Contents

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Physics for IIT-JEE 2012-13: Mechanics I

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Brief Contents

Preface

Since the time the IIT-JEE (Indian Institute of Technology Joint Entrance Examination) started, the examination scheme and the methodology have witnessed many a change. From the lengthy subjective problems of 1950s to the matching column type questions of the present day, the paper-setting pattern and the approach have changed. A variety of questions have been framed to test an aspirant's calibre, aptitude, and attitude for engineering field and profession. Across all these years, however, there is one thing that has not changed about the IIT-JEE, i.e., its objective of testing an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grass-root level.

No subject can be mastered overnight; nor can a subject be mastered just by formulae-based practice. Mastering a subject is an expedition that starts with the basics, goes through the illustrations that go on the lines of a concept, leads finally to the application domain (which aims at using the learnt concept(s) in problem-solving with accuracy) in a highly structured manner.

This series of books is an attempt at coming face-to-face with the latest IIT-JEE pattern in its own format, which is going to be highly advantageous to an aspirant for securing a good rank. A thorough knowledge of the contemporary pattern of the IIT-JEE is a must. This series of books features all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion reason type, matrix match type, or paragraph-based, thought-type questions. Not discounting to need for skilled and guided practice, the material in the book has been enriched with a large number of fully solved concept-application exercises so that every step in learning is ensured for the understanding and application of the subject.

This whole series of books adopts a multi-facetted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant. Each book in the series has a sizeable portion devoted to questions and problems from previous years' IIT-JEE papers, which will help students get a feel and pattern of the questions asked in the examination. The best part about this series of books is that almost all the exercises and problem have been provided with not just answers but also solutions.

Overall the whole content of the book is an amalgamation of the theme of physics with ahead-of-time problems, which an aspirant must follow to accomplish success in IIT-JEE.

B. M. SHARMA

CHAPTER

1

Basic Mathematics

- Elementary Algebra
- Elementary Trigonometry
- Basic Coordinate Geometry
- Differentiation
- Integral of a Function
- Application in Physics

Mathematics is the supporting tool of physics. The elementary knowledge of basic maths is useful in problem solving in physics. Basic knowledge of elementary algebra, trigonometry, coordinate geometry and basic calculus is must before going into the depth of physics.

ELEMENTARY ALGEBRA

Common Formulae

1. $(a+b)^2 = a^2 + b^2 + 2ab$
2. $(a-b)^2 = a^2 + b^2 - 2ab$
3. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
4. $(a+b)(a-b) = a^2 - b^2$
5. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
6. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
7. $(a+b)^2 - (a-b)^2 = 4ab$
8. $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$

Polynomial, Linear, and Quadratic Equations

Real Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x a real variable, then $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is called a *real polynomial.*

Degree or Index of a Polynomial

The highest power appearing in a polynomial is called its degree. For example, $f(x) = x^3 + 8x + 3$ is a polynomial of degree 3.

Students must note here that it is not necessary that the highest power must be of a single variable only. For example, $f(x) = 3x^2y + y^2 + 2$ is a polynomial of degree 3 because of variable y in the term x^2y. We add powers of the variables in a term to find degree of a polynomial irrespective of the nature of variables. Thus, in the present case, x^2y is having power of 2 + 1 = 3 and hence the degree of the given polynomial is 3.

Linear Equations

Equations having polynomials of unit degree are called linear equations, e.g., $x + y = 2$ or $2x + 3 = 5$. Such equations always represent a straight line on a graph.

Quadratic Equations

Equations of second degree are called quadratic equations. The general form of a quadratic equation is as given below:

$$ax^2 + bx + c = 0, \text{ where } a \neq 0.$$

Roots of a Quadratic Equation

Solutions of a quadratic equation are called its roots. Roots are those values of a variable such as x for which the given quadratic equation collapses to zero. As a rule, a quadratic equation always has two roots which may or may not be equal.

Roots of a quadratic equation are generally represented by α and β.

