

The equation of a line which cuts-off intercepts $a$ and $b$, respectively from the $x$ and $y$-axes is

$$\frac{x}{a} + \frac{y}{b} = 1$$

**Proof:** Let $AB$ be the line which cuts-off intercepts $OA = a$ and $OB = b$ on the $x$ and $y$-axes respectively. Let $P(x, y)$ be any point on the line.

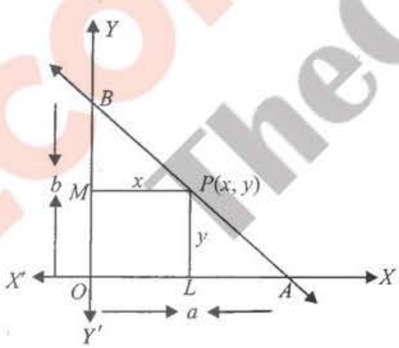



**Fig. 1.37**

From the diagram, we get that

Area of $\Delta OAB$ = Area of $\Delta OPA$ + Area of $\Delta OPB$

$$\Rightarrow \quad \frac{1}{2} OA \times OB = \frac{1}{2} OA \times PL + \frac{1}{2} OB \times PM$$

$$\Rightarrow \quad \frac{1}{2} ab = \frac{1}{2} ay + \frac{1}{2} bx$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is the equation of the line in the intercept form.

**Example 1.51** **Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.**

**Sol.** Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(i)}$$

This passes through (3, 4), therefore

$$\frac{3}{a} + \frac{4}{b} = 1 \qquad \text{(ii)}$$

It is given that $a + b = 14$

$$\Rightarrow \quad b = 14 - a$$

Putting $b = 14 - a$ in (ii), we get

$$\frac{3}{a} + \frac{4}{14 - a} = 1$$

$$\Rightarrow \quad a^2 - 13a + 42 = 0$$

$$\Rightarrow \quad (a - 7)(a - 6) = 0$$

$$\Rightarrow \quad a = 7, 6$$

For $a = 7$, $b = 14 - 7 = 7$

and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of $a$ and $b$ in (i), we get the equations of the lines

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{x}{6} + \frac{y}{8} = 1$$

or $\quad x + y = 7$ and $4x + 3y = 24$

**Example 1.52** **Find the equation of the straight line that**

**i. makes equal intercepts on the axes and passes through the point (2, 3),**

**ii. passes through the point (−5, 4) and is such that the portion intercepted between the axes is divided by the point in the ratio 1 : 2.**

**Sol. i.** Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Since it makes equal intercepts on the coordinates axes, therefore $a = b$.

So, the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ or } x + y = a$$

The line passes through the point (2, 3).

Therefore, $\quad 2 + 3 = a$

$$\Rightarrow \quad a = 5$$

Thus, the equation of the required line is $x + y = 5$.

**ii.** Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Clearly, this line meets the coordinate axes at $A(a, 0)$ and $B(0, b)$, respectively.

The coordinates of the point that divides the line joining $A(a, 0)$ and $B(0, b)$ in the ratio 1 : 2 are

$$\left(\frac{1(0)+2(a)}{1+2}, \frac{1(b)+2(0)}{1+2}\right) = \left(\frac{2a}{3}, \frac{b}{3}\right)$$

It is given that the point $(-5, 4)$ divides $AB$ in the ratio 1 : 2.

Therefore, $2a/3 = -5$ and $b/3 = 4$

$\Rightarrow$ $a = -15/2$ and $b = 12$

Hence, the equation of the required line is

$$-\frac{x}{\frac{15}{2}} + \frac{y}{12} = 1$$

$\Rightarrow$ $8x - 5y + 60 = 0$

### Normal Form or Perpendicular Form of a Line

The equation of the straight line upon which the length of the perpendicular from the origin is $p$ and this perpendicular makes an angle $\alpha$ with +ve direction of $x$-axis is

$$x \cos \alpha + y \sin \alpha = p$$

**Proof:** Let the line $AB$ be such that the length of the perpendicular $OQ$ from the origin $O$ to the line be $p$ and $\angle XOQ = \alpha$.

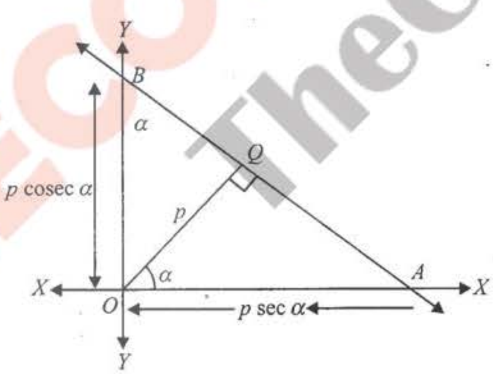



**Fig. 1.38**

From the diagram, using the intercept form, we get

Equation of line $AB$ is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1$$

or $\quad x \cos \alpha + y \sin \alpha = p$

**Example 1.53** **The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of $y$-axis. Find the equation of the line.**

**Sol.** Here $p = 7$ and $\alpha = 30°$

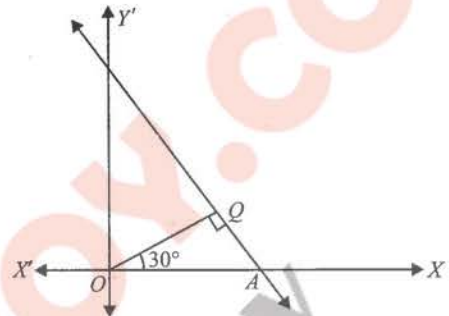



**Fig. 1.39**

Equation of the required line is

$$x \cos 30° + y \sin 30° = 7$$

$\Rightarrow$ $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 7$

$\Rightarrow$ $\sqrt{3}x + y = 14$

## ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN

Let the angle $\theta$ between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\tan \theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

**Proof:** Let $m_1$ and $m_2$ be the slopes of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, respectively.

Then,

$$m_1 = -a_1/b_1 \text{ and } m_2 = -a_2/b_2$$

Now,

$$\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$\Rightarrow$
$$\tan \theta = \left|\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right|$$

$\Rightarrow$
$$\tan \theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

$\Rightarrow$
$$\theta = \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

## Condition for the Lines to be Parallel

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$m_1 = m_2$$

$$\Rightarrow \quad -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

## Condition for the Lines to be Perpendicular

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

$$m_1m_2 = -1 \Rightarrow \left(-\frac{a_1}{b_1}\right) \times \left(-\frac{a_2}{b_2}\right) = -1$$

$$\Rightarrow \quad a_1a_2 + b_1b_2 = 0$$

It follows from the above discussion that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

**i.** Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

**ii.** Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

**iii.** Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

**iv.** Perpendicular, if $a_1a_2 + b_1b_2 = 0$

**Example 1.54** **Find the angle between the pairs of straight lines**

**i.** $x - y\sqrt{3} - 5 = 0$ **and** $\sqrt{3}x + y - 7 = 0$

**ii.** $y = (2 - \sqrt{3})x + 5$ **and** $y = (2 + \sqrt{3})x - 7$

**Sol. i.** The equations of two straight lines are

$$x - y\sqrt{3} - 5 = 0 \qquad \text{(i)}$$

and

$$\sqrt{3}x + y - 7 = 0 \qquad \text{(ii)}$$

Let $m_1$ and $m_2$ be the slopes of these two lines. Then,

$$m_1 = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

and $$m_2 = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

Clearly, $m_1m_2 = -1$. Thus, the two lines are at right angle.

**ii.** Let $m_1$ and $m_2$ be the slopes of the straight lines $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$, respectively.

Then,

$$m_1 = 2 - \sqrt{3} \text{ and } m_2 = 2 + \sqrt{3}$$

Let $\theta$ be the angle between the lines. Then,

$$\tan\theta = \pm\left(\frac{(m_1 - m_2)}{1 + m_1m_2}\right)$$

$$= \pm\left(\frac{(2-\sqrt{3}) - (2+\sqrt{3})}{1 + (2-\sqrt{3})(2+\sqrt{3})}\right)$$

$$= \pm\left(-\frac{2\sqrt{3}}{1 + 4 - 3}\right) = \pm\sqrt{3}$$

So, the acute angle between the lines is given by

$$\tan\theta = |\pm\sqrt{3}| = \sqrt{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

**Example 1.55** **A straight canal is $4\frac{1}{2}$ miles from a place and the shortest route from this place to the canal is exactly north-east. A village is 3 miles north and four east from the place. Does it lie by the nearest edge of the canal?**

**Sol.** Let the given place be $O$. Take this as the origin and the east and north directions through $O$ as the $x$-and $y$-axes, respectively.

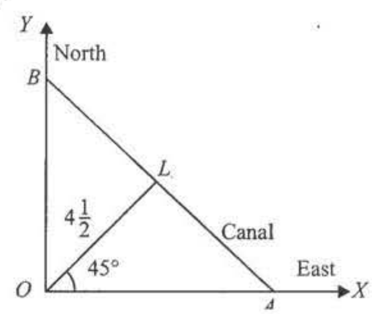



**Fig. 1.40**

Let $AB$ be the nearest edge of the canal. From the question, $OL$ is perpendicular to $AB$ such that $OL = 4\frac{1}{2}$ miles and $\angle LOA = 45°$

So, the equation of the canal is

$$x\cos 45° + y\sin 45° = 4\frac{1}{2}$$

$$\Rightarrow \quad \sqrt{2}(x + y) = 9 \qquad \text{(i)}$$

The position of the village is (4, 3). The village will lie on the edge of the canal if (4, 3) satisfies the Eq. (i).

Clearly, (4, 3) does not satisfy (i). Hence, the village does not lie by the nearer edge of the canal.

**Example 1.56** **Reduce the line $2x - 3y + 5 = 0$ in slope-intercept, intercept, and normal forms.**

**Sol.** $y = \frac{2x}{3} + \frac{5}{3}$, $\tan\theta = m = 2/3$, $c = \frac{5}{3}$

Intercept form:

$$\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1,\ a = -\frac{5}{2},\ b = \frac{5}{3}$$

Normal form:

$$-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$$

$$\sin\alpha = \frac{3}{\sqrt{13}},\ \cos\alpha = -\frac{2}{\sqrt{13}},\ p = \frac{5}{\sqrt{13}}$$

**Example 1.57** **A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line $x = 3$, find the other vertices of the rectangle.**

**Sol.** Let $A \equiv (1, 2)$, $C \equiv (5, 5)$

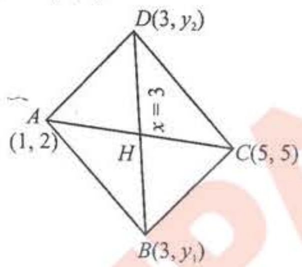



**Fig. 1.41**

Since vertices $B$ and $D$ lie on line $x = 3$; therefore, let $B \equiv (3, y_1)$ and $D \equiv (3, y_2)$.

Now since $AC$ and $BD$ bisect each other, therefore, middle points of $AC$ and $BD$ will be same

$$\frac{y_1 + y_2}{2} = \frac{2+5}{2} \text{ or } y_1 + y_2 = 7 \qquad \text{(i)}$$

Also $\qquad BD^2 = AC^2$

$\therefore \qquad (y_1 - y_2)^2 = (1-5)^2 + (2-5)^2 = 25$

or $\qquad y_1 - y_2 = \pm 5 \qquad$ (ii)

Solving (i) and (ii), we get

$$y_1 = 6, y_2 = 1 \text{ or } y_1 = 1, y_2 = 6$$

Hence, other vertices of the rectangle are (3, 1) and (3, 6).

**Example 1.58** **A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$, find the equation of the other sides of the triangle.**

**Sol.** Given line is

$$x + y - 2 = 0 \qquad \text{(i)}$$

Its slope $\qquad m_1 = -1.$

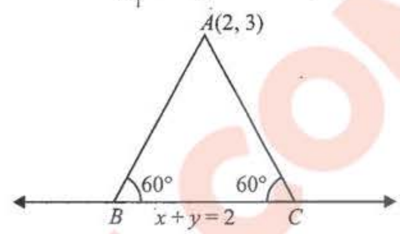



**Fig. 1.42**

Let the slope of the line be $m$ which makes an angle of 60° with line in Eq. (i), then

$$\tan 60° = \left|\frac{m_1 - m}{1 + m_1 m}\right| \text{ or } \sqrt{3} = \left|\frac{-1-m}{1-m}\right|$$

or $\qquad \sqrt{3} = \left|\frac{1+m}{m-1}\right|$ or $\frac{1+m}{m-1} = \pm\sqrt{3}$

or $\qquad 1 + m = \pm\sqrt{3}\,(m-1)$

$\Rightarrow \qquad m = \frac{\sqrt{3}+1}{\sqrt{3}-1}, \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$$= 2+\sqrt{3},\ 2-\sqrt{3}$$

Equation of other two sides of the triangle are

$$y - 3 = (2+\sqrt{3})(x-2)$$

and $\qquad y - 3 = (2-\sqrt{3})(x-2)$

**Example 1.59** **Two consecutive sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation to one diagonal is $11x + 7y = 9$, find the equation of the other diagonal.**

**Sol.**

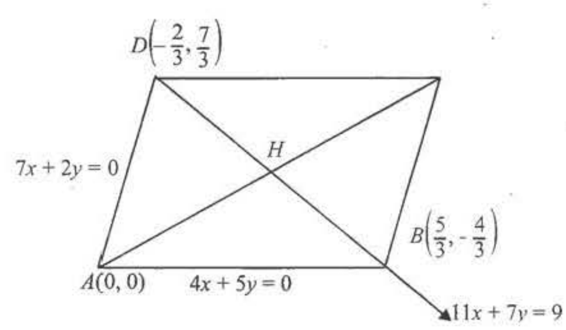



**Fig. 1.43**

Let the equation of sides $AB$ and $AD$ of the parallelogram $ABCD$ be as given in Eqs. (i) and (ii), respectively, i.e.,

$$4x + 5y = 0 \qquad \text{(i)}$$

and $\qquad 7x + 2y = 0 \qquad$ (ii)

Solving (i) and (ii), we have

$$x = 0, y = 0$$

$$\therefore \quad A \equiv (0, 0)$$

Equation of one diagonal of the parallelogram is

$$11x + 7y = 9 \qquad \text{(iii)}$$

Clearly, $A(0, 0)$ does not lie on diagonal as shown in Eq. (iii), therefore Eq. (iii) is the equation of diagonal $BD$.

Solving (i) and (iii), we get $B \equiv \left(\frac{5}{3}, -\frac{4}{3}\right)$

Solving (ii) and (iii), we get $D \equiv \left(-\frac{2}{3}, \frac{7}{3}\right)$

Since $H$ is the middle point of $BD$

$$\therefore \quad H \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$$

Now, equation of diagonal $AC$ which passes through $A(0, 0)$ and $H\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$$y - 0 = \frac{0 - \frac{1}{2}}{0 - \frac{1}{2}}(x - 0) \text{ or } y - x = 0$$

**Example 1.60** **A line $4x + y = 1$ through the point $A(2, -7)$ meets the line $BC$ whose equation is $3x - 4y + 1 = 0$ at the point $B$. Find the equation of the line $AC$, so that $AB = AC$.**

**Sol.**

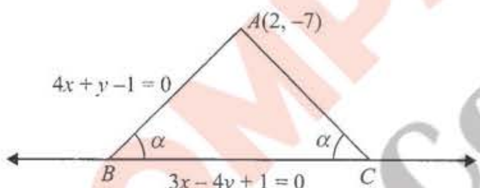



**Fig. 1.44**

Let the equation of $BC$ be

$$3x - 4y + 1 = 0 \qquad \text{(i)}$$

and the equation of $AB$ be

$$4x + y - 1 = 0 \qquad \text{(ii)}$$

Since $AB = AC \;\therefore\; \angle ABC = \angle ACB = \alpha$ (say)

Slope of line $BC = 4/3$ and slope of $AB = -4$.

Let slope of $AC = m$, equating the two values of $\tan\alpha$, we get

$$\left|\frac{-4 - \frac{3}{4}}{1 - 4 \times \frac{3}{4}}\right| = \left|\frac{\frac{3}{4} - m}{1 + \frac{3}{4}m}\right|$$

$$\Rightarrow \quad \pm\frac{19}{8} = \frac{3 - 4m}{4 + 3m}$$

$$\Rightarrow \quad m = 52/89 \text{ or } m = -4$$

Therefore, equation of $AC$ is

$y + 7 = -(52/89)(x - 2)$ or $52x + 89y + 519 = 0$

**Example 1.61** **A variable straight line is drawn through the point of intersection of the straight lines $x/a + y/b = 1$ and $x/b + y/a = 1$ and meets the coordinate axes at $A$ and $B$. Show that the locus of the midpoint of $AB$ is the curve $2xy(a + b) = ab(x + y)$.**

**Sol.**

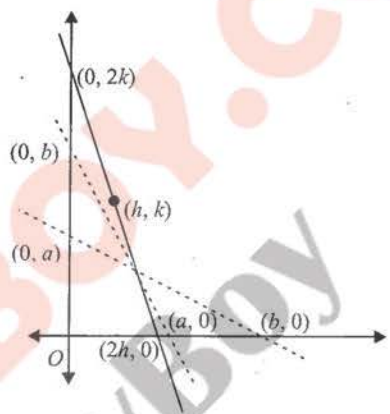



**Fig. 1.45**

Given lines are

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(i)}$$

and

$$\frac{x}{b} + \frac{y}{a} = 1 \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$x = \frac{ab}{a + b} \text{ and } y = \frac{ab}{a + b}$$

Now a variable line passing through these points meets the axis at points $A$ and $B$. Let the midpoint of $AB$ be $(h, k)$ whose locus is to be found.

Then coordinates of $A$ and $B$ are $(2h, 0)$ and $(0, 2k)$.

Now points $A$, $B$ and $[ab/(a + b), ab/(a + b)]$ are collinear.

Then

$$\Delta = \frac{1}{2}\begin{vmatrix} 2h & 0 & 1 \\ 0 & 2k & 1 \\ \frac{ab}{a+b} & \frac{ab}{a+b} & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4hk - 2h\frac{ab}{a + b} - 2k\frac{ab}{a + b} = 0$$

$$\Rightarrow \quad 2xy(a + b) = ab(x + y)$$

**Example 1.62** **If the line $(x/a) + (y/b) = 1$ moves in such a way that $(1/a^2) + (1/b^2) = (1/c^2)$ where $c$ is a constant, prove that the foot of the perpendicular from the origin on the straight line describes the circle $x^2 + y^2 = c^2$.**

**Sol.** Variable line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(i)}$$

Any line perpendicular to Eq. (i) and passing through the origin will be

$$\frac{x}{b}-\frac{y}{a}=0 \qquad \text{(ii)}$$

Now the foot of the perpendicular from the origin to the line Eq. (i) is the point of intersection of Eq. (i) and (ii).

Let be $P(\alpha, \beta)$, then $\frac{\alpha}{a}+\frac{\beta}{b}=1$ (iii)

and

$$\frac{\alpha}{b}-\frac{\beta}{a}=0 \qquad \text{(iv)}$$

Squaring and adding Eqs. (iii) and (iv), we get

$$\alpha^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\beta^2\left(\frac{1}{b^2}+\frac{1}{a^2}\right)=1$$

But

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2} \text{ (given)}$$

Hence $c$ is a constant and $a$, $b$ are parameters (variables). Therefore,

$$(\alpha^2+\beta^2)\frac{1}{c^2}=1.$$

Hence, the locus of $P(\alpha, \beta)$ is

$$x^2+y^2=c^2$$

## Equation of a Line Parallel to a Given Line

The equation of a line parallel to a given line $ax+by+c=0$ is

$$ax+by+\lambda=0$$

where $\lambda$ is *a* constant. Find $\lambda$ by using given condition.

## Equation of a Line Perpendicular to a Given Line

The equation of a line perpendicular to a given line $ax+by+c=0$ is

$$bx-ay+\lambda=0$$

where $\lambda$ is a constant. Find $\lambda$ by using given condition.

**Example 1.63** **Find the equation of the line which is parallel to $3x-2y+5=0$ and passes through the point $(5,-6)$.**

**Sol.** The equation of any line parallel to the line $3x-2y+5=0$ is

$$3x-2y+\lambda=0 \qquad \text{(i)}$$

This passes through $(5,-6)$, therefore we get

$$3\times(5)-2\times(-6)+\lambda=0$$

$$\Rightarrow \quad \lambda=-27$$

Putting $\lambda=-27$ in (i), we get

$$3x-2y-27=0$$

which is the required equation.

**Example 1.64** **Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x+2y+5=0$.**

**Sol.** The equation of a line perpendicular to $3x+2y+5=0$ is

$$2x-3y+\lambda=0 \qquad \text{(i)}$$

This passes through the point (3, 4), therefore we get

$$2\times(3)-3\times(4)+\lambda=0$$

$$\Rightarrow \quad \lambda=6$$

Putting $\lambda=6$ in (i), we get

$$2x-3y+6=0$$

which is the required equation.

**Example 1.65** **Find the coordinates of the foot of the perpendicular drawn from the point $(1,-2)$ on the line $y=2x+1$.**

**Sol.** Let $M$ be the foot of the perpendicular drawn from $P(1,-2)$ on the line $y=2x+1$.

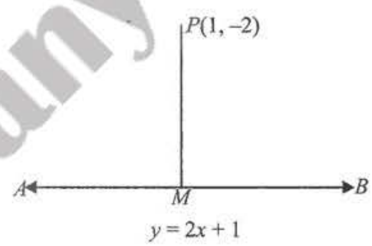



Fig. 1.46

Then $M$ is the point of intersection of $y=2x+1$ and a line passing through $P(1,-2)$ and perpendicular to $y=2x+1$.

The equation of a line perpendicular to $y=2x+1$ or $2x-y+1=0$ is

$$x+2y+\lambda=0 \qquad \text{(i)}$$

This passes through $P(1,-2)$, therefore we get

$$\Rightarrow \quad 1-4+\lambda=0$$

$$\Rightarrow \quad \lambda=3$$

Putting $\lambda=3$ in (i), we get

$$x+2y+3=0$$

Point $M$ is the point of intersection of $2x-y+1=0$ and $x+2y+3=0$

Solving these equations by cross-multiplication, we get

$$\frac{x}{-5}=\frac{y}{-5}=\frac{1}{5}$$

$$\Rightarrow \quad x=-1 \text{ and } y=-1$$

Hence, the coordinates of the foot of the perpendicular are $(-1,-1)$.

**Example 1.66** **Find the image of the point (–8, 12) with respect to the line mirror $4x + 7y + 13 = 0$.**

**Sol.** Let the image of the point $P(-8, 12)$ in the line mirror $AB$ be $Q(\alpha, \beta)$.

Then, $PQ$ is perpendicularly bisected at $R$.

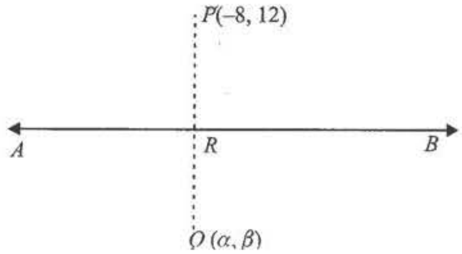



**Fig. 1.47**

The coordinates of $R$ are $\left(\frac{\alpha - 8}{2}, \frac{\beta + 12}{2}\right)$

Since $R$ lies on $4x + 7y + 13 = 0$, we get

$$2\alpha - 16 + (7\beta + 84)/2 + 13 = 0$$

$$\Rightarrow \quad 4\alpha + 7\beta + 78 = 0 \quad \text{(i)}$$

Since $PQ \perp AB$, therefore (slope of $AB$) × (slope of $PQ$) $= -1$

$$\Rightarrow \quad -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1$$

$$\Rightarrow \quad 7\alpha - 4\beta + 104 = 0 \quad \text{(ii)}$$

Solving (i) and (ii), we get

$$\alpha = -16, \beta = -2$$

Hence, the image of $(-8, 12)$ in the line mirror $4x + 7y + 13 = 0$ is $(-16, -2)$.

**Example 1.67** **A ray of the light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected. Find the equation of the line containing the reflected ray.**

**Sol.** Solving the equations of $LM$ and $PA$ coordinates of $A$ can be obtained.

If slope of $AQ$ is determined, then the equation of $AQ$ can be determined.

If slope of $AQ$ is $m$, then equating the two values of $\tan\theta$ (considering the angles between $AL$ and $AP$ and between $AM$ and $AQ$, $m$ can be found.

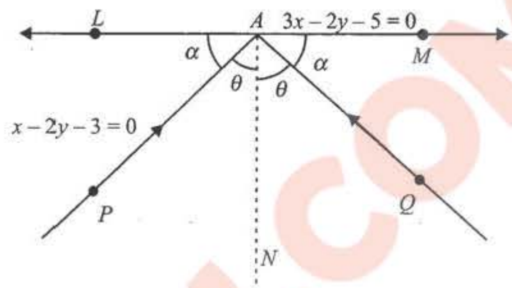



**Fig. 1.48**

Equation of line LM is

$$3x - 2y - 5 = 0 \quad \text{(i)}$$

Equation of $PA$ is

$$x - 2y - 3 = 0 \quad \text{(ii)}$$

Solving (i) and (ii), we get

$$x = 1, y = -1$$

$$\therefore \quad A \equiv (1, -1)$$

Let slope of $AQ = m$, slope of $LM = 3/2$, slope of $PA = 1/2$

Let $\quad \angle LAP = \angle QAM = \alpha$

As $\quad \angle LAP = \alpha$

$$\Rightarrow \quad \tan\alpha = \left|\frac{\frac{3}{2} - \frac{1}{2}}{1 + \frac{3}{2} \times \frac{1}{2}}\right| = \frac{4}{7} \quad \text{(iii)}$$

Again $\quad \angle QAM = \alpha$

$$\therefore \quad \tan\alpha = \left|\frac{m - \frac{3}{2}}{1 + \frac{3}{2}m}\right| = \left|\frac{2m - 3}{2 + 3m}\right| \quad \text{(iv)}$$

From Eq (iii) and (iv), we have

$$\left|\frac{2m - 3}{2 + 3m}\right| = \frac{4}{7}$$

or $$\frac{2m - 3}{2 + 3m} \pm \frac{4}{7}$$

$$\therefore \quad m = \frac{1}{2}, \frac{29}{2}$$

But slope of $AP = \frac{1}{2}$

$\therefore$ Slope of $AQ = \frac{29}{2}$

Now, the equation of $AQ$ will be

$$y + 1 = \frac{29}{2}(x - 1)$$

or $$29x - 2y - 31 = 0$$

**Example 1.68** **A ray of light is sent along the line $2x - 3y = 5$. After refracting across the line $x + y = 1$ it enters**

**the opposite side after turning by 15° away from the line $x + y = 1$. Find the equation of the line along which the refracted ray travels.**

**Sol.**

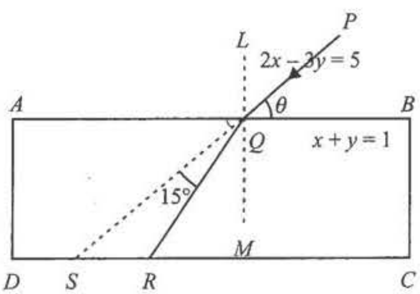



**Fig. 1.49**

Equation of line $AB$ is

$$x + y = 1 \quad \text{(i)}$$

Equation of line $QP$ is

$$2x - 3y = 5 \quad \text{(ii)}$$

$QR$ is the refracted ray. According to question $\angle SQR = 15°$

Solving Eqs. (i) and (ii), we get

$$x = \frac{8}{5} \text{ and } y = -\frac{3}{5}$$

$$\Rightarrow \quad Q \equiv \left(\frac{8}{5}, -\frac{3}{5}\right)$$

$$\text{Slope of } QP = \frac{2}{3}$$

$$\Rightarrow \quad \text{Slope of } QS = \frac{2}{3}$$

Let slope of $QR = m$

But $\angle SQR = 15°$

$$\Rightarrow \quad \tan 15° = \left|\frac{\frac{2}{3} - m}{1 + \frac{2}{3}m}\right|$$

$$\Rightarrow \quad 2 - \sqrt{3} = \left|\frac{2 - 3m}{3 + 2m}\right|$$

$$\Rightarrow \quad \frac{2 - 3m}{3 + 2m} = \pm(2 - \sqrt{3})$$

$$\Rightarrow \quad 2 - 3m = \pm[6 - 3\sqrt{3} + (4 - 2\sqrt{3})m]$$

$$\text{or} \quad 2 - 3m = \begin{cases} 6 - 3\sqrt{3} + (4 - 2\sqrt{3})m \\ -6 + 3\sqrt{3} - (4 - 2\sqrt{3})m \end{cases}$$

$$\Rightarrow \quad m = \frac{3\sqrt{3} - 4}{7 - 2\sqrt{3}}, \frac{3\sqrt{3} - 8}{1 - 2\sqrt{3}}$$

For required line $m = \dfrac{3\sqrt{3} - 8}{1 - 2\sqrt{3}}$

## Concept Application Exercise 1.3

1. Find the angle between lines $x = 2$ and $x - 3y = 6$.
2. If the coordinates of the points $A$, $B$, $C$, and $D$, be $(a, b)$, $(a', b')$, $(-a, b)$, and $(a', -b')$, respectively, then find the equation of the line bisecting the line segments $AB$ and $CD$.
3. If the coordinates of the vertices of the triangle $ABC$ are $(-1, 6)$, $(-3, -9)$, and $(5, -8)$, respectively, then find the equation of the median through $C$.
4. Find the equation of the line perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$ and passing through a point at which it cuts $x$-axis.
5. If the middle points of the sides $BC$, $CA$, and $AB$ of the triangle $ABC$ are $(1, 3)$, $(5, 7)$ and $(-5, 7)$, respectively, then find the equation of the side $AB$.
6. Find the equations of the lines which pass through the origin and are inclined at an angle $\tan^{-1} m$ to the line $y = mx + c$.
7. If $(-2, 6)$ is the image of the point $(4, 2)$ with respect to line $L = 0$, then find the equation of line $L$.
8. If the lines $x + (a - 1)y + 1 = 0$ and $2x + a^2y - 1 = 0$ are perpendicular, then find the values of $a$.
9. Find the area bounded by the curves $x + 2|y| = 1$ and $x = 0$.
10. Find the equation of the straight line passing through the intersection of the lines $x - 2y = 1$ and $x + 3y = 2$ and parallel to $3x + 4y = 0$.
11. A straight line through the point $(2, 2)$ intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points $A$ and $B$. Then find the equation to the line $AB$ so that the triangle $OAB$ is equilateral.
12. Find the equation of the straight line passing through the point $(4, 3)$ and making intercepts on the coordinate axes whose sum is $-1$.
13. A straight line through the point $A(3, 4)$ is such that its intercept between the axis is bisected at $A$. Find its equation.
14. The diagonals $AC$ and $BD$ of a rhombus intersect at $(5, 6)$. If $A \equiv (3, 2)$, then find the equation of diagonal $BD$.
15. If the foot of the perpendicular from the origin to a straight line is at the point $(3, -4)$. Then find the equation of the line.
16. If we reduce $3x + 3y + 7 = 0$ to the form $x \cos\alpha + y \sin\alpha = p$, then find the value of $p$.
17. Find the equation of the straight line which passes through the origin and makes angle 60° with the line $x + \sqrt{3}y + 3\sqrt{3} = 0$.

**18.** A line intersects the straight lines $5x - y - 4 = 0$ and $3x - 4y - 4 = 0$ at $A$ and $B$, respectively. If a point $P(1, 5)$ on the line $AB$ is such that $AP : PB = 2 : 1$ (internally), find point $A$.

**19.** In the given figure $PQR$ is an equilateral triangle and $OSPT$ is a square. If $OT = 2\sqrt{2}$ units, find the equation of lines $OT$, $OS$, $SP$, $QR$, $PR$, and $PQ$.

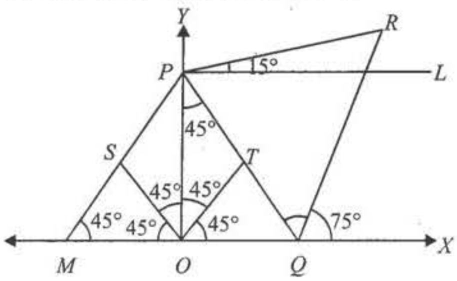



**Fig. 1.50**

**20.** Find the obtuse angle between the lines $x - 2y + 3 = 0$ and $3x + y - 1 = 0$.

**21.** Two fixed points $A$ and $B$ are taken on the coordinates axes such that $OA = a$ and $OB = b$. Two variable point $A'$ and $B'$ are taken on the same axes such that $OA' + OB' = OA + OB$. Find the locus of the point of intersection of $AB'$ and $A'B$.

## DISTANCE FORM OF A LINE (PARAMETRIC FORM)

The equation of the straight line passing through $(x_1, y_1)$ and making an angle $\theta$ with the positive direction of $x$-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

where $r$ is the distance of the point $(x, y)$ on the line from point $(x_1, y_1)$.

**Proof:** Let the given line meets $x$-axis at $A$, $y$-axis at $B$ and passes through the point $Q(x_1, y_1)$.

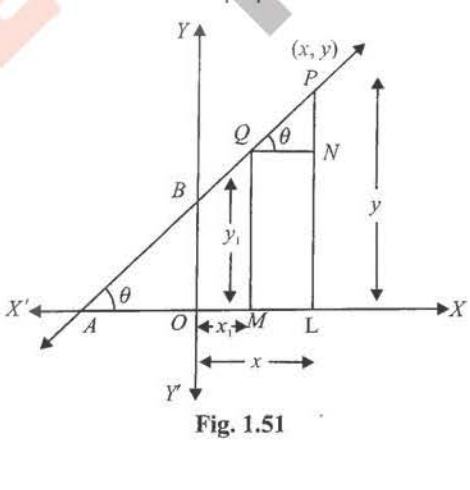



**Fig. 1.51**

Let $P(x, y)$ be any point on the line at a distance $r$ from $Q(x_1, y_1)$ i.e., $PQ = r$.

In $\Delta PQN$, we have

$$\cos\theta = \frac{QN}{PQ} = \frac{x - x_1}{r}$$

$$\Rightarrow \quad \cos\theta = \frac{x - x_1}{r} \quad \text{(i)}$$

and

$$\sin\theta = \frac{PN}{PQ}$$

$$\Rightarrow \quad \sin\theta = \frac{y - y_1}{r} \quad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the required equation of the line in the distance form.

$\Rightarrow \; x - x_1 = r\cos\theta$ and $y - y_1 = r\sin\theta$

$\Rightarrow \; x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$.

Thus, the coordinates of any point on the line at a distance $r$ from the given point $(x_1, y_1)$ are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If $P$ is on the right side of $(x_1, y_1)$, then $r$ is positive and if $P$ is on the left side of $(x_1, y_1)$, then r is negative. Since different values of $r$ determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.

In the parametric form, one can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance $r$ from the point $(x_1, y_1)$ on the line $(x - x_1)/\cos\theta = (y - y_1)/\sin\theta$ there are two points viz., $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

**Example 1.69** **A straight line is drawn through the point $P(2, 3)$ and is inclined at an angle of $30°$ with the $x$-axis. Find the coordinates of two points on it at a distance 4 from $P$.**

**Sol.** Here $(x_1, y_1) = (2, 3)$, $\theta = 30°$, the equation of the line is

$$\frac{x - 2}{\cos 30°} = \frac{y - 3}{\sin 30°}$$

$$\Rightarrow \quad \frac{x - 2}{\frac{\sqrt{3}}{2}} = \frac{y - 3}{\frac{1}{2}}$$

$$\Rightarrow \quad x - 2 = \sqrt{3}(y - 3)$$

$$\Rightarrow \quad x - \sqrt{3}y = 2 - 3\sqrt{3}$$

Points on the line at a distance 4 from $P(2, 3)$ are

$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

or $(2 \pm 4\cos 30°, 3 \pm 4\sin 30°)$

or $(2 \pm 2\sqrt{3}, 3 \pm 2)$ or $(2 + 2\sqrt{3}, 5)$ and $(2 - 2\sqrt{3}, 1)$

**Example 1.70** **Find the equation of the line passing through the point $A(2,3)$ and making an angle of 45° with the $x$-axis. Also determine the length of intercept on it between $A$ and the line $x + y + 1 = 0$.**

**Sol.**

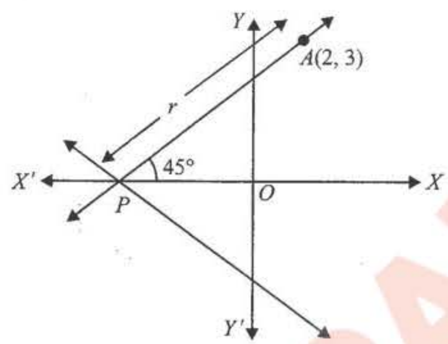



**Fig. 1.52**

The equation of a line passing through $A$ and making an angle of 45° with the $x$-axis is

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°}$$

$$\frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \quad x - y + 1 = 0$$

Suppose this line meets the line $x + y + 1 = 0$ at $P$ such that $AP = r$.

Then, the coordinates of $P$ are given by

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$$

$$\Rightarrow \quad x = 2 + r\cos 45°, y = 3 + r\sin 45°$$

$$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}}, y = 3 + \frac{r}{\sqrt{2}}$$

Thus, the coordinates of $P$ are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

Since $P$ lies on $x + y + 1 = 0$. Therefore,

$$2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$$

$$\Rightarrow \quad \sqrt{2}\, r = -6$$

$$\Rightarrow \quad r = -3\sqrt{2}$$

Therefore, length $AP = |r| = 3\sqrt{2}$

Thus, the length of the intercept $= 3\sqrt{2}$

**Example 1.71** **The line joining two points $A(2, 0)$, $B(3, 1)$ is rotated about $A$ in anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If $B$ goes to $C$ in the new position, what will be the coordinates of $C$?**

**Sol.** The slope $m$ of the line $AB$ is given by

$$m = \frac{1-0}{3-2} = 1$$

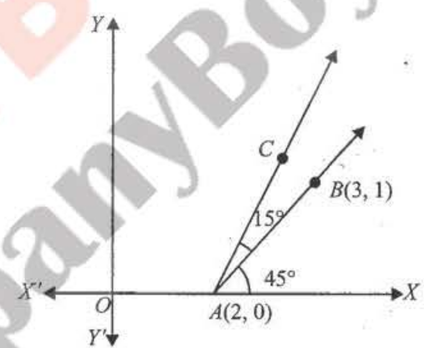



**Fig. 1.53**

So, $AB$ makes an angle of 45° with $x$-axis. Now $AB$ is rotated through 15° in anticlockwise direction and so it makes an angle of 60° with $x$-axis in its new position $AC$.

Clearly $AC$ passes through $A(2, 0)$ and makes an angle of 60° with $x$-axis, therefore the equation of $AC$ is

$$\frac{x-2}{\cos 60°} = \frac{y-0}{\sin 60°}$$

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}}$$

We have

$$AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}.$$

So, the coordinates of $C$ are given by

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = \sqrt{2}$$

$$\Rightarrow \quad x = 2 + \frac{1}{2}\sqrt{2} = 2 + \frac{1}{\sqrt{2}} \text{ and } y = \frac{\sqrt{3}}{2}\sqrt{2} = \frac{\sqrt{6}}{2}$$

Hence, the coordinates of $C$ are $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$.

**Example 1.72** **Find the distance of the point (1, 3) from the line $2x - 3y + 9 = 0$ measured along a line $x - y + 1 = 0$.**

**Sol.** The slope of the line $x - y + 1 = 0$ is 1. So it makes an angle of 45° with $x$-axis.

The equation of a line passing through (1, 3) and making an angle of 45° is

$$\frac{x-1}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

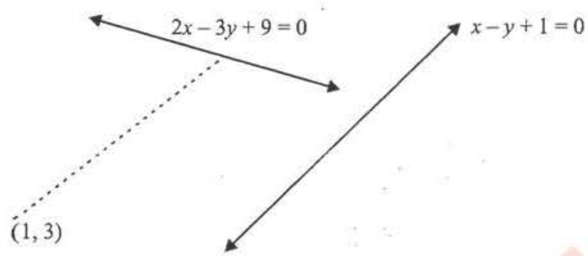



**Fig. 1.54**

Coordinates of any point on this line are $(1 + r\cos 45^\circ, 3 + r\sin 45^\circ) = \left(1 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

If this point lies on the line $2x - 3y + 9 = 0$, then

$$2 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0$$

$$\Rightarrow \qquad r = 2\sqrt{2}$$

Hence, the required distance $= 2\sqrt{2}$

### Concept Application Exercise 1.4

1. Two particles start from the point $(2, -1)$, one moves 2 units along the line $x + y = 1$ and the other 5 units along the line $x - 2y = 4$. If the particles move towards increasing, then find their new positions.
2. Find the distance between $A(2, 3)$ on the line of gradient 3/4 and the point of intersection $P$ of this line with $5x + 7y + 40 = 0$.
3. The centre of a square is at the origin and one vertex is $A(2, 1)$. Find the coordinates of other vertices of the square.

## CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if they pass through a common point, i.e., they meet at a point.

Thus, if three line are concurrent the point of intersection of two lines lies on the third line. Let the three concurrent lines be

$$a_1x + b_1y + c_1 = 0 \qquad \text{(i)}$$
$$a_2x + b_2y + c_2 = 0 \qquad \text{(ii)}$$
$$a_3x + b_3y + c_3 = 0 \qquad \text{(iii)}$$

Then the point of intersection of Eqs. (i) and (ii) must lie on the third.

The coordinates of the point of intersection of Eqs. (i) and (ii) are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

This point lies on line (iii). Therefore, we get

$$\Rightarrow \quad a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

**Alternative Method:**

Three lines $L_1 \equiv a_1x + b_1y + c_1 = 0$; $L_2 \equiv a_2x + b_2y + c_2 = 0$; $L_3 \equiv a_3x + b_3y + c_3 = 0$ are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero at the same time so that $\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = 0$, i.e., $\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$.

**Example 1.73** **Find the value of $\lambda$, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33$ and $2x - 3y + \lambda = 0$ are concurrent.**

**Sol.** The given lines are concurrent if

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) = 0$$

$$\Rightarrow \quad -\lambda - 7 = 0 \Rightarrow \lambda = -7$$

**Alternative Method:** The given equations are

$$3x - 4y - 13 = 0 \qquad \text{(i)}$$
$$8x - 11y - 33 = 0 \qquad \text{(ii)}$$
$$\text{and} \quad 2x - 3y + \lambda = 0 \qquad \text{(iii)}$$

Solving Eqs. (i) and (ii), we get

$$x = 11 \text{ and } y = 5$$

Thus, (11, 5) is the point of intersection of Eqs. (i) and (ii).

The given lines will be concurrent if they pass through the common point, i.e., the point of intersection of any two lines lies on the third.

Therefore, (11, 5) lies on Eq. (iii),

i.e., $2 \times 11 - 3 \times 5 + \lambda = 0$

$\Rightarrow \qquad \lambda = -7$

**Example 1.74** **If the lines $a_1x + b_1y + 1 = 0$, $a_2x + b_2y + 1 = 0$ and $a_3x + b_3y + 1 = 0$ are concurrent, then show that the points $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$ are collinear.**

**Sol.** The given lines are

$$a_1x + b_1y + 1 = 0 \quad \text{(i)},$$
$$a_2x + b_2y + 1 = 0 \quad \text{(ii)}$$
and $$a_3x + b_3y + 1 = 0 \quad \text{(iii)}$$

If these lines are concurrent, we must have

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

which is the condition of collinearity of three points $(a_1, b_1)$, $(a_2, b_2)$, and $(a_3, b_3)$.

Hence, if the given lies are concurrent, the given points are collinear.

## DISTANCE OF A POINT FROM A LINE

The length of the perpendicular from a point $(x_1, y_1)$ to a line $ax + by + c = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

**Proof:** The line $ax + by + c = 0$ meets $x$-axis at $A\left(\frac{-c}{a}, 0\right)$ and $y$-axis at $B\left(0, -\frac{c}{b}\right)$.

Let $P(x_1, y_1)$ be the point. Draw $PN \perp AB$.

Now, area of $\Delta PAB$

$$= \frac{1}{2}\left|x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right| = \left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| \quad \text{(i)}$$

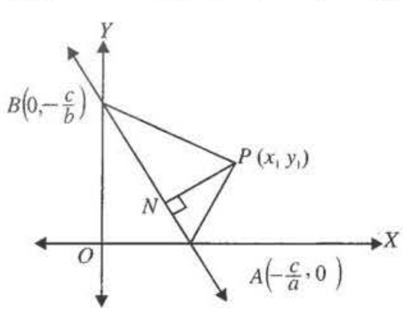



**Fig. 1.55**

Also, area of $\Delta PAB$

$$= \frac{1}{2} AB \times PN$$

$$= \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \times PN$$

$$= \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN \quad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| = \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN$$

$$\Rightarrow \qquad PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

### Distance between Two Parallel lines

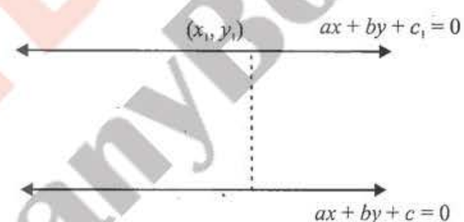



**Fig. 1.56**

Distance of $a$ point $(x_1, y_1)$ from the line $ax + by + c = 0$ is

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Now point $(x_1, y_1)$ lies on $a_1x + b_1y + c_1 = 0$, then

$$ax_1 + by_1 + c_1 = 0$$

$$\Rightarrow \qquad ax_1 + by_1 = -c_1$$

$$\Rightarrow \qquad p = \frac{|c - c_1|}{\sqrt{a^2 + b^2}}$$

**Example 1.75** **If $p$ is the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.**

**Sol.** The given line is

$$bx + ay - ab = 0 \quad \text{(i)}$$

It is given that $p$ = length of the perpendicular from the origin to Eq. (i). That is,

$$p = \frac{|b(0) + a(0) - ab|}{\sqrt{b^2 + a^2}}$$

$$= \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad p^2 = \frac{a^2b^2}{a^2+b^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

**Example 1.76** **Find the points on $y$-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.**

**Sol.** Let the required point be $P(0, \alpha)$. It is given that the length of the perpendicular from $P(0, \alpha)$ on $4x - 3y - 12 = 0$ is 3

$$\Rightarrow \quad \left|\frac{4(0) - 3\alpha - 12}{\sqrt{4^2 + (-3)^2}}\right| = 3$$

$$\Rightarrow \quad |3\alpha + 12| = 15$$

$$\Rightarrow \quad |\alpha + 4| = 5$$

$$\Rightarrow \quad \alpha + 4 = \pm 5$$

$$\Rightarrow \quad \alpha = 1, -9$$

Hence, the required points are (0, 1) and (0, − 9).

**Example 1.77** **Two sides of a squares lie on the lines $x + y = 1$ and $x + y + 2 = 0$. What is its area?**

**Sol.** Clearly, the length of the side of the square is equal to the distance between the parallel lines.

$$x + y - 1 = 0 \qquad \text{(i)}$$

and

$$x + y + 2 = 0 \qquad \text{(ii)}$$

Hence side length is $\frac{|2-(-1)|}{\sqrt{(1+1)}} = \frac{3}{\sqrt{2}}$

$\Rightarrow$ Area of square is $\frac{9}{2}$

**Example 1.78** **Find the coordinates of a point on $x + y + 3 = 0$, whose distance from $x + 2y + 2 = 0$ is $\sqrt{5}$.**

**Sol.** Putting $x = t$ in $x + y + 3 = 0$, we get $y = -3 - t$. So, let the required point be $(t, -3 - t)$. This point is at a distance of $\sqrt{5}$ units from $x + 2y + 2 = 0$. Therefore,

$$\left|\frac{t - 6 - 2t + 2}{\sqrt{1^2 + 2^2}}\right| = \sqrt{5}$$

$$\Rightarrow \quad \left|\frac{-t-4}{\sqrt{5}}\right| = \sqrt{5}$$

$$\Rightarrow \quad t + 4 = \pm 5$$

$$\Rightarrow \quad t = 1, -9$$

Hence, the required points are (1, − 4) and (−9, 6).

**Example 1.79** **Find the equations of straight lines passing through (− 2, − 7) and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.**

**Sol.**

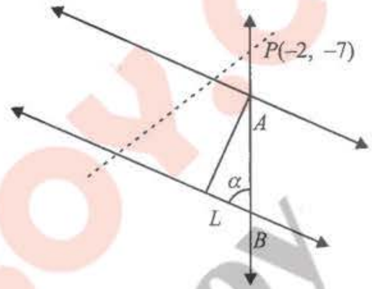



**Fig. 1.57**

Given lines are

$$4x + 3y - 12 = 0 \qquad \text{(i)}$$

and

$$4x + 3y - 3 = 0 \qquad \text{(ii)}$$

Given $\quad AB = 3$

$AL$ = distance between parallel lines Eq. (i) and Eq. (ii)

$$= \frac{|-12+3|}{\sqrt{4^2+3^2}} = \frac{9}{5}$$

From $\Delta\, ALB$, we get

$$LB^2 = AB^2 - AL^2$$

$$= 3^2 - 81/25 = (9 \times 16/25)$$

$$\therefore \quad LB = 12/5$$

$\tan\alpha = AL/LB = 3/4$. Also $\tan\theta$ = slope of line (i) $= -4/3$. Let Slope of $PB$ is $m$.

Now $\quad \tan\alpha = \frac{3}{4} = \left|\frac{m - \left(-\frac{4}{3}\right)}{1 + m\left(-\frac{4}{3}\right)}\right|$

$$\Rightarrow \quad m = \infty \text{ or } m = -\frac{7}{24}$$

$\Rightarrow$ Equation of line is $x + 2 = 0$ and $y + 7 = \frac{-7}{24}(x + 2)$

or $x + 2 = 0$ and $7x + 24y + 182 = 0$

**Example 1.80** **Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $\frac{2c^2}{(ab)}$.**

**Sol.**

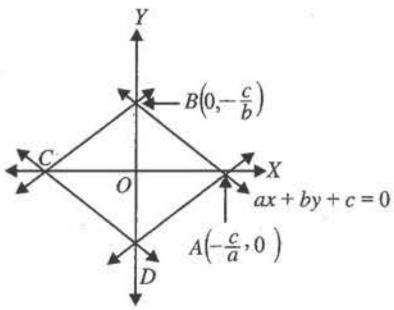



**Fig. 1.58**

Clearly from the above figure,

Area of parallelogram

$$= 4 \times (\text{area of } \Delta AOB)$$

$$= 4 \times \frac{1}{2}\left|\frac{c}{a}\right|\left|\frac{c}{b}\right|$$

$$= 2\frac{c^2}{ab}$$

### Concept Application Exercise 1.5

1. Find the equation of a straight line passing through the point $(-5, 4)$ and which cuts off an intercept of $\sqrt{2}$ units between the lines $x + y + 1 = 0$ and $x + y - 1 = 0$.
2. Find the ratio in which the line $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$.
3. Find the equations of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.
4. If $p$ and $p'$ are the distances of origin from the lines $x \sec\alpha + y \operatorname{cosec}\alpha = k$ and $x\cos\alpha - y\sin\alpha = k\cos 2\alpha$, then prove that $4p^2 + p'^2 = k^2$.

## POSITION OF POINTS RELATIVE TO A LINE

Let the equation of the given line be

$$ax + by + c = 0 \qquad \text{(i)}$$

and let the coordinates of the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The coordinates of the point $P$ which divides the line joining $P$ and $Q$ in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) \qquad \text{(ii)}$$

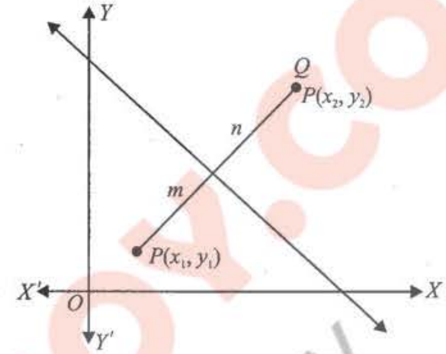



**Fig. 1.59**

If this point lies on Eq. (i), then

$$a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c = 0$$

$$\Rightarrow \quad m(ax_2 + by_2 + c) + n(ax_1 + by_1 + c) = 0$$

$$\Rightarrow \quad \frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} \qquad \text{(iii)}$$

If point $R$ is between the points $P$ and $Q$, i.e., points $P$ and $Q$ are on the opposite sides of the given line, then the ratio $m : n$ is positive.

So, from Eq. (iii), we get

$$-\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$$

$$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs

If the points $R$ is not between $P$ and $Q$, i.e., points $P$ and $Q$ are on the same side of the given line, then the ratio $m : n$ is negative.

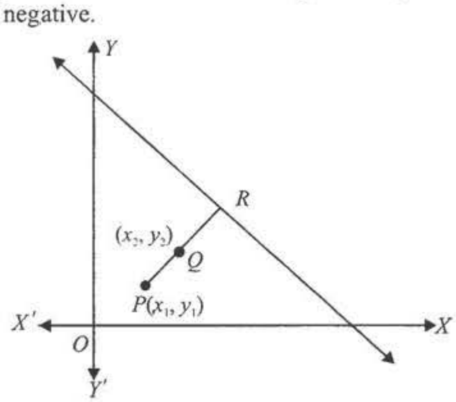



**Fig. 1.60**

So, from Eq. (iii), we get

$$-\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$$

$$\Rightarrow \quad \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign.

Thus, the two points $(x_1, y_1)$ and $(x_2, y_2)$ are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according as the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

**Example 1.81** **Are the points (3, 4) and (2, − 6) on the same or opposite sides of the line $3x - 4y = 8$?**

**Sol.** Let $L = 3x - 4y - 8$. Then the value of $L$ at (3, 4) is $L_1 = -15$ and the value of $L$ at (2, − 6) is $L_2 = 22$.

Since $L_1$ and $L_2$ are of opposite signs, therefore the two points are on the opposite sides of the given line.

**Example 1.82** **If the point $(a, a)$ is placed in between the lines $|x + y| = 4$, then find the values of $a$.**

**Sol.** Lines $x + y = 4$ and $x + y = -4$ is parallel and points (2, 2) and (−2, −2) lie on these lines.

If point $(a, a)$ lies between the lines, then $a > -2$ and $a < 2$ i.e. $-2 < a < 2$.

$$\Rightarrow \quad |a| < 2$$

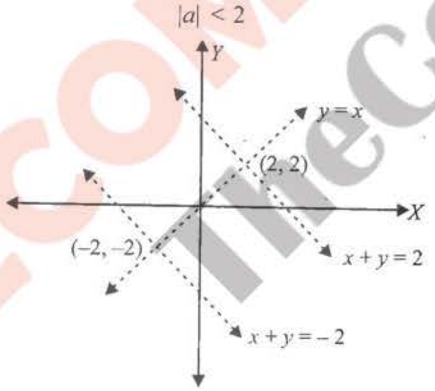



**Fig. 1.61**

**Example 1.83** **Find the range of values of the ordinate of a point moving on the line $x = 1$, and always remaining in the interior of the triangle formed by the lines $y = x$, the $x$-axis and $x + y = 4$.**

**Sol.** From the figure, $y$-coordinates of $P$ vary from 0 to 1. So, to be an interior point, $0 <$ ordinate of $P < 1$.

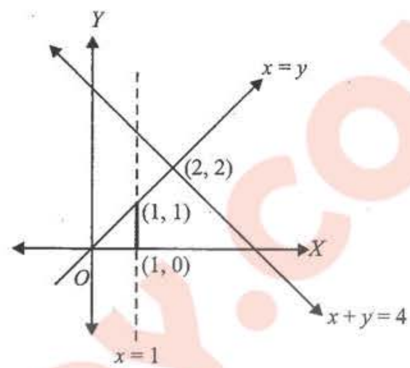



**Fig. 1.62**

**Example 1.84** **If the point $P(a^2, a)$ lies in the region corresponding to the acute angle between the lines $2y = x$ and $4y = x$, then find the values of $a$.**

**Sol.** Acute angle is formed by lines in first and third quadrants

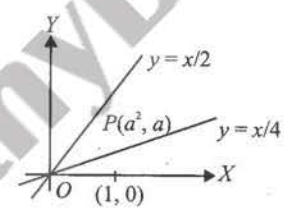



**Fig. 1.63**

But $a^2 > 0$, hence point $P(a^2, a)$ lies in first quadrant.

We have $a - (a^2/4) > 0$ and $a - (a^2/2) < 0$ ($\because$ (1, 0) and $P$ lies on same side of $x - 2y = 0$ and (1, 0) and $P$ lies opposite sides of $x - 4y = 0$)

$$\Rightarrow \quad 0 < a < 4 \text{ and } a \in (-\infty, 0) \cup (2, \infty)$$

$$\Rightarrow \quad a \in (2, 4)$$

## Concept Application Exercise 1.6

1. The point (8, −9) with respect to the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ lies on the
   - **a.** same side of the lines
   - **b.** different of sides the line
   - **c.** one of the line
   - **d.** none of these
2. Which pair of points lies on the same side of $3x - 8y - 7 = 0$?
   - **a.** (0, −1) and (0, 0)
   - **b.** (4, −3) and (0, 1)
   - **c.** (−3, −4) and (1, 2)
   - **d.** (−1, −1) and (3, 7)
3. Find the range of $\alpha$ for which the points $(\alpha, 2 + \alpha)$ and $(3\alpha/2, \alpha^2)$ lie on the opposite sides of the line $2x + 3y = 6$.

## EQUATIONS OF BISECTORS OF THE ANGLES BETWEEN THE LINES

### Angle Bisectors

A bisector of angle between lines $L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$ is the locus of a point, which moves such that the length of perpendiculars drawn from it to the two given lines, are equal.

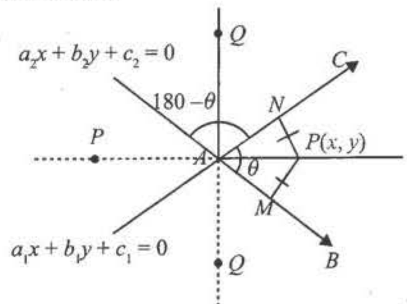



**Fig. 1.64**

From Fig. $PN = PM$

Then equations of the bisectors are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

#### i. *To determine the acute angle bisector and the obtuse angle bisector*

$AP$ is the bisector of an acute angle if,

$\tan(\angle PAN) = \tan(\theta/2)$ is such that $|\tan \theta/2| < 1$

$\Rightarrow \quad \theta/2 < \pi/4$

$\Rightarrow \quad \theta < \pi/2$

$AP$ is an obtuse angle bisector if,

$\tan(\angle PAN) = \tan(\theta/2)$ is such that $|\tan \theta/2| > 1$

$\Rightarrow \quad \theta/2 > \pi/4$

$\Rightarrow \quad \theta > \pi/2$

#### ii. *To determine the bisector of the angle contaning the origin and that of the angle not containing the origin*

Rewrite the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ such that the constant terms $c_1, c_2$ are positive.

Then, we have

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

it gives the equation of the bisector of the angle containing the origin and

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

gives the equation of the bisector of the angle not containing the origin.

**Proof :**

$L_1 : a_1x + b_1y + c_1 = 0$

and $L_2 : a_2x + b_2y + c_2 = 0$ (where $c_1, c_2 > 0$)

Since any point $P(h, k)$ and $O$ (origin) are on the same side of the line $L_1 = 0$ and $L_2 = 0$ or on opposite sides of $L_1 = 0$ and $L_2 = 0$ $P(h, k)$ will give the same sign with respect to $L_1 = 0$ and $L_2 = 0$ ($\because c_1, c_2 > 0$)

Therefore, the equation $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

gives the bisector of that angle which contains the origin.

Now, consider the point $Q(h, k)$ on other bisector.

Note that $Q$ and $O$ are on different side of $L_2 = 0$, whereas $Q$ and $O$ are on the same side of $L_1 = 0$ (or $Q$ and $O$ lie on same side of $L_2 = 0$ and on different sides of $L_1 = 0$)

$\frac{a_1h + b_1k + c_1}{\sqrt{a_1^2 + b_1^2}}$ and $\frac{a_2h + b_2k + c_2}{\sqrt{a_2^2 + b_2^2}}$ have opposite signs.

Therefore, the equation $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

will give the bisector of that angle which does not contain the origin.

Bisector of the angle containing the point $(\alpha, \beta)$ is

$$\therefore \quad \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

if $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ have the same sign.

### Shortcut Method for Identifying Acute and Obtuse Angle Bisectors

Equations of the bisectors of the lines $L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$

$(a_1b_2 \neq a_2b_1)$ where $c_1 > 0$ and $c_2 > 0$ are

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

| Conditions | Acute angle bisector | Obtuse angle bisector |
|---|---|---|
| $a_1a_2+b_1b_2>0$ | $-$ | $+$ |
| $a_1a_2+b_1b_2<0$ | $+$ | $-$ |

**Note:**

*A line which is equally inclined to given two lines is parallel to the angle bisectors of the given lines.*

**Example 1.85** **For the straight lines $4x+3y-6=0$ and $5x+12y+9=0$, find the equation of the**

**i. bisector of the obtuse angle between them,**

**ii. bisector of the acute angle between them,**

**iii. bisector of the angle which contains (1, 2).**

**Sol.** Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}}=\pm\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$\Rightarrow \quad 9x-7y-41=0$

and $\quad 7x+9y-3=0$

If $\theta$ is the angle between the line $4x+3y-6=0$ and the bisector $9x-7y-41=0$, then

$$\tan\theta=\left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(-\frac{4}{3}\right)\frac{9}{7}}\right|=\frac{11}{3}>1$$

Hence,

i. The bisector of the obtuse angle is $9x-7y-41=0$.

ii. The bisector of the acute angle is $7x+9y-3=0$.

iii. The bisector of the angle containing the point (1, 2).