Let $ax^2 + bx + c = 0$ be a quadratic equation. Then:

1. Its roots are $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$; $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.
2. Hence, its solution is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
3. Sum of its roots is given by $\alpha + \beta = \dfrac{-b}{a}$.
4. Product of its roots is given by $\alpha\beta = \dfrac{c}{a}$.
5. Difference of its roots is given by $\alpha - \beta = \dfrac{\sqrt{b^2 - 4ac}}{a}$.

Binomial Expression and Theorem

An algebraic expression containing two terms is called a *binomial expression.*

For example, $(a + b)$, $(2x - 3y)$, $\left(x + \dfrac{1}{y}\right)$, $\left(x + \dfrac{3}{x}\right)$, etc., are binomial expressions.

Binomial Theorem for Positive Integral Index

The general form of a binomial expression is $(x + a)^n$, where n is any positive integer (called index) and x and a are real numbers.

Binomial theorem states:

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1} \cdot a^1 + {}^nC_2x^{n-2} \cdot a^2 + \cdots + {}^nC_rx^{n-r} \cdot a^r + \cdots + {}^nC_n \cdot a^n$$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ and $n! = n(n-1)(n-2)\cdots 3 \times 2 \times 1$, the product of first n natural numbers [e.g., $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$].

Important Points

- Total number of terms in the expansion = $(n + 1)$.
- In every successive term in the expansion, the power of x goes on decreasing by 1 and that of a increasing by 1, so that the sum of the powers of x and a in each term is always equal to n.
- ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients.
- The expression $n!$ is read as "factorial n". So,

  ${}^nC_1 = \dfrac{n!}{1!(n-1)!} = \dfrac{n(n-1)(n-2)\cdots 3 \times 2 \times 1}{1(n-1)(n-2)\cdots 3 \times 2 \times 1} = n.$

  Similary, ${}^nC_2 = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)}{2 \times 1}, \ldots, {}^nC_n = 1.$

Binomial Theorem for Any Index

If n is positive, negative or fraction and x is any real number such that $-1 < x < 1$, i.e., x lies between -1 and $+1$, then according to the binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \infty \text{ terms.}$$

Note:

- *If n is a positive integer, then the expansion will have $(n+1)$ terms.*
- *If n is a negative integer or a fraction, then the number of terms in the expansion will be infinite, i.e, there will be no last term.*
- *If $|x| \ll 1$, then only first two terms of the expansion are significant. It is so because the values of the second and higher order terms become very small and can be neglected. Thus, in this case, the binomial expansion reduces to the following simplified forms when $|x| \ll 1$:*

$$(1+x)^n = 1 + nx, \qquad (1+x)^{-n} = 1 - nx,$$
$$(1-x)^n = 1 - nx, \quad \text{and} \quad (1-x)^{-n} = 1 + nx.$$

Illustration 1.1 **Calculate $(1001)^{1/3}$.**

Sol. We can write 1001 as: $1001 = 1000\left(1 + \frac{1}{1000}\right)$, so that we have

$$(1001)^{1/3} = \left[1000\left(1+\frac{1}{1000}\right)\right]^{1/3} = 10\left[1+\frac{1}{1000}\right]^{1/3}$$
$$= 10(1+0.001)^{1/3} = 10(1 + \frac{1}{3} \times 0.001)$$
$$= 10.003333$$

Illustration 1.2 **Expand $(1+x)^{-3}$.**

Sol. $$(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-3-1)x^2}{2!} + \frac{(-3)(-3-1)(-3-2)}{3!}x^3 + \cdots$$
$$= 1 - 3x + \frac{12}{2}x^2 - \frac{60}{3\times 2}x^3 + \cdots$$
$$= 1 - 3x + 6x^2 - 10x^3 + \cdots$$

Concept Application Exercise 1.1

1. Expand $(1+x)^{-2}$.
2. Using binomial expansion, simplify the following expression: $Q\left[\left(1+\frac{\Delta x}{x}\right)^3 - 1\right]$, assuming Δx to be small in comparison to x.