For the point (1, 2), we have

$$4x+3y-6=4(1)+3(2)-6>0$$

$$5x+12y+9=5(1)+12(2)+9>0$$

Hence, the equation of the bisector of the angle containing the point (1, 2) is

$$\frac{4x+3y-6}{5}=\frac{5x+12y+9}{13}$$

$\Rightarrow \quad 9x-7y-41=0$

**Example 1.86** **Find the equation of the bisector of the obtuse angle between the lines $3x-4y+7=0$ and $12x+5y-2=0$.**

**Sol.** Firstly, make the constant terms $(c_1, c_2)$ positive

$$3x-4y+7=0$$

and $\quad -12x-5y+2=0$

$\Rightarrow \quad a_1a_2+b_1b_2=(3)(-12)+(-4)(-5)$

$$=-36+20=-16$$

Hence, "$-$" sign gives the obtuse bisector.

$\Rightarrow$ Obtuse bisector is

$$\frac{(3x-4y+7)}{\sqrt{(3)^2+(-4)^2}}=-\frac{(-12x-5y+2)}{\sqrt{(-12)^2+(-5)^2}}$$

$\Rightarrow \quad 13(3x-4y+7)=-5(-12x-5y+2)$

$\Rightarrow \quad 21x+77y-101=0$

is the obtuse angle bisector

**Example 1.87** **The vertices of a triangle $ABC$ are (1, 1), (4, – 2) and (5, 5) respectively. Then find the equation of perpendicular dropped from $C$ to the internal bisector of angle $A$.**

**Sol.** The internal bisector $AD$ of angle $BAC$ will divide the opposite side $BC$ in the ratio of arms of the angle i.e., $AB : AC$ or $3\sqrt{2} : 4\sqrt{2}$ or 3 : 4.

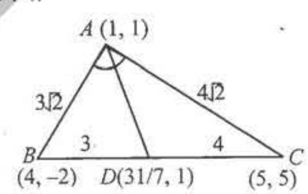



**Fig. 1.65**

Hence, the point $D$ on $BC$ by section formula is (31/7, 1) and $A$ is (1, 1).

Therefore, slope of $AD = 0$. Hence, slope of $CL$ which is perpendicular from $C$ on $AD$ is $\infty$.

Therefore, equation of $CL$ is $x-5=0$.

**Example 1.88** **Two equal sides of an isosceles triangle are $7x-y+3=0$ and $x+y-3=0$ and its third side passes through the point (1, –10). Determine the equation of the third side.**

**Sol.** Since triangle is isosceles, the third side is equally inclined to the lines $7x-y+3=0$ and $x+y-3=0$.

Hence, the third side is parallel to angle bisectors of the given lines.

The equations of the two bisectors of the angle are

$$\frac{7x-y+3}{\sqrt{50}}=\pm\frac{x+y-3}{\sqrt{2}}$$

or $\quad 3x + y - 3 = 0 \quad$ (i)

and $\quad x - 3y + 9 = 0 \quad$ (ii)

The line through $(1, -10)$ parallel to Eq. (i) is $3x + y + 7 = 0$ and parallel to Eq. (ii) is $x - 3y - 31 = 0$.

## Image of a Point with Respect to the Line Mirror

Let the image of $A(x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$ be $B(x_2, y_2)$ then it is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

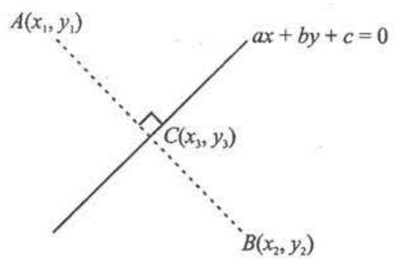



**Fig. 1.66**

Also let the foot of perpendicular from point $A(x_1, y_1)$ on the line $ax + by + c = 0$ be $C(x_3, y_3)$ which is given by

$$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

**Proof:**

Equation of line $AB$ is

$$ax + by + c = 0 \quad \text{(i)}$$

Let $\quad P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$

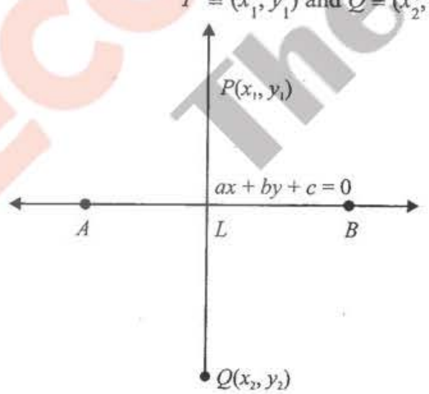



**Fig. 1.67**

Slope of $AB = -\frac{a}{b}$ and slope of $PQ = (y_2 - y_1)/(x_2 - x_1)$

Since $PQ \perp AB$, we get

$$\left(-\frac{a}{b}\right)\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -1$$

or $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = k \text{ (say)} \quad \text{(ii)}$$

From Eq. (ii), $x_2 - x_1 = ka$ and $y_2 - y_1 = kb$

Now,

$$a(x_2 - x_1) + b(y_2 - y_1) = k(a^2 + b^2)$$

or $$ax_2 + by_2 - (ax_1 + by_1) = k(a^2 + b^2)$$

or $$ax_2 + by_2 + c - (ax_1 + by_1 + c) = k(a^2 + b^2) \quad \text{(iii)}$$

Let $L$ be the point of intersection of lines $AB$ and $PQ$, then

$$L \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Since $L$ lies on line Eq. (i),

$$\Rightarrow \quad a\left(\frac{x_1 + x_2}{2}\right) + b\left(\frac{y_1 + y_2}{2}\right) + c = 0$$

or $$ax_1 + by_1 + c + ax_2 + by_2 + c = 0$$

or $$ax_2 + by_2 + c = -(ax_1 + by_1 + c)$$

Therefore, from Eq. (iii), we get

$$-2(ax_1 + by_1 + c) = k(a^2 + b^2)$$

or $$k = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2} \quad \text{(iv)}$$

From Eqs. (ii) and (iv), we have

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

**Example 1.89** **Find the image of the point $(4, -13)$ in the line $5x + y + 6 = 0$.**

**Sol.**

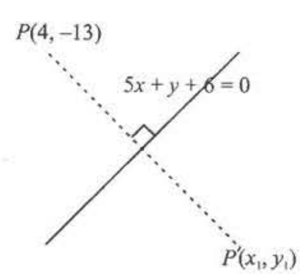



**Fig. 1.68**

Let $P'(x_1, y_1)$ be the image of $(4, -13)$ with respect to the line mirror $5x + y + 6 = 0$, then

$$\frac{x_1-4}{5}=\frac{y_1+13}{1}=\frac{-2[5(4)-13+6]}{(5^2+1^2)}$$

$$\Rightarrow \quad \frac{x_1-4}{5}=\frac{y_1+13}{1}=-\frac{26}{26}$$

$$\Rightarrow \quad x_1-4=-5 \text{ and } y_1+13=-1$$

$$\Rightarrow \quad x_1=-1 \text{ and } y_1=-14$$

Hence, $P'(-1, -14)$.

**Example 1.90** **Find the foot of the perpendicular from the point (2, 4) upon $x + y = 1$.**

**Sol.** Required foot $(h, k)$ of the perpendicular is given by

$$\frac{h-2}{1}=\frac{k-4}{1}=\frac{-(2+4-1)}{1+1}=\frac{-5}{2}$$

$$h = 2-(5/2) = -1/2 \text{ and } k = 4-(5/2) = 3/2$$

Required foot is $(-1/2, 3/2)$.

**Example 1.91** **In triangle $ABC$, equation of the right bisectors of the sides $AB$ and $AC$ are $x + y = 0$ and $y - x = 0$ respectively. If $A \equiv (5, 7)$ then find the equation of side $BC$.**

**Sol.** '$B$' and '$C$' will be the image of $A$ in $y + x = 0$ and $y - x = 0$ respectively.

Thus $\quad B \equiv (-7, -5), C \equiv (7, 5)$.

Hence, equation of $BC$ is

$$y-5=\frac{-5-5}{-7-7}(x-7) \text{ i.e., } 14y = 10x.$$

## FAMILY OF STRAIGHT LINES

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

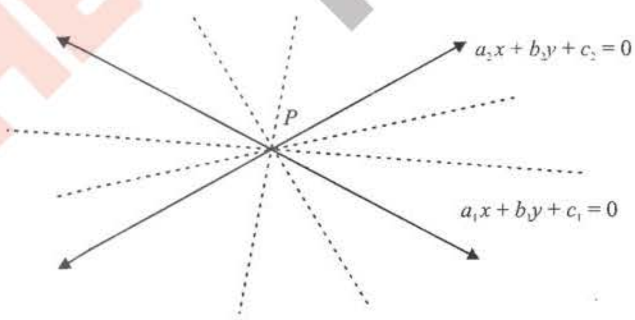



Fig. 1.69

Then, the general equation of any straight line passing through the point of intersection of lines $L_1$ and $L_2$ is given by $L_1 + \lambda L_2 = 0$

where $\lambda \in R$.

These lines form a family of straight line from point $P$.

Also this general equation satisfies point of intersection of $L_1$ and $L_2$ for any value of $\lambda$.

**Note:**

- *Consider variable straight line $ax + by + c = 0$, where $a$, $b$, $c$ are real. This lines do not form a family of concurrent straight line, as for different values of $a$, $b$, $c$ the lines not necessarily concurrent. But if $a$, $b$, $c$ are related to each other by any linear relation, i.e., $al + bm + cn = 0$, where $l, m, n$ are constants, then lines are concurrent for different values of $a$, $b$, $c$. The given variable line can be adjusted as $ax + by - (al + bm)/n = 0$. This is, $a[x - (l/n)] + b[y - (m/n)] = 0$, which passes through point $(l/n, m/n)$ for different values of $a$ and $b$.*
- *A straight line is such that the algebraic sum of the perpendiculars drawn upon it from any number of fixed points is zero; then the line always passes through a fixed point.*

  ***Proof:*** Let the fixed points be $(x_r, y_r)$; $r = 1, 2, 3, \ldots, n$ and the given line be

$$ax + by + c = 0 \qquad \text{(i)}$$

*Given,*

$$\sum_{r=1}^{n} \frac{ax_r + by_r + c}{\sqrt{a^2+b^2}} = 0;$$

$$\text{or} \quad \sum_{r=1}^{n} (ax_r + by_r + c) = 0$$

$$\Rightarrow \quad a\sum_{r=1}^{n} x_r + b\sum_{r=1}^{n} y_r + \sum_{r=1}^{n} c = 0$$

$$\Rightarrow a(x_1 + x_2 + \cdots + x_n) + b(y_1 + y_2 + \cdots + y_n) + cn = 0$$

$$\Rightarrow a\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) + b\left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right) + c = 0 \text{ (ii)}$$

*From Eq. (ii) it is clear that Eq. (i) passes through the fixed point*

$$\Rightarrow \left(\frac{x_1 + x_2 + \cdots + x_n}{n}, \frac{y_1 + y_2 + \cdots + y_n}{n}\right)$$

**Example 1.92** **Show that the straight lines $x(a + 2b) + y(a + 3b) = a + b$ for different values of $a$ and $b$ pass through a fixed point. Find that point.**

**Sol.** Given equation is

$$x(a+2b)+y(a+3b) = a+b$$

or $\quad a(x+y-1)+b(2x+3y-1) = 0 \quad$ (i)

Both $a$ and $b$ cannot be simultaneously zero, therefore at least one of $a$ and $b$ will be non-zero. Let $a \neq 0$.

Now Eq. (i) can be written as

$$x+y-1+(b/a)(2x+3y-1) = 0$$

or $\quad x+y-1+\lambda(2x+3y-1) = 0, \quad$ (ii)

where $\quad \lambda = b/a$

From Eq. (ii) it is clear that Eq. (ii) passes through the point of intersection of lines

$$x+y-1 = 0 \quad \text{(iii)}$$

and

$$2x+3y-1 = 0 \quad \text{(iv)}$$

Solving Eqs. (iii) and (iv), we get $x = 2, y = -1$.

Hence, lines represented by Eq. (i) pass through the fixed point $(2, -1)$ for all values of $a$ and $b$.

**Example 1.93** **If $a$, $b$, $c$ are in AP, then prove that $ax + by + c = 0$ will always pass through a fixed point. Find that fixed point.**

**Sol.** Since $a$, $b$, $c$ are in $AP$, therefore $2b = a + c$. Putting the value of $b$ in $ax + by + c = 0$, we get

$$ax+\left(\frac{a+c}{2}\right)y+c = 0$$

$$\Rightarrow \quad a(2x+y)+c(y+2) = 0$$

$$\Rightarrow (2x+y)+\left(\frac{c}{a}\right)(y+2) = 0$$

This equation represents a family of straight lines passing through the intersection of $2x + y = 0$ and $y + 2 = 0$, i.e; $(1, -2)$.

**Alternative method:**

Given line is

$$ax+by+c = 0 \quad \text{(i)}$$

where $a$, $b$, $c$ are in AP then

$$a-2b+c = 0 \quad \text{(ii)}$$

Now comparing ratio of coefficient of $a$, $b$ and $c$ in Eq. (i) and (ii), we have

$$\frac{x}{1} = \frac{y}{-2} = \frac{1}{1}$$

$$\Rightarrow \quad x = 1 \text{ and } y = -2$$

**Example 1.94** **Find the straight line passing through the point of intersection of $2x+ 3y + 5 = 0$, $5x - 2y - 16 = 0$ and through the point $(-1, 3)$.**

**Sol.** The equation of any line through the point of intersection of the given line is

$$2x+3y+5+\lambda(5x-2y-16) = 0 \quad \text{(i)}$$

But the required line passes through $(-1, 3)$, hence we get

$$-2+9+5+\lambda(-5-6-16) = 0$$

Hence, $\lambda = 4/9$. Use this value of $\lambda$ in Eq. (i) and the required line is

$$9(2x+3y+5)+4(5x-2y-16) = 0$$

or on simplification, we get

$$2x+y-1 = 0$$

**Example 1.95** **Consider a family of straight lines $(x + y) + \lambda(2x - y + 1) = 0$. Find the equation of the straight line belonging to this family that is farthest from $(1, -3)$.**

**Sol.**

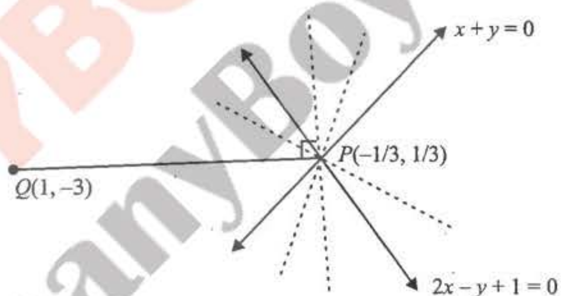



**Fig. 1.70**

Given family is concurrent at '$P$' $(-1/3, 1/3)$. If $Q \equiv (1, -3)$, then $m_{PQ} = [-3-(1/3)]/[1+(1/3)] = -5/2$.

Now, member of the family that is farthest from '$Q$' will have its slope as $2/5$. ($\because$ line will be $\perp$ to $PQ$)

As a result, we get

$$\frac{2}{5} = -\frac{(1+2\lambda)}{(1-\lambda)}$$

$$\Rightarrow \quad \lambda = -\frac{7}{8}$$

Thus, the equation of required line is

$$(x+y)-\frac{7}{8}(2x-y+1) = 0 \text{ i.e. } 15y-6x-7 = 0$$

**Example 1.96** **Find the values of non-negative real numbers $h_1$, $h_2$, $h_3$, $k_1$, $k_2$, $k_3$ such that the algebraic sum of the perpendiculars drawn from points $(2, k_1)$, $(3, k_2)$, $(7, k_3)$, $(h_1, 4)$, $(h_2, 5)$, $(h_3, -3)$ on a variable line passing through $(2, 1)$ is zero.**

**Sol.** Let the equation of variable line be $ax + by + c = 0$, it is given that

$$\sum_{i=1}^{6} \frac{ax_i+by_i+c}{\sqrt{a^2+b^2}} = 0$$

$$\Rightarrow \quad a\left(\frac{\Sigma x_i}{6}\right) + b\left(\frac{\Sigma y_i}{6}\right) + c = 0$$

So, the fixed point must be $\Sigma x_i/6, \Sigma y_i/6$. But fixed point is (2, 1) so $(2 + 3 + 7 + h_1 + h_2 + h_3)/6 = 2$.

$$\Rightarrow \quad h_1 + h_2 + h_3 = 0$$

$$\Rightarrow \quad h_1 = 0, h_2 = 0, h_3 = 0$$

(as $h_1, h_2, h_3$ are non-negative)

Similarly, we get

$$\frac{k_1 + k_2 + k_3 + 4 + 5 - 3}{6} = 1$$

$$\Rightarrow \quad k_1 = k_2 = k_3 = 0$$

### Concept Application Exercise 1.7

1. If $a$ and $b$ are two arbitrary constants, then prove that the straight line $(a - 2b)x + (a + 3b)y + 3a + 4b = 0$ will pass through a fixed point. Find that point.
2. If $a, b, c$ are in harmonic progression, then the straight line $(x/a) + (y/b) + (l/c) = 0$ always passes through a fixed point, then find that point.
3. If algebraic sum of distances of a variable line from points (2, 0), (0, 2) and (– 2, – 2) is zero, then the line passes through the fixed point is

   **a.** (–1 –1) **b.** (0, 0)

   **c.** (1, 1) **d.** (2, 2)
4. Consider the family of lines $5x + 3y - 2 + \lambda_1(3x - y - 4) = 0$ and $x - y + 1 + \lambda_2(2x - y - 2) = 0$. Find the equation of a straight line that belongs to both the families.

## PAIR OF STRAIGHT LINES

If separate equations of two lines be $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$, then their combined equation will be

$$(ax + by + c)(a_1x + b_1y + c_1) = 0$$

and conversely if the combined equation of two lines be $(ax + by + c)(a_1x + b_1y + c_1) = 0$, then their separate equations will be $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$.

For example, the combined equation of lines $2x + y + 3 = 0$ and $x - y + 4 = 0$ is $(2x + y + 3)(x - y + 4) = 0$ or $2x^2 - xy - y^2 + 11x + y - 12 = 0$.

Separate equations of lines represented by the equation $6x^2 + 5xy - 4y^2 = 0$, i.e., $(2x - y)(3x + 4y) = 0$ are $2x - y = 0$ and $3x + 4y = 0$

**Note:**

*In order to find the combined equation of two lines, make R.H.S. of equation of straight lines equal to zero and then multiply the two equations.*

### Homogeneous Equation

An equation (whose R.H.S. is zero) in which the sum of the powers of $x$ and $y$ in every term is the same, say $n$, is called a homogeneous equation of $n$th degree in $x$ and $y$.

Thus, $ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of second degree and $cx^3 + dxy^2 + ey^3 = 0$ is a homogeneous equation of third degree in $x$ and $y$.

### Pair of Straight Lines through the Origin

To show that any homogeneous equation of second degree in $x$ and $y$ represents two straight lines through the origin.

Lets consider a general homogeneous equation of second degree in $x$ and $y$ as

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{(i)}$$

Dividing both sides by $x^2$, we get

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \qquad \text{(ii)}$$

Since Eq. (ii) is an equation of second degree in $y/x$, it has two roots. Let the roots be $m_1$ and $m_2$. If $\alpha, \beta$ be the roots of equation $ax^2 + bx + c = 0$, then $ax^2 + bx + c = a(x - \alpha) \times (x - \beta)$. Therefore, Eqs. (ii) can be written as

$$b\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0$$

or $\quad b(y - m_1x)(y - m_2x) = 0$

Thus, Eq. (i) represents two straight lines $y - m_1x = 0$ and $y - m_2x = 0$ both of which pass through the origin.

Comparing $b(y - m_1x)(y - m_2x) = 0$ with Eq. (i), we have

$$\frac{bm_1m_2}{a} = -\frac{b(m_1 + m_2)}{2h} = \frac{1}{1}$$

$$\Rightarrow \quad m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}$$

### An Angle between the Line Represented by $ax^2 + 2hxy + by^2 = 0$

Let $\theta$ be the angle between the lines, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$= \pm \frac{\sqrt{\frac{4h^2}{b^2} - 4\frac{a}{b}}}{1 + (a/b)}$$

$$= \pm \frac{\sqrt{4h^2 - 4ab}}{a + b}$$

or $$\tan\theta = \pm \frac{2\sqrt{h^2 - ab}}{a + b}$$

Acute angle between the lines is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

**Note:**

- *If the lines are perpendicular, then $\theta = 90°$*

$$\Rightarrow \quad \cot\theta = 0$$

$$\Rightarrow \quad \frac{a + b}{2\sqrt{h^2 - ab}} = 0$$

or $a + b = 0$ *which is the required condition or* (*coeff. of* $x^2$) + (*coeff. of* $y^2$) = 0

- *If the lines are parallel, then $\theta = 0$*

$$\Rightarrow \quad \tan\theta = 0$$

$$\Rightarrow \quad \frac{2\sqrt{h^2 - ab}}{a + b} = 0$$

$$\Rightarrow \quad \sqrt{h^2 - ab} = 0$$

$$\Rightarrow \quad h^2 = ab$$

*which is the condition for the lines to be parallel (coincident).*

*Since both the lines pass through the origin, therefore they will be coincident if parallel.*

## Bisectors of Angle between the Lines Represented by $ax^2 + 2hxy + by^2 = 0$

Let the given equation represent the straight lines

$$y - m_1x = 0 \qquad \text{(i)}$$

$$y - m_2x = 0 \qquad \text{(ii)}$$

then

$$m_1 + m_2 = -2h/b \text{ and } m_1m_2 = a/b \qquad \text{(iii)}$$

The equations to the bisectors of the angles between the straight lines in Eqs. (i) and (ii) are

$$\frac{y - m_1x}{\sqrt{1 + m_1^2}} = \frac{y - m_2x}{\sqrt{1 + m_2^2}}$$

and $$\frac{y - m_1x}{\sqrt{1 + m_1^2}} = -\frac{y - m_2x}{\sqrt{1 + m_2^2}}$$

Therefore, the combined equation of the bisectors is

$$\left\{\frac{y - m_1x}{\sqrt{1 + m_1^2}} - \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right\}\left\{\frac{y - m_1x}{\sqrt{1 + m_1^2}} + \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right\} = 0$$

or $$\frac{(y - m_1x)^2}{1 + m_1^2} - \frac{(y - m_2x)^2}{1 + m_2^2} = 0$$

or $$(1 + m_2^2)(y^2 - 2m_1xy + m_1^2x^2) - (1 + m_1^2)(y^2 - 2m_2xy + m_2^2x^2) = 0$$

or $$(m_1^2 - m_2^2)(x^2 - y^2) + 2(m_1m_2 - 1)(m_1 - m_2)xy = 0$$

or $$(m_1 + m_2)(x^2 - y^2) + 2(m_1m_2 - 1)xy = 0$$

Hence, by Eq. (iii), we get

$$\frac{-2h}{b}(x^2 - y^2) + 2\left(\frac{a}{b} - 1\right)xy = 0,$$

or $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

## General Equation of the Second Degree

We have proved that homogeneous equation of the second degree in $x$ and $y$ represents two straight lines passing through the origin. But every quadratic equation in $x$ and $y$ may not always represent two straight lines. The most general form of a quadratic equation in $x$ and $y$ is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Since it is an equation in $x$ and $y$, therefore it must represent the equation of a locus in a plane. It may represent a pair of straight lines, circle, or other curves in different case. Now we will consider the case when the above equation represents two straight lines.

## Condition for General 2nd Degree Equation in $x$ and $y$ Represent Pair of Straight Lines

The given equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

If $a \neq 0$, then writing Eq. (i) as a quadratic equation in $x$ we get

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0$$

Solving, we have

$$x = \frac{-2(hy + g) \pm \sqrt{4(hy + g)^2 - 4a(by^2 + 2fy + c)}}{2a}$$

$$= \frac{-(hy + g) \pm \sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}}{a}$$

Equation (i) will represent two straight lines if left-hand side of Eq. (i) can be resolved into two linear factors; therefore, the expression under the square roots should be a perfect square.

Hence,

$$4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0 \quad \text{(ii)}$$

(as $Ax^2 + Bx + C$ is a perfect square if $B^2 - 4AC = 0$)

$$\Rightarrow \quad g^2h^2 + a^2f^2 - 2afgh - h^2g^2 + abg^2 + ach^2 - a^2bc = 0$$

$$\Rightarrow \quad a(af^2 + bg^2 + ch^2 - 2fgh - abc) = 0$$

or

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{(iii)}$$

which is the required condition.

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, then the equation of the pair of lines through the origin and parallel to them is $ax^2 + 2hxy + by^2 = 0$

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two lines, then the left-hand sides can be resolved into two linear factors.

Let the factors be $(l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$

Multiplying we see that $(l_1x + m_1y)(l_2x + m_2y)$, the terms of second degree, must be identical with $ax^2 + 2hxy + by^2$. Therefore, $ax^2 + 2hxy + by^2 = 0$ is identical with $(l_1x + m_1y) \times (l_2x + m_2y) = 0$.

Thus, equation $ax^2 + 2hxy + by^2$ represents two lines $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$, which are parallel to lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$, respectively.

**Note:**

*The angle between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is equal to the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$. That is,*

$$\tan^{-1}\left(\pm \frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

## Point of Intersection of Pair of Straight Lines

The point of intersection of the two lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$$

The point of intersection can also be determined with the help of partial differentiation as follows.

Let $\phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Differentiating $\phi$ with respect to $x$, keeping $y$ constant, we get

$$\frac{\partial \phi}{\partial x} = 2ax + 2hy + 2g$$

Similarly, differentiating $\phi$ with respect to $y$, keeping $x$ constant, we get

$$\frac{\partial \phi}{\partial y} = 2hx + 2by + 2f$$

For point of intersection, we get

$$\frac{\partial \phi}{\partial x} = 0 \text{ and } \frac{\partial \phi}{\partial y} = 0$$

Thus, we have $ax + hy + g = 0$ and $hx + by + f = 0$.

Solving the two equations, we get

$$\frac{x}{fh - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2}$$

$$\therefore \quad (x, y) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$$

**Example 1.97** **Find the value of $\lambda$ if $2x^2 + 7xy + 3y^2 + 8x + 14y + \lambda = 0$ represents a pair of straight lines.**

**Sol.** The given equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, if $abc + 2fgh - af^2 - bg^2 - bc^2 = 0$, i.e., if

$$6\lambda + 2(7)(4)\left(\frac{7}{2}\right) - 2(7)^2 - 3(4)^2 - \lambda\left(\frac{7}{2}\right)^2 = 0$$

$$\Rightarrow \quad 6\lambda + 196 - 98 - 48 - \frac{49\lambda}{4} = 0$$

$$\Rightarrow \quad \frac{49\lambda}{4} - 6\lambda = 196 - 146 = 50$$

$$\Rightarrow \quad \frac{25\lambda}{4} = 50$$

$$\Rightarrow \quad \lambda = \frac{200}{25} = 8$$

**Example 1.98** **If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is $\tan^{-1}(m)$, then find the value of $m$.**

**Sol.** The angle between the lines $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is given by

$$\tan\theta = \tan^{-1}\left(\pm \frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

where $a = 2$, $b = 3$, $h = 5/2$

$$\Rightarrow \quad \tan\theta = \frac{\pm 2\sqrt{\frac{25}{4} - 6}}{2 + 3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\pm\frac{1}{5}\right) \Rightarrow m = \pm\frac{1}{5}$$

**Example 1.99** **Find the value of '$a$' for which the lines represented by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular.**

**Sol.** The lines given by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular if $a + 2 = 0$, i.e., $a = -2$.

**Example 1.100** **If $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ bisect angles between each other, then find the condition.**

**Sol.** The equation of the bisectors of the angles between the lines $x^2 - 2pxy - y^2 = 0$ is

$$\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-p}$$

$$\Rightarrow \quad \frac{x^2 - y^2}{2} = \frac{-xy}{p}$$

$$\Rightarrow \quad px^2 + 2xy - py^2 = 0$$

This is same as $x^2 - 2qxy - y^2 = 0$, therefore

$$\frac{p}{1} = \frac{2}{-2q} = \frac{-p}{-1}$$

$$\Rightarrow \quad pq + 1 = 0$$

**Example 1.101** **If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is rotated about the origin through 90°, then find the equations in the new position.**

**Sol.** Let $y = m_1 x$, $y = m_2 x$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$. Then, we have

$$m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

Let $y = m'_1 x$ and $y = m'_2 x$ be new positions of $y = m_1 x$ and $y = m_2 x$, respectively.

Then, $y = m_1 x$ is perpendicular to $y = m'_1 x$

$$\therefore \quad m_1 m'_1 = -1$$

$$\Rightarrow \quad m'_1 = -\frac{1}{m_1}. \text{ Similarly, } m'_2 = -\frac{1}{m_2}$$

So, the new lines are $y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$ and so their combined equation is

$$(m_1 y + x)(m_2 y + x) = 0$$

$$\Rightarrow m_1 m_2 y^2 + x^2 + xy(m_1 + m_2) = 0$$

$$\Rightarrow \frac{a}{b}y^2 + x^2 + xy\left(\frac{-2h}{b}\right) = 0$$

$$\Rightarrow \quad bx^2 - 2hxy + ay^2 = 0$$

**Example 1.102** **If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the $y$-axis, then prove that $2fgh = bg^2 + ch^2$.**

**Sol.** We must have

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \qquad \text{(i)}$$

Let the point of intersection on the $y$-axis be $(0, \lambda)$. Then, from $\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$, we have

$$\therefore \quad fh = bg \qquad \text{(ii)}$$

The equation of the y-axis is x = 0. Solving this and the equation of the pair of lines, we get

$$by^2 + 2fy + c = 0$$

which must have equal roots.

$$\therefore \quad f^2 = bc \qquad \text{(iii)}$$

Equation (ii) can be written as $fgh = bg^2$

Putting $f^2 = bc$, $fgh = bg^2$ in Eq. (i), we get

$$bg^2 = ch^2$$

$$\therefore \quad 2fgh = bg^2 + ch^2$$

**Example 1.103** **Find the distance between the pair of parallel lines $x^2 + 4xy + 4y^2 + 3x + 6y - 4 = 0$.**

**Sol.** Given lines are

$$(x + 2y)^2 + 3(x + 2y) - 4 = 0$$

Therefore, $$x + 2y = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = -4, 1$$

Therefore, lines are

$$x + 2y + 4 = 0$$

$$x + 2y - 1 = 0$$

$$\text{Required distance} = \frac{|4-(-1)|}{\sqrt{1+4}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

## Combined Equation of Pair of Lines Joining Origin and the Points of Intersection of a Curve and a Line

Lets find the equation of the straight lines joining the origin and the points of intersection of the curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

and the line

$$lx + my + n = 0 \qquad \text{(ii)}$$

Equation (ii) can be written as

$$lx + my = -n \text{ or } \frac{lx + my}{-n} = 1 \qquad \text{(iii)}$$

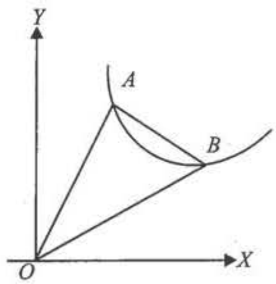



**Fig. 1.71**

Let $A$ and $B$ be the points of intersection of the (ii) and Eq. (i)

In order to make Eq. (i) homogeneous with the help of Eq. (iii), we write Eq. (i) as

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)1 + c(1)^2 = 0$$

$$\Rightarrow ax^2 + 2hxy + by^2 + (2gx + 2fy) \times \left(\frac{lx + my}{-n}\right)$$

$$+ c\left(\frac{lx + my}{-n}\right)^2 = 0 \qquad \text{(iv)}$$

Since coordinates of $A$ and $B$ satisfy Eqs. (i) and (ii), therefore these satisfy Eqs. (iv) which is formed by combining Eqs. (i) and (ii).

Thus, Eq. (iv) is a locus through points $A$ and $B$.

Also Eq. (iv) being a homogeneous equation of second degree in $x$ and $y$ represents two straight lines through the origin.

Hence, Eq. (iii) is the joint equation of $OA$ and $OB$.

**Example 1.104** **Prove that the straight lines joining the origin to the points of intersection of the straight line $hx + ky = 2hk$ and the curve $(x - k)^2 + (y - h)^2 = c^2$ are right angles, if $h^2 + k^2 = c^2$.**

**Sol.** Making the equation of the curve homogeneous with the help of that of the line, we get

$$x^2 + y^2 - 2(kx + hy)\left(\frac{hx + ky}{2hk}\right) + (h^2 + k^2 - c^2)\left(\frac{hx + ky}{2hk}\right)^2 = 0$$

This is the equation of the pair of lines joining the origin to the points of intersection of the given line and the curve. These will be at right angles if

coefficient of $x^2$ + coefficient of $y^2 = 0$, i.e.,

$$(h^2 + k^2)(h^2 + k^2 - c^2) = 0$$

$$\Rightarrow \qquad h^2 + k^2 = c^2 \ (\because h^2 + k^2 \neq 0)$$

**Example 1.105** **Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}(2\sqrt{2}/3)$.**

**Sol.** Equation of the given curve is

$$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$$

and equation of the given straight line is

$$y - 3x = 2$$

$$\therefore \qquad \frac{y - 3x}{2} = 1$$

Making Eq. (i) homogeneous equation of the second degree in $x$ any $y$ with the help of Eq. (i), we have

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y - 3x}{2}\right) + 8y\left(\frac{y - 3x}{2}\right) - 11\left(\frac{y - 3x}{2}\right)^2 = 0$$

or $7x^2 - 2xy - y^2 = 0$

This is the equation of the lines joining the origin to the point of intersection of Eqs. (i) and (ii).

Comparing Eqs. (iii) with the equation $ax^2 + 2hxy + by^2 = 0$, we get

$a = 7$, $b = -1$ and $2h = -2$, i.e., $h = 1$

If $\theta$ be the acute angle between pair of lines as shown in Eq. (iii), then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{1 + 7}}{7 - 1}\right|$$

$$= \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

$$\therefore \qquad \theta = \tan^{-1}\frac{2\sqrt{2}}{3}$$

**Example 1.106** **If the lines joining origin and point of intersection of curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ are mutually perpendicular, then prove that $g(a_1 + b_1) = g_1(a + b)$.**

**Sol.** $\qquad ax^2 + 2hxy + by^2 = -2gx$

$$a_1x^2 + 2h_1xy + b_1y^2 = -2g_1x$$

$$\Rightarrow \qquad \frac{ax^2 + 2hxy + by^2}{a_1x^2\, 2h_1xy + b_1y^2} = \frac{g}{g_1}$$

$$(ag_1 - a_1g)x^2 + 2(hg_1 - h_1g)xy + (bg_1 - b_1g)y^2 = 0$$

The lines will be perpendicular if $ag_1 - a_1g + bg_1 - b_1g = 0$

i.e., $\qquad$ if $(a + b)g_1 = (a_1 + b_1)g$

## Concept Application Exercise 1.8

1. Find the combined equation of the pair of lines through the point (1, 0) and parallel to the lines represented by $2x^2 - xy - y^2 = 0$.

2. If the slope of one line is double the slope of another line and the combined equation of the pair of lines is $(x^2/a) + (2xy/h) + (y^2/b) = 0$, then find the ratio $ab : h^2$.

3. Show that straight lines $(A^2 - 3B^2)x^2 + 8AB\ xy\ (B^2 - 3A^2)y^2 = 0$ form with the line $Ax + By + C = 0$ an equilateral triangle of area $C^2/\sqrt{3(A^2+B^2)}$.

4. If one of the lines denoted by the line pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between coordinate axes, then prove that $(a + b)^2 = 4h^2$.

5. Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the coordinates of their point of intersection and also the angle between them.

6. A line $L$ passing through the point (2,1) intersects the curve $4x^2 + y^2 - x + 4y - 2 = 0$ at the points $A$ and $B$. If the lines joining origin and the points $A$, $B$ are such that the coordinate axes are the bisectors between them, then find the equation of line $L$.

7. If the equation $x^2 + (\lambda + \mu)xy + \lambda\mu y^2 + x + \mu y = 0$ represents two parallel straight lines, then prove that $\lambda = \mu$.

8. If one of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between positive direction of the axes, then find the relation for $a$, $b$, and $h$.

9. If the pair of lines $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ are rotated about the origin by $\pi/6$ in the anticlockwise sense, then find the equation of the pair in the new position.

10. If the equation $2x^2 + kxy + 2y^2 = 0$ represents a pair of real and distinct lines, then find the values of $k$.

11. Find the point of intersection of the pair and straight lines represented by the equations $6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$.

12. Find the angle between the lines represented by $x^2 + 2xy \sec\theta + y^2 = 0$.

# EXERCISES

## Subjective Type

*Solutions on page 1.70*

1. Show that the reflection of the line $ax + by + c = 0$ in the line $x + y + 1 = 0$ is the line $bx + ay + (a + b - c) = 0$, where $a \neq b$.

2. A straight line cuts intercepts from the axis of coordinates, the sum of the reciprocals of which is a constant. Show that it always passes through a fixed point.

3. If the equal sides $AB$ and $AC$ of a right angled isosceles triangle be produced to $P$ and $Q$ so that $BP \times CQ = AB^2$, then show that the line $PQ$ always passes through a fixed point.

4. Straight lines $y = mx + c_1$ and $y = mx + c_2$, where $m \in R^+$, meet the $x$-axis at $A_1$ and $A_2$, respectively, and $y$-axis at $B_1$ and $B_2$, respectively. It is given that points $A_1$, $A_2$, $B_1$ and $B_2$ are concyclic. Find the locus of intersection of lines $A_1B_2$ and $A_2B_1$.

5. A line $L_1 \equiv 3y - 2x - 6 = 0$ is rotated about its point of intersection with $y$-axis in clockwise direction to make it $L_2$ such that the area formed by $L_1$, $L_2$, $x$-axis and line $x = 5$ is 49/3 sq. units if its point of intersection with $x = 5$ lies below $x$-axis. Find the equation of $L_2$.

6. Find the locus of the circumcentre of a triangle whose two sides are along the coordinate axes and the third side passes through the point of intersection of the lines $ax + by + c = 0$ and $lx + my + n = 0$.

7. A regular polygon has two of its consecutive diagonals as the lines $\sqrt{3}x + y = \sqrt{3}$ and $2y = \sqrt{3}$. Point $(1, c)$ is one of its vertices. Find the equation of the sides of the polygon and also find the coordinates of the vertices.

8. A diagonal of rhombus $ABCD$ is member of both the families of lines $(x + y - 1) + \lambda(2x + 3y - 2) = 0$ and $(x - y + 2) + \lambda(2x - 3y + 5) = 0$ and one of the vertices of the rhombus is (3, 2). If the area of the rhombus is $12\sqrt{5}$ sq. units, then find the remaining vertices of the rhombus.

9. A triangle has the lines $y = m_1x$ and $y = m_2x$ as two of its sides, with $m_1$, $m_2$ being roots of the equation $bx^2 + 2hx + a = 0$. If $H(a, b)$ is the orthocentre of the triangle, then show that equation of the third side is $(a + b)(ax + by) = ab(a + b - 2h)$.

10. Let $A = (6, 7)$, $B = (2, 3)$ and $C = (-2, 1)$ be the vertices of a triangle. Find the point $P$ in the interior of the triangle such that $\Delta PBC$ is an equilateral triangle.

11. Find all the values of $\theta$ for which the point $(\sin^2\theta, \sin\theta)$ lies inside the square formed by the lines $xy = 0$ and $4xy - 2x - 2y + 1 = 0$.

12. Two sides of a triangle have the joint equation $x^2 - 2xy - 3y^2 + 8y - 4 = 0$. The third side, which is variable, always passes through the point $(5, -1)$. Find the range of the values of the slope of the third side, so that the origin is an interior point of the triangle.

13. Let $ABC$ be a given isosceles triangle with $AB = AC$. Sides $AB$ and $AC$ are extended up to $E$ and $F$, respectively, such that $BE \times CF = AB^2$. Prove that the line $EF$ always passes through a fixed point.