ELEMENTARY TRIGONOMETRY

System of Measurement of an Angle

There are different types of measurement of an angle.

1. *Sexagesimal system:* In this system,
   1 right angle = 90° (degree); 1 degree = 60′ (minutes)
   1 minute = 60″ (seconds)
2. *Centesimal system:* In this system,
   1 right angle = 100 grades (100 g)
   1 grade = 100 minutes (100′)
   1 minute = 100 seconds (100″)
3. *Circular system:* In this system, angle is measured in radian.
   π radians = 180°
   Consider a particle moves from a position P to position Q along a circle of radius r with centre at O (see Fig. 1.1). Then,

$$\text{Angle } \theta = \frac{\text{Arc length } PQ}{\text{Radius of circle}} = \frac{s}{r} \Rightarrow s = r\theta$$

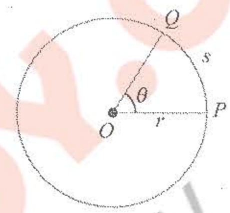


Fig. 1.1

If the length of arc PQ = radius of circle r, then $\theta = 1$ radian.

Radian

When a body completes one revolution, then $\theta = 2\pi$ rad.

$\therefore \quad 2\pi$ rad $= 360°$ or 2×3.14 rad $= 360°$

$\Rightarrow \quad 1 \text{ rad} = \frac{360°}{2 \times 3.14} = 57.3°$

Four Quadrants and Sign Conventions

Consider two mutually perpendicular lines intersecting at O. These two mutually perpendicular lines divide the plane into four parts called quadrants (see Fig. 1.2).

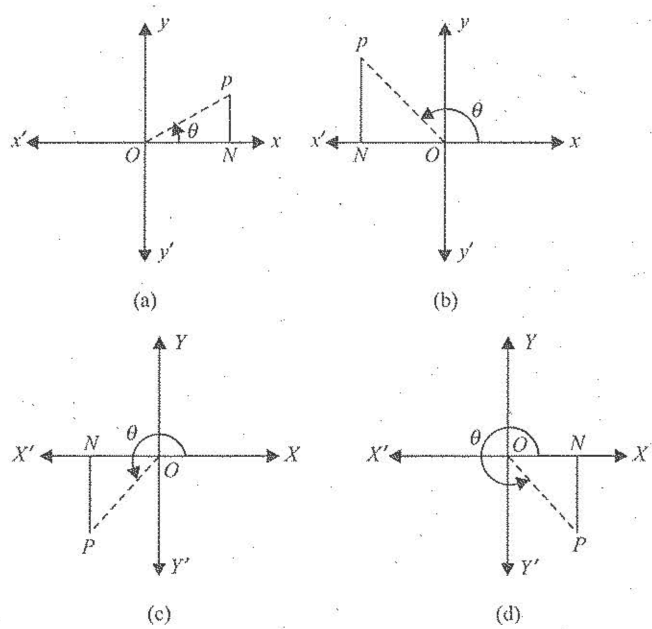


Fig. 1.2

Points to Remember

1. To determine the sign of a trigonometrical ratio in any quadrant, OP will be taken as positive in all four quadrants.
2. In first quadrant, all trigonometrical ratios are positive (see Fig. 1.3).

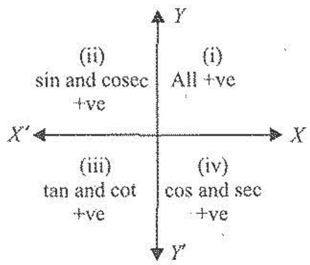
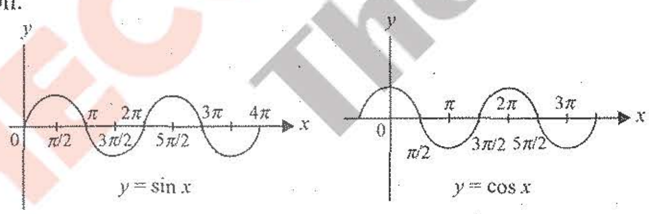


Fig. 1.3

3. In second quadrant, only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive.
4. In third quadrant, only $\tan\theta$ and $\cot\theta$ are positive.
5. In fourth quadrant, only $\cos\theta$ and $\sec\theta$ are positive.
6. The value of $\sin\theta$ and $\cos\theta$ are such that $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$.
7. But $\tan\theta$ and $\cot\theta$ can take any real value.