14. Let $L_1 = 0$ and $L_2 = 0$ be two fixed lines. A variable line is drawn through the origin to cut the two lines at $R$ and $S$. $P$ is a point on the line $AB$ such that $(m + n)/OP = m/OR + n/OS$. Show that the locus of $P$ is a straight line passing through the point of intersection of the given lines ($R$, $S$, $P$ are on the same side of $O$).

15. A variable line cuts $n$ given concurrent straight lines at $A_1, A_2, \ldots, A_n$ such that $\sum_{i=1}^{n} \frac{1}{OA_i}$ is a constant. Show that it always passes through a fixed point, $O$ being the point of intersection of the lines.

16. Two sides of a rhombus lying in the first quadrant are given by $3x - 4y = 0$ and $12x - 5y = 0$. If the length of the longer diagonal is 12, then find the equations of the other two sides of rhombus.

17. If $D$, $E$, $F$ are three points on the sides $BC$, $AC$ and $AB$ of a triangle $ABC$ such that $AD$, $BE$ and $CF$ are concurrent, then show that $BD \times CE \times AF = DC \times EA \times FB$.

18. Let the sides of a parallelogram be $U = a$, $U = b$, $V = a'$ and $V = b'$ where $U = lx + my + n$, $V = l'x + m'y + n'$. Show that the equation of the diagonal through the point of intersection of $U = a$ and $V = a'$ and $U = b$ and $U = b'$ is given by

$$\begin{vmatrix} U & V & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix} = 0$$

19. Consider two lines $L_1$ and $L_2$ given by $x - y = 0$, and $x + y = 0$, respectively, and a moving point $P(x, y)$. Let $d(P, L_i)$, $i = 1, 2$ represents the distance of point '$P$' from the line $L_i$. If point '$P$' moves in certain region '$R$' in such a way that $2 \le d(P, L_1) + d(P, L_2) \le 4$. Find the area of region $R$.

20. If $(x, y)$ and $(X, Y)$ be the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux + vy$, where $u$ and $v$ are independent of $x$ and $y$, becomes $VX + UY$, show that $u^2 + v^2 = U^2 + V^2$.

21. Consider two lines $L_1$ and $L_2$ given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, respectively, where $c_1, c_2 \neq 0$, intersecting at point $P$. A line $L_3$ is drawn through the origin meeting the lines $L_1$ and $L_2$ at $A$ and $B$, respectively, such that $PA = PB$. Similarly, one more line $L_4$ is drawn through the origin meeting the lines $L_1$ and $L_2$ at $A_1$ and $B_1$, respectively, such that $PA_1 = PB_1$. Obtain the combined equation of lines $L_3$ and $L_4$.

22. Show that if any line through the variable point $A(k + 1, 2k)$ meets the lines $7x + y - 16 = 0$, $5x - y - 8 = 0$, $x - 5y + 8 = 0$ at $B$, $C$, $D$, respectively, then $AC$, $AB$ and $AD$ are in harmonic progression. (The three lines lie on the same side of point $A$.)

23. Show that the lines $4x + y - 9 = 0$, $x - 2y + 3 = 0$, $5x - y - 6 = 0$ make equal intercepts on any line of slope 2.

24. Having given the bases and the sum of the areas of a number of triangles which have a common vertex, show that the locus of the vertex is a straight line.

25. Find the locus of the point at which two given portions of the straight line subtend equal angle.

26. A right angled triangle $ABC$ having $C$ as right angle is of given magnitude and the angular points $A$ and $B$ slide along two given perpendicular axes. Show that the locus of $C$ is the pair of straight lines whose equations are $y = \pm (b/a)\,x$.

27. Let $2x + 3y = 6$ be a line meeting the coordinate axes at $A$ and $B$, respectively. A variable line $x/a + y/b = 1$ meets the axes at $P$ and $Q$, respectively, in such a way that the lines $BP$ and $AQ$ always meet at right angle at $R$. Find the locus of the orthocentre of the triangle $ARB$.

28. If $\theta$ is an angle between the lines given by the equation $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$, then find the equation of the line passing through the point of intersection of these lines and making an angle $\theta$ with the positive $x$-axis.

## Objective Type

*Solutions on page* 1.79

**Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* is correct. Find the correct answer.**

1. Equations of diagonals of square formed by lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are
   **a.** $y = x, y + x = 1$ **b.** $y = x, x + y = 2$
   **c.** $2y = x, y + x = 1/3$ **d.** $y = 2x, y + 2x = 1$

2. If each of the points $(x_1, 4)$, $(-2, y_1)$ lies on the line joining the points $(2, -1)$, $(5, -3)$, then the point $P(x_1, y_1)$ lies on the line
   **a.** $6(x + y) - 25 = 0$ **b.** $2x + 6y + 1 = 0$
   **c.** $2x + 3y - 6 = 0$ **d.** $6(x + y) + 25 = 0$

3. If
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$
then the two triangles with vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ and $(a_1, b_1)$, $(a_2, b_2)$, $(a, b_3)$ are

a. equal in area  b. similar
c. congruent  d. none of these

4. The area of a parallelogram formed by the lines $ax \pm bx \pm c = 0$ is
a. $c^2/(ab)$  b. $2c^2/(ab)$
c. $c^2/2ab$  d. none of these

5. The straight line $ax + by + c = 0$ where $abc \neq 0$ will pass through the first quadrant if
a. $ac > 0, bc > 0$  b. $c > 0$ and $bc < 0$
c. $bc > 0$ and/or $ac > 0$  d. $ac < 0$ and/or $bc < 0$

6. If the point $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of $(x_1, y_1)$ and $(x_2, y_2)$ internally, then
a. $t < 0$  b. $0 < t < 1$
c. $t > 1$  d. $t = 1$

7. A square of side a lies above the $x$-axis and has one vertex at the origin. The side passing trough the origin makes an angle $\alpha (0 < \alpha < \pi/4)$ with the positive direction of $x$-axis. The equation of its diagonal not passing through the origin is
a. $y(\cos\alpha + \sin\alpha) + x(\sin\alpha - \cos\alpha) = a$
b. $y(\cos\alpha + \sin\alpha) + x(\sin\alpha + \cos\alpha) = a$
c. $y(\cos\alpha + \sin\alpha) + x(\cos\alpha - \sin\alpha) = a$
d. $y(\cos\alpha - \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$

8. If sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is
a. a square  b. a circle
c. a straight line  d. two intersecting lines

9. The area enclosed by $2|x| + 3|y| \leq 6$ is
a. 3 sq. units  b. 4 sq. units
c. 12 sq. units  d. 24 sq. units

10. If two vertices of a triangle are $(-2, 3)$ and $(5, -1)$, orthocentre lies at the origin and centroid on the line $x + y = 7$, then the third vertex lies at
a. (7, 4)  b. (8, 14)
c. (12, 21)  d. none of these

11. $OPQR$ is a square and $M$, $N$ are the middle points of the sides $PQ$ and $QR$, respectively, then the ratio of the areas of the square and the triangle $OMN$ is
a. 4:1  b. 2:1
c. 8:3  d. 7:3

12. The vertices of a triangle are $(pq, 1/(pq))$, $(qr, 1/(qr))$ and $(rq, 1/(rp))$ where $p, q, r$ are the roots of the equation $y^3 - 3y^2 + 6y + 1 = 0$. The coordinates of its centroid are
a. (1, 2)  b. (2, −1)
c. (1, − 1)  d. (2, 3)

13. The lines $y = m_1x$, $y = m_2x$ and $y = m_3x$ make equal intercepts on the line $x + y = 1$, then
a. $2(1 + m_1)(1 + m_3) = (1 + m_2)(2 + m_1 + m_3)$
b. $(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$
c. $(1 + m_1)(1 + m_2) = (1 + m_3)(2 + m_1 + m_3)$
d. $2(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$

14. The condition on a and $b$, such that the portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y = 0$ and $x + by = 0$, subtends a right angle at the origin is
a. $a = b$  b. $a + b = 0$
c. $a = 2b$  d. $2a = b$

15. One diagonal of a square is along the line $8x - 15y = 0$ and one of its vertex is (1, 2). Then the equations of the sides of the square passing through this vertex are
a. $23x + 7y = 9$, $7x + 23y = 53$
b. $23x - 7y + 9 = 0$, $7x + 23y + 53 = 0$
c. $23x - 7y - 9 = 0$, $7x + 23y - 53 = 0$
d. none of these

16. If $u = a_1x + b_1y + c_1 = 0$, $v = a_2x + b_2y + c_2 = 0$ and $a_1/a_2 = b_1/b_2 = c_1/c_2$, then the curve $u + kv = 0$ is
a. the same straight line $u$
b. different straight line
c. not a straight line
d. none of these

17. The point $A(2, 1)$ is translated parallel to the line $x - y = 3$ by a distance 4 units. If the new position $A'$ is in third quadrant, then the coordinates of $A'$ are
a. $(2 + 2\sqrt{2}, 1 + 2\sqrt{2})$  b. $(-2 + \sqrt{2}, -1 - 2\sqrt{2})$
c. $(2 - 2\sqrt{2}, 1 - 2\sqrt{2})$  d. none of these

18. The area of the triangle formed by the lines $y = ax$, $x + y - a = 0$ and the $y$-axis is equal to
a. $\dfrac{1}{2|1 + a|}$  b. $\dfrac{a^2}{|1 + a|}$
c. $\dfrac{1}{2}\left|\dfrac{a}{1 + a}\right|$  d. $\dfrac{a^2}{2|1 + a|}$

19. An equation of a line through the point (1, 2) whose distance from the point (3, 1) has the greatest value is
a. $y = 2x$  b. $y = x + 1$
c. $x + 2y = 5$  d. $y = 3x - 1$

20. A rectangle $ABCD$ has its side $AB$ parallel to line $y = x$ and vertices $A$, $B$ and $D$ lie on $y = 1$, $x = 2$ and $x = -2$, respectively. Locus of vertex '$C$' is
a. $x = 5$  b. $x - y = 5$
c. $y = 5$  d. $x + y = 5$

21. The centroid of an equilateral triangle is (0, 0). If two vertices of the triangle lie on $x + y = 2\sqrt{2}$ then one of them will have its coordinates

a. $(\sqrt{2}+\sqrt{6}, \sqrt{2}-\sqrt{6})$ b. $(\sqrt{2}+\sqrt{3}, \sqrt{2}-\sqrt{3})$
c. $(\sqrt{2}+\sqrt{5}, \sqrt{2}-\sqrt{5})$ d. none of these

**22.** Vertices of a variable triangle are (3, 4), ($5\cos\theta$, $5\sin\theta$) and ($5\sin\theta$, $-5\cos\theta$), where $\theta \in R$. Locus of its orthocentre is

a. $(x+y-1)^2+(x-y-7)^2=100$
b. $(x+y-7)^2+(x-y-1)^2=100$
c. $(x+y-7)^2+(x+y-1)^2=100$
d. $(x+y-7)^2+(x-y+1)^2=100$

**23.** The line $x/3 + y/4 = 1$ meets the $y$-axis and $x$-axis at $A$ and $B$, respectively. A square $ABCD$ is constructed on the line segment $AB$ away from the origin. The coordinates of the vertex of the square farthest from the origin are

a. (7, 3) b. (4, 7)
c. (6, 4) d. (3, 8)

**24.** A line of fixed length 2 units moves so that its ends are on the positive $x$-axis and that part of the line $x + y = 0$ which lies in the second quadrant. Then the locus of the midpoint of the line has the equation

a. $x^2+5y^2+4xy-1=0$ b. $x^2+5y^2+4xy+1=0$
c. $x^2+5y^2-4xy-1=0$ d. $4x^2+5y^2+4xy+1=0$

**25.** Consider the points $A(0, 1)$ and $B(2, 0)$, and $P$ be a point on the line $4x + 3y + 9 = 0$. Coordinates of $P$ such that $|PA - PB|$ is maximum are

a. (– 12/5, 17/5) b. (– 84/5, 13/5)
c. (– 6/5, 17/5) d. (0, – 3)

**26.** Consider points $A(3, 4)$ and $B(7, 13)$. If $P$ be a point on the line $y = x$ such that $PA + PB$ is minimum, then coordinates of $P$ are

a. (12/7, 12/7) b. (13/7, 13/7)
c. (31/7, 31/7) d. (0, 0)

**27.** A line '$L$' is drawn from $P(4, 3)$ to meet the lines $L_1$ and $L_2$ given by $3x + 4y + 5 = 0$ and $3x + 4y + 15 = 0$ at points $A$ and $B$, respectively. From '$A$', a line perpendicular to $L$ is drawn meeting the line $L_2$ at $A_1$. Similarly from point '$B$', a line perpendicular to $L$ is drawn meeting the line $L_1$ at $B_1$. Thus a parallelogram $AA_1BB_1$ is formed. Then the the equation of '$L$' so that the area of the parallelogram $AA_1BB_1$ is least is

a. $x-7y+17=0$ b. $7x+y+31=0$
c. $x-7y-17=0$ d. $x+7y-31=0$

**28.** Line $L$ has intercepts $a$ and $b$ on the coordinate axes. When the axes are rotated through a given angle keeping the origin fixed, the same line $L$ has intercepts $p$ and $q$. Then,

a. $a^2+b^2=P^2+q^2$ b. $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$
c. $a^2+p^2=b^2+q^2$ d. $\frac{1}{a^2}+\frac{1}{p^2}=\frac{1}{b^2}+\frac{1}{q^2}$

**29.** If the straight lines $2x + 3y - 1 = 0$, $x + 2y - 1 = 0$ and $ax + by - 1 = 0$ form a triangle with origin as orthocentre, then $(a, b)$ is given by

a. (6, 4) b. (–3, 3)
c. (–8, 8) d. (0, 7)

**30.** A light ray emerging from the point source placed at $P(2, 3)$ is reflected at a point '$Q$' on the $y$-axis and then passes through the point $R(5, 10)$. Coordinates of '$Q$' are

a. (0, 3) b. (0, 2)
c. (0, 5) d. none of these

**31.** A straight line passing through $P(3, 1)$ meets the coordinate axes at $A$ and $B$. It is given that distance of this straight line from the origin '$O$' is maximum. Area of triangle $OAB$ is equal to

a. 50/3 sq. units b. 25/3 sq. units
c. 20/3 sq. units d. 100/3 sq. units

**32.** Let $O$ be the origin. If $A(1, 0)$ and $B(0, 1)$ and $P(x, y)$ are points such that $xy > 0$ and $x + y < 1$, then

a. $P$ lies either inside the triangle $OAB$ or in the third quadrant
b. $P$ cannot lie inside the triangle $OAB$
c. $P$ lies inside the triangle $OAB$
d. $P$ lies in the first quadrant only

**33.** The graph $y^2 + 2xy + 40|x| = 400$ divides the plane into regions. Then the area of bounded region is

a. 200 sq. units b. 400 sq. units
c. 800 sq. units d. 500 sq. units

**34.** In a triangle $ABC$, $A \equiv (\alpha, \beta)$, $B \equiv (2, 3)$ and $C \equiv (1, 3)$ and point $A$ lies on line $y = 2x + 3$ where $\alpha \in I$. Area of $\Delta ABC$, $\Delta$, is such that $[\Delta] = 5$. Possible coordinates of $A$ are (where $[\cdot]$ represents greatest integer function)

a. (2, 3) b. (5, 13)
c. (–5, –7) d. (–3, –5)

**35.** If $a/bc - 2 = \sqrt{b/c} + \sqrt{c/b}$, where $a, b, c > 0$, then family of lines $\sqrt{a}x + \sqrt{b}y + \sqrt{c} = 0$ passes though the fixed point given by

a. (1, 1) b. (1, – 2)
c. (–1, 2) d. (–1, 1)

**36.** $P(m, n)$ (where $m$, $n$ are natural numbers) is any point in the interior of the quadrilateral formed by the pair of lines $xy = 0$ and the lines $2x + y - 2 = 0$ and $4x + 5y = 20$. The possible number of positions of the point $P$ is

a. 7 b. 5
c. 4 d. 6

**37.** If the area of the rhombus enclosed by the lines $lx \pm my \pm n = 0$ be 2 sq. units, then

a. $l, m, n$ are in G.P. b. $l, n, m$ are in G.P.
c. $lm = n$ d. $ln = m$

38. $P$, $Q$, $R$ and $S$ are the points of intersection with the coordinate axes of the lines $px + qy = pq$ and $qx + py = pq$, then $(P, Q > 0)$
a. $P, Q, R, S$ from a parallelogram
b. $P, Q, R, S$ from a rhombus
c. $P, Q, R, S$ are concyclic
d. none of these

39. In a triangle $ABC$ if $A \equiv (1, 2)$ and internal angle bisectors through $B$ and $C$ are $y = x$ and $y = -2x$, then the inradius $r$ of the $\Delta ABC$ is
a. $1/\sqrt{3}$ b. $1/\sqrt{2}$
c. $2/3$ d. none of these

40. In a triangle $ABC$, the bisectors of angles $B$ and $C$ lie along the lines $x = y$ and $y = 0$. If $A$ is (1, 2), then the equation of line $BC$ is
a. $2x + y = 1$ b. $3x - y = 5$
c. $x - 2y = 3$ d. $x + 3y = 1$

41. If the vertices of a triangle are $(\sqrt{5}, 0)$, $(\sqrt{3}, \sqrt{2})$ and $(2, 1)$, then the orthocentre of the triangle is
a. $(\sqrt{5}, 0)$ b. $(0, 0)$
c. $(\sqrt{5} + \sqrt{3} + 2, \sqrt{2} + 1)$ d. none of these

42. One of the diagonals of a square is the portion of the line $x/2 + y/3 = 2$ intercepted between the axes. Then the extremities of the other diagonal are
a. $(5, 5), (-1, 1)$ b. $(0, 0), (4, 6)$
c. $(0, 0), (-1, 1)$ d. $(5, 5), (4, 6)$

43. Two sides of a triangle are along the coordinate axes and the medians through the vertices (other than the origin) are mutually perpendicular. The number of such triangles is/are
a. zero b. two c. four d. infinite

44. A rectangular billiard table has vertices at $P$ (0, 0), $Q$ (0, 7), $R$ (10, 7) and $S$ (10, 0). A small billiard ball starts at $M$ (3, 4) and moves in a straight line to the top of the table, bounces to the right side of the table, then comes to rest at $N$ (7, 1). The $y$-coordinate of the point where it hits the right side, is
a. 3.7 b. 3.8 c. 3.9 d. 4

45. In $\Delta ABC$ the coordinates of the vertex $A$ are $(4, -1)$ and lines $x - y - 1 = 0$ and $2x - y = 3$ are internal bisectors of angles $B$ and $C$. Then, radius of incircle of triangle $ABC$ is
a. $4/\sqrt{5}$ b. $3/\sqrt{5}$
c. $6/\sqrt{5}$ d. $7/\sqrt{5}$

46. Distance of origin from line $(1 + \sqrt{3})y + (1 - \sqrt{3})x = 10$ along the line $y = \sqrt{3}x + k$ is
a. $5/\sqrt{2}$ b. $5\sqrt{2} + k$
c. 10 d. 0

47. If it is possible to draw a line which belongs to all the given family of lines $y - 2x + 1 + \lambda_1(2y - x - 1) = 0$, $3y - x - 6 + \lambda_2(y - 3x + 6) = 0$, $ax + y - 2 + \lambda_3(6x + ay - a) = 0$, then
a. $a = 4$ b. $a = 3$
c. $a = -2$ d. $a = 2$

48. If the equation of any two diagonals of a regular pentagon belongs to family of lines $(1 + 2\lambda)y - (2 + \lambda)x + 1 - \lambda = 0$ and their lengths are $\sin 36°$, then locus of centre of circle circumscribing the given pentagon (the triangles formed by these diagonals with sides of pentagon have no side common) is
a. $x^2 + y^2 - 2x - 2y + 1 + \sin^2 72° = 0$
b. $x^2 + y^2 - 2x - 2y + \cos^2 72° = 0$
c. $x^2 + y^2 - 2x - 2y + 1 + \cos^2 72° = 0$
d. $x^2 + y^2 - 2x - 2y + \sin^2 72° = 0$

49. $ABC$ is a variable triangle such that $A$ is (1, 2), $B$ and $C$ lie on line $y = x + \lambda$ (where $\lambda$ is a variable), then locus of the orthocentre of triangle $ABC$ is
a. $(x - 1)^2 + y^2 = 4$ b. $x + y = 3$
c. $2x - y = 0$ d. none of these

50. Locus of the image of the point (2, 3) in the line $(x - 2y + 3) + \lambda(2x - 3y + 4) = 0$ is $(\lambda \in R)$
a. $x^2 + y^2 - 3x - 4y - 4 = 0$
b. $2x^2 + 3y^2 + 2x + 4y - 7 = 0$
c. $x^2 + y^2 - 2x - 4y + 4 = 0$
d. none of these