The Graphs of sin and cos Functions

The function $y = \sin x$, where x is any dimensionless quantity, is called a sine function. The argument x is usually measured in radian. The function $y = \sin x$ is plotted in Fig. 1.4(a). The maximum positive and negative values of a sine function are +1 and 1, respectively. Between $x = 0$ and $x = \pi$, the function is positive, the peak value of +1 occurs at $x = \frac{\pi}{2}$. Similarly, for the interval $x = \pi$ to $x = 2\pi$, the function is negative and the negative peak occurs at $x = \frac{3\pi}{2}$. The sine function is a periodic function, with a period of 2π. That is, the pattern of the graph repeats itself after an interval of 2π. Mathematically, it may be stated as $y = \sin x = \sin(2\pi + x) = \sin(4\pi + x) = \cdots$ and so on.



(a) (b)

Fig. 1.4

If the graph of the sine function is displaced to the left through $\frac{\pi}{2}$, we get the graph of cosine function: $y = \cos x$ as shown in Fig. 1.4(b). The cosine function is also a periodic function with a period of 2π.

Some Important Trigonometric Formulae

1. **a.** $\sin^2\theta + \cos^2\theta = 1$
   b. $1 + \tan^2\theta = \sec^2\theta$
2. Addition and subtraction formulae:
   a. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
   b. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
   c. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
3. Multiple formulae:
   a. $\sin 2A = 2 \sin A \cos A$
   b. $\cos 2A = \cos^2 A - \sin^2 A$
   c. $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
   d. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Trigonometrical Ratios of Allied Angles

The angles whose sum or difference with angle θ is zero or a multiple of 90° are called angles allied to θ. Commonly used T-ratios of some of the allied angles are given below.

1. $\sin(-\theta) = -\sin\theta$
   $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
   $\cos(-\theta) = \cos\theta$
   $\sec(-\theta) = \sec\theta$
   $\tan(-\theta) = -\tan\theta$
   $\cot(-\theta) = -\cot\theta$

Note:
- ***As angle $-\theta$ lies in the fourth quadrant, only $\cos\theta$ and $\sec\theta$ are positive, e.g., $\sin(-30°) = -\sin 30°$ and $\cos(-30°) = +\cos 30°$.***

2. $\sin(90° - \theta) = \cos\theta$
   $\operatorname{cosec}(90° - \theta) = \sec\theta$
   $\cos(90° - \theta) = \sin\theta$
   $\sec(90° - \theta) = \operatorname{cosec}\theta$
   $\tan(90° - \theta) = \cot\theta$
   $\cot(90° - \theta) = \tan\theta$

Note:
- ***As angle $(90° - \theta)$ lies in first quadrant, all T-ratios are positive, e.g., $\sin 30° = \sin(90° - 60°) = \cos 60°$.***

3. $\sin(90° + \theta) = \cos\theta$
   $\operatorname{cosec}(90° + \theta) = \sec\theta$
   $\cos(90° + \theta) = -\sin\theta$
   $\sec(90° + \theta) = -\operatorname{cosec}\theta$
   $\tan(90° + \theta) = -\cot\theta$
   $\cot(90° + \theta) = -\tan\theta$

Note:
- ***As angle $(90° + \theta)$ lies in 2nd quadrant, therefore only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive, e.g., $\sin 120° = \sin(90° + 30°) = \cos 30°$.***

4. $\sin(180° - \theta) = \sin\theta$
   $\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta$
   $\cos(180° - \theta) = -\cos\theta$
   $\sec(180° - \theta) = -\sec\theta$
   $\tan(180° - \theta) = -\tan\theta$
   $\cot(180° - \theta) = -\cot\theta$