51. If one side of a rhombus has end points (4, 5) and (1, 1) then the maximum area of the rhombus is
a. 50 sq. units b. 25 sq. units
c. 30 sq. units d. 20 sq. units

52. The equations of the sides of a triangle are $x + y - 5 = 0$, $x - y + 1 = 0$ and $y - 1 = 0$. Then the coordinates of the circumcentre are
a. (2, 1) b. (1, 2)
c. (2, −2) d. 1, −2)

53. Two vertices of a triangle are (4, −3) and (−2, 5). If the orthocentre of the triangle is at (1, 2), then the third vertex is
a. (−33, −26) b. (33, 26)
c. (26, 33) d. none of these

54. If $\sum_{i=1}^{4} (x_i^2 + y_i^2) \le 2x_1x_3 + 2x_2x_4 + 2y_2y_3 + 2y_1y_4$ the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are
a. the vertices of a rectangle

**b.** collinear

**c.** trapezium

**d.** none of these

**55.** Locus of a point that is equidistant from the lines $x + y - 2\sqrt{2} = 0$ and $x + y - \sqrt{2} = 0$ is

**a.** $x + y - 5\sqrt{2} = 0$ **b.** $x + y - 3\sqrt{2} = 0$

**c.** $2x + 2y - 3\sqrt{2} = 0$ **d.** $2x + 2y - 5\sqrt{2} = 0$

**56.** If the intercept made on the line $y = mx$ by lines $y = 2$ and $y = 6$ is less than 5, then the range of values of $m$ is

**a.** $(-\infty, -4/3) \cup (4/3, +\infty)$

**b.** $(-4/3, 4/3)$

**c.** $(-3/4, 4/3)$

**d.** none of these

**57.** If $P\,(1 + t/\sqrt{2},\ 2 + t/\sqrt{2})$ be any point on a line, then the range of the values of $t$ for which the point $P$ lies between the parallel lines $x + 2y = 1$ and $2x + 4y = 15$ is

**a.** $-4\sqrt{2}/3 < t < 5\sqrt{2}/6$

**b.** $0 < t < 5\sqrt{2}/6$

**c.** $4\sqrt{2}/ < t < 0$

**d.** none of these

**58.** $P$ is a point on the line $y + 2x = 1$ and, $Q$ and $R$ are two points on the line $3y + 6x = 6$ such that triangle $PQR$ is an equilateral triangle. The length of the side of the triangle is

**a.** $2/\sqrt{5}$ **b.** $3/\sqrt{5}$

**c.** $4/\sqrt{5}$ **d.** none of these

**59.** The coordinates of two consecutive vertices $A$ and $B$ of a regular hexagon $ABCDEF$ are (1, 0) and (2, 0), respectively. The equation of the diagonal $CE$ is

**a.** $\sqrt{3}x + y = 4$ **b.** $x + \sqrt{3}y + 4 = 0$

**c.** $x + \sqrt{3}y = 4$ **d.** none of these

**60.** The foot of the perpendicular on the line $3x + y = \lambda$ drawn from the origin is $C$. If the line cuts the $x$-axis and $y$-axis at $A$ and $B$, respectively, then $BC:CA$ is

**a.** 1:3 **b.** 3:1

**c.** 1:9 **d.** 9:1

**61.** If $x_1, x_2, x_3$ as well as $y_1, y_2, y_3$ are in G.P. with same common ratio, then the points $P\,(x_1, y_1)$, $Q\,(x_2, y_2)$ and $R\,(x_3, y_3)$

**a.** lie on a straight line **b.** lie on an ellipse

**c.** lie on a circle **d.** are vertices of a triangle

**62.** If the quadrilateral formed by the lines $ax + by + c = 0$, $a'x + b'y + c = 0$, $ax + by + c' = 0$, $a'x + b'y + c' = 0$ have perpendicular diagonals, then

**a.** $b^2 + c^2 = b'^2 + c'^2$ **b.** $c^2 + a^2 = c'^2 + a'^2$

**c.** $a^2 + b^2 = a'^2 + b'^2$ **d.** none of these

**63.** The straight lines $7x - 2y + 10 = 0$ and $7x + 2y - 10 = 0$ form an isosceles triangle with the line $y = 2$. Area of this triangle is equal to

**a.** 15/7 sq. units **b.** 10/7 sq. units

**c.** 18/7 sq. units **d** none of these

**64.** $A$ rectangle $ABCD$, where $A \equiv (0, 0)$, $B \equiv (4, 0)$, $C \equiv (4, 2)$, $D \equiv (0, 2)$, undergoes the following transformations successively: (i) $f_1(x, y) \to (y, x)$, (ii) $f_2(x, y) \to (x + 3y, y)$, (iii) $f_3(x, y) \to ((x - y)/2, (x + y)/2)$. The final figure will be

**a.** a square **b.** a rhombus

**c.** a rectangle **d.** a parallelogram

**65.** $\theta_1$ and $\theta_2$ are the inclination of lines $L_1$ and $L_2$ with $x$-axis. If $L_1$ and $L_2$ pass through $P\,(x_1, y_1)$, then equation of one of the angle bisector of these lines is

**a.** $\dfrac{x - x_1}{\cos\left(\dfrac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\sin\left(\dfrac{\theta_1 - \theta_2}{2}\right)}$

**b.** $\dfrac{x - x_1}{-\sin\left(\dfrac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\cos\left(\dfrac{\theta_1 - \theta_2}{2}\right)}$

**c.** $\dfrac{x - x_1}{\sin\left(\dfrac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\cos\left(\dfrac{\theta_1 - \theta_2}{2}\right)}$

**d.** $\dfrac{x - x_1}{-\sin\left(\dfrac{\theta_1 - \theta_2}{2}\right)} = \dfrac{y - y_1}{\cos\left(\dfrac{\theta_1 + \theta_2}{2}\right)}$

**66.** A light ray coming along the line $3x + 4y = 5$ gets reflected from the line $ax + by = 1$ and goes along the line $5x - 12y = 10$. Then,

**a.** $a = 64/115$, $b = 112/15$

**b.** $a = 14/15$, $b = -8/115$

**c.** $a = 64/115$, $b = -8/115$

**d.** $a = 64/15$, $b = 14/15$

**67.** Line $ax + by + p = 0$ makes angle $\pi/4$ with $x \cos\alpha + y \sin\alpha = p$, $p \in R^+$. If these lines and the line $x \sin\alpha - y\cos\alpha = 0$ are concurrent, then

**a.** $a^2 + b^2 = 1$ **b.** $a^2 + b^2 = 2$

**c.** $2(a^2 + b^2) = 1$ **d.** none of these

**68.** A line is drawn perpendicular to line $y = 5x$, meeting the coordinate axes at $A$ and $B$. If the area of triangle $OAB$ is 10 sq. units where '$O$' is the origin, then the equation of drawn line is

**a.** $3x - y - 9$ **b.** $x - 5y = 10$

**c.** $x + 4y = 10$ **d.** $x - 4y = 10$

**69.** If $x - 2y + 4 = 0$ and $2x + y - 5 = 0$ are the sides of a isosceles triangle having area 10 sq. units the equation of third side is

**a.** $3x - y = -9$ **b.** $3x - y + 11 = 0$
**c.** $x - 3y = 19$ **d.** $3x - y + 15 = 0$

**70.** The number of values of '$a$' for which the lines

$2x + y - 1 = 0,$
$ax + 3y - 3 = 0,$ and
$3x + 2y - 2 = 0$
are concurrent is

**a.** 0 **b.** 1 **c.** 2 **d.** infinite

**71.** The extremities of the base of an isosceles triangle are (2, 0) and (0, 2). If the equation of one of the equal side is $x = 2$, then equation of other equal side is

**a.** $x + y = 2$ **b.** $x - y + 2 = 0$
**c.** $y = 2$ **d.** $2x + y = 2$

**72.** If the equation of the locus of a point equidistant from the points $(a_1, b_1)$ and $(a_2, b_2)$ is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of $c$ is

**a.** $a_1^2 - a_2^2 + b_1^2 - b_2^2$ **b.** $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$
**c.** $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ **d.** $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$

**73.** The line $PQ$ whose equation is $x - y = 2$ cuts the $x$-axis at $P$ and $Q$ is (4, 2). The line $PQ$ is rotated about $P$ through 45° in the anticlockwise direction. The equation of the line $PQ$ in the new position is

**a.** $y = -\sqrt{2}$ **b.** $y = 2$ **c.** $x = 2$ **d.** $x = -2$

**74.** The equation of straight line passing through $(-a, 0)$ and making the triangle with axes of area '$T$' is

**a.** $2Tx + a^2y + 2aT = 0$ **b.** $2Tx - a^2y + 2aT = 0$
**c.** $2Tx - a^2y - 2aT = 0$ **d.** none of these

**75.** If the equation of base of an equilateral triangle is $2x - y = 1$ and the vertex is $(-1, 2)$, then the length of the sides of the triangle is

**a.** $\sqrt{\frac{20}{3}}$ **b.** $\frac{2}{\sqrt{15}}$ **c.** $\sqrt{\frac{8}{15}}$ **d.** $\sqrt{\frac{15}{2}}$

**76.** The number of integral values of $m$, for which the $x$-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is

**a.** 2 **b.** 0 **c.** 4 **d.** 1

**77.** Equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the $x$-axis is

**a.** $x\sqrt{3} + y + 8 = 0$ **b.** $x\sqrt{3} - y = 8$
**c.** $x\sqrt{3} - y = 8$ **d.** $x - \sqrt{3}y + 8 = 0$

**78.** The equation to the straight line passing through the point $(a \cos^3 \theta, a \sin^3 \theta)$ and perpendicular to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ is

**a.** $x \cos \theta - y \sin \theta = a \cos 2\theta$
**b.** $x \cos \theta + y \sin \theta = a \cos 2\theta$
**c.** $x \sin \theta + y \cos \theta = a \cos 2\theta$
**d.** none of these

**79.** The line parallel to the $x$-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$, is

**a.** above the $x$-axis at a distance of 3/2 units from it
**b.** above the $x$-axis at a distance of 2/3 units from it
**c.** below the $x$-axis at a distance of 3/2 units from it
**d.** below the $x$-axis at a distance of 2/3 units from it

**80.** If a straight line through origin bisects the line passing through the given points $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$, then the lines

**a.** are perpendicular
**b.** are parallel
**c.** have an angle between them of $\pi/4$
**d.** none of these

**81.** The straight lines $4ax + 3by + c = 0$, where $a + b + c = 0$, are concurrent at the point

**a.** (4, 3) **b.** (1/4, 1/3)
**c.** (1/2, 1/3) **d.** none of these

**82.** The straight lines $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by - 1 = 0$ are concurrent, if the straight line $35x - 22y + 1 = 0$ passes through the point

**a.** $(a, b)$ **b.** $(b, a)$ **c.** $(-a, -b)$ **d.** none of these

**83.** The coordinates of the foot of the perpendicular from the point (2, 3) on the line $-y + 3x + 4 = 0$ are given by

**a.** (37/10, – 1/10) **b.** (– 1/10, 37/10)
**c.** (10/37, – 10) **d.** (2/3, – 1/3)

**84.** If the extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$ and the equation of one of the sides is $x = 2a$, then the area of the triangle is

**a.** $5a^2$ sq. units **b.** $5a^2/2$ sq. units
**c.** $25a^2/2$ sq. units **d.** none of these

**85.** The combined equation of straight lines that can be obtained by reflecting the lines $y = |x - 2|$ in the $y$-axis is

**a.** $y^2 + x^2 + 4x + 4 = 0$ **b.** $y^2 + x^2 - 4x + 4 = 0$
**c.** $y^2 - x^2 + 4x - 4 = 0$ **d.** $y^2 - x^2 - 4x - 4 = 0$

**86.** If the slope of one line represented by $a^3x^2 - 2hxy + b^3y^2 = 0$ is square of the slope of another line, then

**a.** $h = 2ab(a + b)$ **b.** $h = ab(a + b)$
**c.** $3h = 2ab(a + b)$ **d.** $2h = ab(a + b)$

**87.** $A \equiv (-4, 0)$, $B \equiv (4, 0)$. $M$ and $N$ are the variable points of $y$-axis such that $M$ lies below $N$ and $MN = 4$. Line joining $AM$ and $BN$ intersect at '$P$'. Locus of '$P$' is

**a.** $2xy - 16 - x^2 = 0$ **b.** $2xy + 16 - x^2 = 0$

**c.** $2xy + 16 + x^2 = 0$ **d.** $2xy - 16 + x^2 = 0$

**88.** Let $A_r$, $r = 1, 2, 3, \ldots$ be points on the number line such that $OA_1, OA_2, OA_3, \ldots$ are in G.P. where $O$ is the origin and the common ratio of the G.P. be a positive proper fraction. Let $M_r$ be the middle point of the line segment $A_rA_{r+1}$. Then the value of $\sum_{r=1}^{\infty} OM_r$ is equal to

**a.** $\dfrac{OA_1(OSA_1 - OA_2)}{2(OA_1 + OA_2)}$ **b.** $\dfrac{OA_1(OA_1 - OA_2)}{2(OA_1 + OA_2)}$

**c.** $\dfrac{OA_1}{2(OA_1 - OA_2)}$ **d.** $\infty$

**89.** The number of triangles that the four lines $y = x + 3$, $y = 2x + 3$, $y = 3x + 2$ and $y + x = 3$ form is

**a.** 4 **b.** 2 **c.** 3 **d.** 1

**90.** Let $A = (3, -4)$, $B = (1, 2)$, let $P = (2k - 1, 2k + 1)$ be a variable point such that $PA + PB$ is the minimum. Then $k$ is

**a.** 7/9 **b.** 0 **c.** 7/8 **d.** none of these

**91.** $L_1$ and $L_2$ are two lines. If the reflection of $L_1$ in $L_2$ and the reflection of $L_2$ in $L_1$ coincide, then the angle between the lines is

**a.** 30° **b.** 60° **c.** 45° **d.** 90°

**92.** If the straight lines $x + y - 2 = 0$, $2x - y + 1 = 0$ and $ax + by - c = 0$ are concurrent, then the family of lines $2ax + 3by + c = 0$ ($a, b, c$ are nonzero) is concurrent at

**a.** $(2, 3)$ **b.** $(1/2, 1/3)$

**c.** $(-1/6, -5/9)$ **d.** $(2/3, -7/5)$

**93.** Two medians drawn from acute angles of a right angled triangle intersect at an angle $\pi/6$. If the length of the hypotenuse of the triangle is 3 units, then area of the triangle (in sq. units) is

**a.** $\sqrt{3}$ **b.** 3 **c.** $\sqrt{2}$ **d.** 9

**94.** A variable line $x/a + y/b = 1$ moves in such a way that harmonic mean of $a$ and $b$ is 8. Then the least area of triangle made by the line with the coordinate axes is

**a.** 8 sq. unit **b.** 16 sq. unit

**c.** 32 sq. unit **d.** 64 sq. unit

**95.** The number of integral points $(x, y)$ (i.e., $x$ and $y$ both are integers) which lie in the first quadrant but not on the coordinate axes and also on the straight line $3x + 5y = 2007$ is equal to

**a.** 133 **b.** 135 **c.** 138 **d.** 140

**96.** The number of straight lines equidistant from three non-collinear points in the plane of the points is equal to

**a.** 0 **b.** 1 **c.** 2 **d.** 3

**97.** The locus of the point which moves such that its distance from the point $(4, 5)$ is equal to its distance from the line $7x - 3y - 13 = 0$ is

**a.** a straight line **b.** a circle

**c.** a parabola **d.** an ellipse

**98.** In $\Delta ABC$ if orthocentre be $(1, 2)$ and circumcentre be $(0, 0)$, then centroid of $\Delta ABC$ is

**a.** $(1/2, 2/3)$ **b.** $(1/3, 2/3)$

**c.** $(2/3, 1)$ **d.** none of these

**99.** The line $x + 3y - 2 = 0$ bisects the angle between a pair of straight lines of which one has equation $x - 7y + 5 = 0$. The equation of the other line is

**a.** $3x + 3y - 1 = 0$ **b.** $x - 3y + 2 = 0$

**c.** $5x + 5y - 3 = 0$ **d.** none of these

**100.** Through a point $A$ on the $x$-axis a straight line is drawn parallel to $y$-axis so as to meet the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ in $B$ and $C$. If $AB = BC$, then

**a.** $h^2 = 4ab$ **b.** $8h^2 = 9ab$ **c.** $9h^2 = 8ab$ **d.** $4h^2 = ab$

**101.** If $A(1, p^2)$, $B(0, 1)$ and $C(p, 0)$ are the coordinates of three points, then the value of $p$ for which the area of the triangle $ABC$ is minimum is

**a.** $1/\sqrt{3}$ **b.** $-1/\sqrt{3}$

**c.** $1/\sqrt{2}$ **d.** none of these

**102.** $m, n$ are integers with $0 < n < m$. $A$ is the point $(m, n)$ on the Cartesian plane. $B$ is the reflection of $A$ in the line $y = x$. $C$ is the reflection of $B$ in the $y$-axis, $D$ is the reflection of $C$ in the $x$-axis and $E$ is the reflection of $D$ in the $y$-axis. The area of the pentagon $ABCDE$ is

**a.** $2m(m + n)$ **b.** $m(m + 3n)$

**c.** $m(2m + 3n)$ **d.** $2m(m + 3n)$

**103.** If the ends of the base of an isosceles triangle are at $(2, 0)$ and $(0, 1)$ and the equation of one side is $x = 2$, then the orthocentre of the triangle is

**a.** $(3/2, 3/2)$ **b.** $(5/4, 1)$ **c.** $(3/4, 1)$ **d.** $(4/3, 7/12)$

**104.** Vertices of a parallelogram $ABCD$ are $A$ (3, 1), $B$ (13, 6), $C$ (13, 21) and $D$ (3, 16). If a line passing through the origin divides the parallelogram into two congruent parts, then the slope of the line is

**a.** 11/12 **b.** 11/8 **c.** 25/8 **d.** 13/8

**105.** The distance between the two parallel lines is 1 unit. $A$ point '$A$' is chosen to lie between the lines at a distance '$d$' from one of them. Triangle $ABC$ is equilateral with $B$ on one line and $C$ on the other parallel line. The length of the side of the equilateral triangle is

**a.** $(2/3)\sqrt{d^2 + d + 1}$ **b.** $2\sqrt{(d^2 - d + 1)/3}$

c. $2\sqrt{d^2-d+1}$  d. $\sqrt{d^2-d+1}$

**106.** Given $A$ (0, 0) and $B$ $(x, y)$ with $x \in (0, 1)$ and $y > 0$. Let the slope of the line $AB$ equal to $m_1$. Point $C$ lies on the line $x = 1$ such that the slope of $BC$ equal to $m_2$ where $0 < m_2 < m_1$. If the area of the triangle $ABC$ can be expressed as $(m_1 - m_2) f(x)$, then the largest possible value of $x$ is

a. 1  b. 1/2  c. 1/4  d. 1/8

**107.** If the vertices $P$ and $Q$ of a triangle $PQR$ are given by (2, 5) and (4, −11), respectively, and the point $R$ moves along the line $N$ given by $9x + 7y + 4 = 0$, then the locus of the centroid of the triangle $PQR$ is a straight line parallel to

a. $PQ$  b. $QR$  c. $RP$  d. $N$

**108.** Given $A \equiv (1, 1)$ and $AB$ is any line through it cutting the $x$-axis in $B$. If $AC$ is perpendicular to $AB$ and meets the $y$-axis in $C$, then the equation of locus of midpoint $P$ of $BC$ is

a. $x + y = 1$  b. $x + y = 2$

c. $x + y = 2xy$  d. $2x + 2y = 1$

**109.** The number of possible straight lines, passing through (2, 3) and forming a triangle with coordinate axes, whose area is 12 sq. units, is

a. one  b. two  c. three  d. four

**110.** In a triangle $ABC$, if $A$ is (2, −1), and $7x - 10y + 1 = 0$ and $3x - 2y + 5 = 0$ are equations of an altitude and an angle bisector, respectively, drawn from $B$, then equation of $BC$ is

a. $x + y + 1 = 0$  b. $5x + y + 17 = 0$

c. $4x + 9y + 30 = 0$  d. $x - 5y - 7 = 0$

**111.** A triangle is formed by the lines $x + y = 0$, $x - y = 0$ and $lx + my = 1$. If $l$ and $m$ vary subject to the condition $l^2 + m^2 = 1$, then the locus of its circumcentre is

a. $(x^2 - y^2)^2 = x^2 + y^2$  b. $(x^2 + y^2)^2 = (x^2 - y^2)$

c. $(x^2 + y^2) = 4x^2y^2$  d. $(x^2 - y^2)^2 = (x^2 + y^2)^2$

**112.** The line $x + y = p$ meets the $x$- and $y$-axes at $A$ and $B$, respectively. $A$ triangle $APQ$ is inscribed in the triangle $OAB$, $O$ being the origin, with right angle at $Q$. $P$ and $Q$ lie, respectively, on $OB$ and $AB$. If the area of the triangle $APQ$ is $3/8^{\text{th}}$ of the area of the triangle $OAB$, then $AQ/BQ$ is equal to

a. 2  b. 2/3  c. 1/3  d. 3

**113.** $A$ is a point on either of two lines $y + \sqrt{3}|x| = 2$ at a distance of $4\sqrt{3}$ units from their point of intersection. The coordinates of the foot of perpendicular from $A$ on the bisector of the angle between them are

a. $(2/\sqrt{3}, 2)$  b. (0, 0)

c. $(2/\sqrt{3}, 2)$  d. (0, 4)