Note:
- ***As angle $(180° - \theta)$ lies in 2^{nd} quadrant, therefore, only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive, e.g., $\sin 150° = \sin(180° - 30°) = \sin 30°$.***

5. $\sin(180° + \theta) = -\sin\theta$
   $\cos(180° + \theta) = -\cos\theta$
   $\tan(180° + \theta) = \tan\theta$

Note:
- ***As angle $(180° + \theta)$ lies in 3^{rd} quadrant, therefore only $\tan\theta$ is positive, e.g., $\tan 210° = \tan(180° + 30°) = \tan 30°$.***

6. $\sin(270° + \theta) = -\cos\theta$
   $\cos(270° + \theta) = \sin\theta$
   $\tan(270° + \theta) = -\cot\theta$

Note:
- ***As angle $(270° + \theta)$ lies in the 4^{th} quadrant, therefore only $\cos\theta$ is positive, e.g., $\sin 300° = \sin(270 + 30°) = -\cos 30°$.***

7. $\sin(360° - \theta) = -\sin\theta$
   $\cos(360° - \theta) = \cos\theta$
   $\tan(360° - \theta) = -\tan\theta$

Note:
- ***As angle $(360° - \theta)$ lies in the 4^{th} quadrant, therefore only $\cos\theta$ is positive.***

8. $\sin(360° + \theta) = \sin\theta$
   $\cos(360° + \theta) = \cos\theta$
   $\tan(360° + \theta) = \tan\theta$

Note:
- ***As angle $(360° + \theta)$ lies in the first quadrant, therefore all the T-ratios are positive, e.g., $\sin 400° = \sin(360° + 40°) = \sin 40°$.***

Illustration 1.3 **Given that $\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$, determine the values of $\sin 60°$, $\sin 120°$, $\sin 240°$, $\sin 300°$, and $\sin(-30°)$.**

Sol.

1. $\sin 60° = ?$
   First, we should determine the quadrant in which 60° lies. It is obviously first quadrant. Then, we should recall whether sin in first quadrant is positive or negative. "All Silver Tea Cups" tells us that all the TRs are positive in the first quadrant, therefore, $\sin 60°$ must be positive.
   Now, we should write 60° in such a way that it is ±30° with any of the two axes (the horizontal XOX' and the vertical YOY'). So, we can write $\sin 60° = \sin(90° - 30°)$.
   Now, we can recall from the TRs on the previous page that $\sin(90° - \theta) = \cos\theta$
   $\Rightarrow \quad \sin 60° = \sin(90° - 30°) = \cos 30° = \sqrt{3}/2$.
2. Similarly, we can find out the value of $\sin 120°$. This angle lies in second quadrant. In second quadrant, sin is positive. Therefore, $\sin 120°$ = some positive value.
   $\Rightarrow \sin 120° = \sin(90° + 30°) = \cos 30° = \dfrac{\sqrt{3}}{2}$
3. $\sin 240°$ lies in third quadrant. So, it should be negative.
   $\Rightarrow \sin 240° = \sin(270° - 30°) = -\cos 30° = -\dfrac{\sqrt{3}}{2}$
4. $\sin 300°$ lies in fourth quadrant where sin is negative.
   $\Rightarrow \sin 300° = \sin(270° + 30°) = -\cos 30° = -\dfrac{\sqrt{3}}{2}$
5. $\sin(-30°)$ lies in fourth quadrant, so it should be negative.
   $\Rightarrow \sin(-30°) = -\sin 30° = -\dfrac{1}{2}$

Inverse Trigonometric Functions

Inverse trigonometric functions are also called anti-trigonometric functions. These are represented by putting a superscript '-1' on the corresponding trigonometric function whose inverse are to be obtained, e.g.,

∴ inverse of $\sin\theta = x$, means $\theta = \sin^{-1} x$

It is read as "sine inverse x". Just as trigonometric operation on any angle gives a particular value, inverse trigonometric operation on any value (or number) will return its corresponding angle.