**114.** A pair of perpendicular straight lines is drawn through the origin forming with the line $2x + 3y = 6$ an isosceles triangle right angled at the origin. The equation to the line pair is

a. $5x^2 - 24xy - 5y^2 = 0$  b. $5x^2 - 26xy - 5y^2 = 0$

c. $5x^2 + 24xy - 5y^2 = 0$  d. $5x^2 + 26xy - 5y^2 = 0$

**115.** Points $A$ and $B$ are in the first quadrant; point '$O$' is the origin. If the slope of $OA$ is 1, slope of $OB$ is 7 and $OA = OB$, then the slope of $AB$ is

a. −1/5  b. −1/4

c. −1/3  d. −1/2

**116.** $OPQR$ is a square and $M$, $N$ are the midpoints of the sides $PQ$ and $QR$, respectively. If the ratio of the areas of the square and the triangle $OMN$ is $\lambda : 6$, then $\lambda/4$ is equal to

a. 2  b. 4

c. 2  d. 16

**117.** A triangle $ABC$ with vertices $A(-1, 0)$, $B(-2, 3/4)$ and $C(-3, -7/6)$ has its orthocentre $H$. Then the orthocentre of triangle $BCH$ will be

a. (−3, −2)  b. (1, 3)

c. (−1, 2)  d. none of these

**118.** If a pair of perpendicular straight lines drawn through the origin forms an isosceles triangle with the line $2x + 3y = 6$, then area of the triangle so formed is

a. 36/13  b. 12/17

c. 13/5  d. 17/13

**119.** The image of $P(a, b)$ in the line $y = -x$ is $Q$ and the image of $Q$ in the $y = x$ is $R$. Then the midpoint of $PR$ is

a. $(a + b, b + a)$  b. $((a + b)/2, (b + 2)/2)$

c. $(a - b, b - a)$  d. (0, 0)

**120.** Let $ABC$ be a triangle. Let $A$ be the point (1, 2), $y = x$ be the perpendicular bisector of $AB$ and $x - 2y + 1 = 0$ be the angle bisector of $\angle C$. If equation of $BC$ is given by $ax + by - 5 = 0$, then the value of $a + b$ is

a. 1  b. 2  c. 3  d. 4

**121.** If in triangle $ABC$, $A \equiv (1, 10)$, circumcentre $\equiv (-1/3, 2/3)$ and orthocentre $\equiv (11/4, 4/3)$, then the coordinates of mid-point of side opposite to $A$ is

a. (1, −11/3)  b. (1, 5)

c. (1, −3)  d. (1, 6)

**122.** In the $\Delta ABC$, the coordinates of $B$ are (0, 0), $AB = 2$, $\angle ABC = \pi/3$ and the middle point of $BC$ has the coordinates (2, 0). The centroid of the triangle is

a. $(1/2, \sqrt{3}/2)$  b. $(5/3, 1/\sqrt{3})$

c. $(4 + \sqrt{3}/3, 1/3)$  d. none of these

**123.** A beam of light is sent along the line $x - y = 1$, which after refracting from the $x$-axis enters the opposite side by turning through 30° towards the normal at the point of incidence on the $x$-axis. Then the equation of the refracted ray is

**a.** $(2-\sqrt{3})x - y = 2+\sqrt{3}$
**b.** $(2+\sqrt{3})x - y = 2+\sqrt{3}$
**c.** $(2-\sqrt{3})x + y = (2+\sqrt{3})$
**d.** $y = (2-\sqrt{3})(x-1)$

**124.** The equation of the straight line which passes through the point $(-4, 3)$ such that the portion of the line between the axes is divided internally by the point in the ratio 5 : 3 is

**a.** $9x - 20y + 96 = 0$ **b.** $9x + 20y = 24$
**c.** $20x + 9y + 53 = 0$ **d.** none of these

**125.** $ABCD$ is a square $A \equiv (1, 2)$, $B \equiv (3, -4)$. If line $CD$ passes through $(3, 8)$ then midpoint of $CD$ is

**a.** $(2, 6)$ **b.** $(6, 2)$ **c.** $(2, 5)$ **d.** $(24/5, 1/5)$

**126.** If the equations $y = mx + c$ and $x \cos\alpha + y \sin\alpha = p$ represent the same straight line, then

**a.** $p = c\sqrt{1+m^2}$ **b.** $c = p\sqrt{1+m^2}$
**c.** $cp = \sqrt{1+m^2}$ **d.** $p^2 + c^2 + m^2 = 1$

**127.** The equation of the bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y = 1$ is

**a.** $x + y + 5 = 0$ **b.** $x - y + 1 = 0$
**c.** $x - y = 5$ **d.** none of these

**128.** The equation of the line segment $AB$ is $y = x$. If $A$ and $B$ lie on the same side of the line mirror $2x - y = 1$, then the image of $AB$ has the equation

**a.** $x + y = 2$ **b.** $8x + y = 9$
**c.** $7x - y = 6$ **d.** none of these

**129.** The line $L_1 \equiv 4x + 3y - 12 = 0$ intersects the $x$- and the $y$-axes at $A$ and $B$, respectively. A variable line perpendicular to $L_1$ intersects the $x$- and the $y$-axes at $P$ and $Q$, respectively. Then the locus of the circumcentre of triangle $ABQ$ is

**a.** $3x - 4y + 2 = 0$ **b.** $4x + 3yy + 7 = 0$
**c.** $6x - 8y + 7 = 0$ **d.** none of these

**130.** Consider 3 lines as follows.

$L_1$: $5x - y + 4 = 0$
$L_2$: $3x - y + 5 = 0$
$L_3$: $x + y + 8 = 0$

If these lines enclose a triangle $ABC$ and sum of the squares of the tangent to the interior angles can be expressed in the form $p/q$ where $p$ and $q$ are relatively prime numbers, then the value of $p + q$ is

**a.** 500 **b.** 450
**c.** 230 **d.** 465

**131.** If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ ($a, b, c$ being distinct and different from 1) are concurrent, then $\left(\frac{1}{1-a}\right)+\left(\frac{1}{1-b}\right)+\left(\frac{1}{1-c}\right)=$

**a.** 0 **b.** 1
**c.** $1/(a+b+c)$ **d.** none of these

**132.** If the pairs of lines $x^2 + 2xy + ay^2 = 0$ and $ax^2 + 2xy + y^2 = 0$ have exactly one line in common, then the joint equation of the other two lines is given by

**a.** $3x^2 + 8xy - 3y^2 = 0$ **b.** $3x^2 + 10xy + 3y^2 = 0$
**c.** $y^2 + 2xy - 3x^2 = 0$ **d.** $x^2 + 2xy - 3y^2 = 0$

**133.** If the origin is shifted to the point $(ab/(a-b), 0)$ without rotation, then the equation $(a-b)(x^2+y^2) - 2abx = 0$ becomes

**a.** $(a-b)(x^2+y^2) - (a+b)xy + abx = a^2$
**b.** $(a+b)(x^2+y^2) = 2ab$
**c.** $(x^2+y^2) = (a^2+b^2)$
**d.** $(a-b)^2(x^2+y^2) = a^2b^2$

**134.** The straight lines represented by $(y - mx)^2 = a^2(1+m^2)$ and $(y - nx)^2 = a^2(1+n^2)$ form a

**a.** rectangle **b.** rhombus
**c.** trapezium **d.** none of these

**135.** The condition that one of the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ may coincide with one of those given by the equation $a'x^2 + 2h'xy + b'y^2 = 0$ is

**a.** $(ab' - a'b)^2 = 4(ha' - h'a)(bh' - b'h)$
**b.** $(ab' - a'b)^2 = (ha' - h'a)(bh' - b'h)$
**c.** $(ha' - h'a)^2 = 4(ab' - a'b)(bh' - b'h)$
**d.** $(bh' - b'h)^2 = 4(ab' - a'b)(ha' - h'a)$

**136.** The angle between the pair of lines whose equation is $4x^2 + 10xy + my^2 + 5x + 10y = 0$ is

**a.** $\tan^{-1}(3/8)$
**b.** $\tan^{-1}(3/4)$
**c.** $\tan^{-1}(2\sqrt{25-4m}/m+4)$, $m \in R$
**d.** none of these

**137.** If one of the lines of $my^2 + (1-m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then $m$ is

**a.** 3 **b.** 2 **c.** $-1/2$ **d.** $-1$

**138.** The pair of lines represented by $3ax^2 + 5xy + (a^2-2)y^2 = 0$ are perpendicular to each other for

**a.** two values of $a$ **b.** $\cdot\, a$
**c.** for one value of $a$ **d.** for no value of $a$

139. If two of the lines represented by $x^4 + x^3y + cx^2y^2 - xy^3 + y^4 = 0$ bisect the angle between the other two, then the value of $c$ is

**a.** 0 **b.** $-1$ **c.** 1 **d.** $-6$

140. If the lines represented by the equation $3y^2 - x^2 + 2\sqrt{3}x - 3 = 0$ are rotated about the point $(\sqrt{3}, 0)$ through an angle 15°, one clockwise direction and other in anti clockwise direction, so that they become perpendicular, then the equation of the pair of lines in the new position is

**a.** $y^2 - x^2 + 2\sqrt{3}x + 3 = 0$ **b.** $y^2 - x^2 + 2\sqrt{3}x - 3 = 0$

**c.** $y^2 - x^2 - 2\sqrt{3}x + 3 = 0$ **d.** $y^2 - x^2 + 3 = 0$

141. If the equation of the pair of straight lines passing through the point (1, 1), one making an angle $\theta$ with the positive direction of $x$-axis and the other making the same angle with the positive direction of $y$-axis, is $x^2 - (a+2)xy + y^2 + a(x+y-1) = 0$, $a \neq -2$, then the value of $\sin 2\theta$ is

**a.** $a - 2$ **b.** $a + 2$

**c.** $2/(a+2)$ **d.** $2/a$

142. Equation of a line which is parallel to the line common to the pair of lines given by $6x^2 - xy - 12y^2 = 0$ and $15x^2 + 14xy - 8y^2 = 0$ and at a distance 7 from it is

**a.** $3x - 4y = -35$ **b.** $5x - 2y = 7$

**c.** $3x + 4y = 35$ **d.** $2x - 3y = 7$

143. The equation $x^2y^2 - 9y^2 + 6x^2y + 54y = 0$ represents

**a.** a pair of straight lines and a circle

**b.** a pair of straight lines and a parabola

**c.** a set of four straight lines forming a square

**d.** none of these

144. The combined equation of the lines $l_1, l_2$ is $2x^2 + 6xy + y^2 = 0$ and that of the lines $m_1, m_2$ is $4x^2 + 18xy + y^2 = 0$. If the angle between $l_1$ and $m_2$ be $\alpha$, then the angle between $l_2$ and $m_1$ will be

**a.** $\pi/2 - \alpha$ **b.** $2\alpha$

**c.** $\pi/4 + \alpha$ **d.** $\alpha$

145. The equation $x - y = 4$ and $x^2 + 4xy + y^2 = 0$ represent the sides of

**a.** an equilateral triangle **b.** a right angled triangle

**c.** an isosceles triangle **d.** none of these

146. The equation $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$ represent

**a.** two pairs of perpendicular straight lines

**b.** two pairs of parallel straight lines

**c.** two pairs of straight lines which are equally inclined to each other

**d.** none of these

147. The distance between the two lines represented by the equation $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ is

**a.** 8/5 **b.** 6/5

**c.** 11/5 **d.** none of these

## Multiple Correct Answers Type

Solutions on page 1.100

**Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.**

1. If $P$ is a point $(x, y)$ on the line $y = -3x$ such that $P$ and the point (3, 4) are on the opposite sides of the line $3x - 4y = 8$, then

**a.** $x > 8/15$ **b.** $x > 8/5$

**c.** $y < -8/5$ **d.** $y < -8/15$

2. If $(-6, -4), (3, 5), (-2, 1)$ are the vertices of a parallelogram, then remaining vertex can be

**a.** $(0, -1)$ **b.** $(7, 9)$

**c.** $(-1, 0)$ **d.** $(-11, -8)$

3. The lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ are the sides of a square. Equation of the remaining side of the square can be

**a.** $2x - y + 6 = 0$ **b.** $2x - y + 8 = 0$

**c.** $2x - y - 10 = 0$ **d.** $2x - y - 14 = 0$

4. Let $O \equiv (0, 0)$, $A \equiv (0, 4)$, $B \equiv (6, 0)$. '$P$' be a moving point such that the area of triangle $POA$ is two times the area of triangle $POB$. Locus of '$P$' will be straight line whose equation can be

**a.** $x + 3y = 0$ **b.** $x + 2y = 0$

**c.** $2x - 3y = 0$ **d.** $3y - x = 0$

5. If $(\alpha, \alpha^2)$ lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$, $5x - 6y - 1 = 0$, then

**a.** $2\alpha + 3\alpha^2 - 1 > 0$ **b.** $\alpha + 2\alpha^2 - 3 < 0$

**c.** $\alpha + 2\alpha^2 - 3 < 0$ **d.** $6\alpha^2 - 5\alpha + 1 > 0$

6. Angles made with $x$-axis by a straight line drawn through (1, 2) so that it intersects $x + y = 4$ at a distance $\sqrt{6}/3$ from (1, 2) are

**a.** 105° **b.** 75° **c.** 60° **d.** 15°

7. Given three straight lines $2x + 11y - 5 = 0$, $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$. Then,

**a.** they form a triangle

**b.** they are concurrent

**c.** one line bisects the angle between the other two

**d.** two of them are parallel

8. If $(-4, 0)$ and $(1, -1)$ are two vertices of a triangle of area 4 sq. units, then its third vertex lies on

a. $y = x$
b. $5x + y + 12 = 0$
c. $x + 5y - 4 = 0$
d. $x + 5y + 12 = 0$

**9.** The equation $x^3 + x^2y - xy^2 = y^3$ represents
a. three real straight lines
b. lines in which two of them are perpendicular to each other
c. lines in which two of them are coincident
d. none of these

**10.** The straight line represented by $x^2 + mxy - 2y^2 + 3y - 1 = 0$ meets at
a. $(-1/3, 2/3)$
b. $(-1/3, -2/3)$
c. $(1/3, 2/3)$
d. none of these

**11.** If $m_1$ and $m_2$ are the roots of the equation $x^2 - ax - a - 1 = 0$, then the area of the triangle formed by the three straight lines $y = m_1x$, $y = m_2x$ and $y = a$ $(a \neq -1)$ is
a. $\dfrac{a^2(a+2)}{2(a+1)}$ if $a > -1$
b. $\dfrac{-a^2(a+2)}{2(a+1)}$ if $a < -1$
c. $\dfrac{-a^2(a+2)}{2(a+1)}$ if $-2 < a - 1$
d. $\dfrac{a^2(a+2)}{2(a+1)}$ if $a < -2$

**12.** If the points $(a^3/(a-1), (a^2-3)/(a-1))$, $(b^3/(b-1), (b^2-3)/(b-1))$ and $(c^3/(c-1), (c^2-3)/(c-1))$, where $a, b, c$ are different from 1, lie on the line $lx + my + n = 0$, then
a. $a + b + c = -\dfrac{m}{l}$
b. $ab + bc + ca = \dfrac{n}{l}$
c. $abc = \dfrac{(m+n)}{l}$
d. $abc - (bc + ca + ab) + 3(a + b + c) = 0$

**13.** If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then $m$ is
a. 1
b. 2
c. $-1/2$
d. $-1$

**14.** If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of measure 45° at the major segment of the circle, then the value of $m$ is
a. 2
b. 1
c. $-1$
d. none of these

**15.** Two sides of a rhombus $OABC$ (lying entirely in first quadrant or third quadrant) of area equal to 2 sq. units are $y = x/\sqrt{3}$, $y = \sqrt{3}\,x$. Then possible coordinates of $B$ is/are ($O$ being origin)
a. $(1 + \sqrt{3},\ 1 + \sqrt{3})$
b. $(-1 - \sqrt{3}, -1 - \sqrt{3})$
c. $(3 + \sqrt{3}, 3 + \sqrt{3})$
d. $(\sqrt{3} - 1, \sqrt{3} - 1)$

**16.** If $(x/a) + (y/b) = 1$ and $(x/c) + (y/d) = 1$ intersect the axes at four concyclic points and $a^2 + c^2 = b^2 + d^2$, then these lines can intersect at $(a, b, c, d > 0)$
a. $(1, 1)$
b. $(1, -1)$
c. $(2, -2)$
d. $(3, 3)$

**17.** The straight line $3x + 4y - 12 = 0$ meets the coordinate axes at $A$ and $B$. An equilateral triangle $ABC$ is constructed. The possible coordinates of vertex '$C$' are
a. $\left(2\left(1 - \dfrac{3\sqrt{3}}{4}\right), \dfrac{3}{2}\left(1 - \dfrac{4}{\sqrt{3}}\right)\right)$
b. $(-2(1 + \sqrt{3}), 3/2\,(1 - \sqrt{3}))$
c. $(2(1 + \sqrt{3}), 3/2\,(1 + \sqrt{3}))$
d. $\left(2\left(1 + \dfrac{3\sqrt{3}}{4}\right), \dfrac{3}{2}\left(1 + \dfrac{4}{\sqrt{3}}\right)\right)$

**18.** The equation of the lines passing through the point $(1, 0)$ and at a distance $\sqrt{3}/2$ from the origin are
a. $\sqrt{3}x + y - \sqrt{3} = 0$
b. $x + \sqrt{3}y - \sqrt{3} = 0$
c. $\sqrt{3}x - y - \sqrt{3} = 0$
d. $x - \sqrt{3}y - \sqrt{3} = 0$

**19.** The sides of a triangle are the straight lines $x + y = 1$, $7y = x$ and $\sqrt{3}y + x = 0$. Then which of the following is an interior point of the triangle?
a. circumcentre
b. centroid
c. incentre
d. orthocentre

In an obtused-angle triangle orthocentre and cicumcentre are exterior to the triangle.

**20.** If $(x, y)$ be a variable point on the line $y = 2x$ lying between the lines $2(x + 1) + y = 0$ and $x + 3(y - 1) = 0$, then
a. $x \in (-1/2, 6/7)$
b. $x \in (-1/2, 3/7)$
c. $y \in (-1, 3/7)$
d. $y \in (-1, 6/7)$

**21.** The $x$-coordinates of the vertices of a square of unit area are the roots of the equation $x^2 - 3|x| + 2 = 0$ and the $y$-coordinates of the vertices are the roots of the equation $y^2 - 3y + 2 = 0$. Then the possible vertices of the square is/are
a. $(1, 1), (2, 1), (2, 2), (1, 2)$
b. $(-1, 1), (-2, 1), (-2, 2), (-1, 2)$
c. $(2, 1), (1, -1), (1, 2), (2, 2)$
d. $(-2, 1), (-1, -1), (-1, 2), (-2, 2)$

**22.** The area of triangle $ABC$ is 20 cm². The coordinates of vertex $A$ are $(-5, 0)$ and those of $B$ are $(3, 0)$. The vertex $C$ lies on the line $x - y = 2$. The coordinates of $C$ are
a. $(5, 3)$
b. $(-3, -5)$
c. $(-5, -7)$
d. $(7, 5)$

**23.** If the straight line $ax + cy = 2b$ where $a, b, c > 0$ makes a triangle of area 2 sq. units with coordinate axes, then
a. $a, b, c$ are in G.P.
b. $a, -b, c$ are in G.P.
c. $a, 2b, c$ are in G.P.
d. $a, -2b, c$ are in G.P.