Properties of Inverse Trigonometric Functions:

1. $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1} x) = x$; provided $-\pi/2 \le \theta \le \pi/2$ and $-1 \le x \le 1$.
2. $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1} x) = x$; provided $0 \le \theta \le \pi$ and $-1 \le x \le 1$.
3. $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1} x) = x$; provided $-\pi/2 \le \theta \le \pi/2$ and $-\infty \le x \le \infty$.

Illustration 1.4 **Find the value of $\sin^{-1} 1$.**

Sol. Let $y = \sin^{-1} 1 = \sin^{-1}(\sin\pi/2) = \pi/2$
$[\because \sin\pi/2 = 1 \text{ and } \sin^{-1}(\sin\theta) = \theta \text{ for } -\pi/2 \le \theta \le \pi/2]$

Illustration 1.5 **Find the value of $\cos^{-1}(-1/2)$.**

Sol. Let $y = \cos^{-1}(-1/2) = \cos^{-1}\left(\cos\dfrac{2\pi}{3}\right) = \dfrac{2\pi}{3}$
$[\because \cos(2\pi/3) = -1/2 \text{ and } \cos^{-1}(\cos\theta) = \theta \text{ for } 0 \le \theta \le \pi]$

BASIC COORDINATE GEOMETRY

If you have to specify the position of a point in space, how will you do it? This is the easiest application of coordinate geometry. We can give, assign or find out exact numerical values of the position of points, lines, curves, slopes, etc. All this is done with the help of coordinate systems. There are many types of coordinate systems such as rectangular, polar, spherical, cylindrical, etc. It is generally the right handed rectangular axes coordinate system which you will be using in physics. This system consists of:

1. Origin 2. Axis or Axes

If the point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position; if it is in a plane, two coordinates are required; if it is in space, three coordinates are needed.

Origin

This is any fixed point which is convenient to you. Say, in a room, you can consider any corner of the room as the origin. On a sheet of paper, you can mark any point on it and consider it as the origin. All measurements are taken basically with respect to this point called origin.

Axis or Axes

Any fixed direction passing through the origin and convenient to you can be taken as an axis. If the position of a point or positions of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two axes are needed. These are generally called x and y axes. If the points are distributed in a space, three axes are taken which are called x, y, and z-axes. If x, y, and z-axes are mutually perpendicular, the system is called rectangular axes coordinate system.

Important Points

1. Origin can be any fixed point convenient to you. It is denoted by O.
2. x-axis is any fixed direction passing through the origin and convenient to you (Fig. 1.5). Thus, it is not at all necessary that the (so called) horizontal line passing through the origin is x-axis.

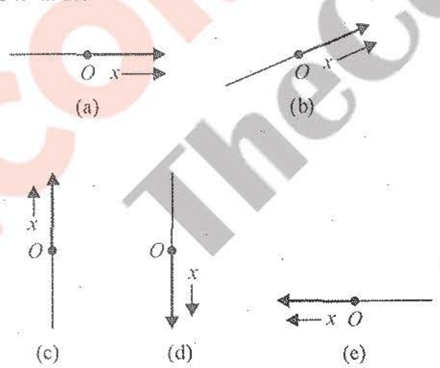


Fig. 1.5

3. Unless otherwise explicitly mentioned, all angles are always measured from the direction of x-axis (called the positive direction of x-axis). Positive angles are measured in anticlockwise direction and negative in clockwise direction.
4. y-axis is any fixed direction passing through the origin perpendicular to the x-axis, convenient to you. Perpendicular means making an angle of $+90°$ with the positive direction of x-axis. Students may feel that once the origin and x-axis have been fixed, the position of y-axis also gets fixed accordingly. But, it is not the case. y-axis can be any fixed direction which is in the plane passing through the origin and the x-axis and perpendicular to x-axis.
Thus, x and y-axes can be any direction as shown in Fig. 1.6.

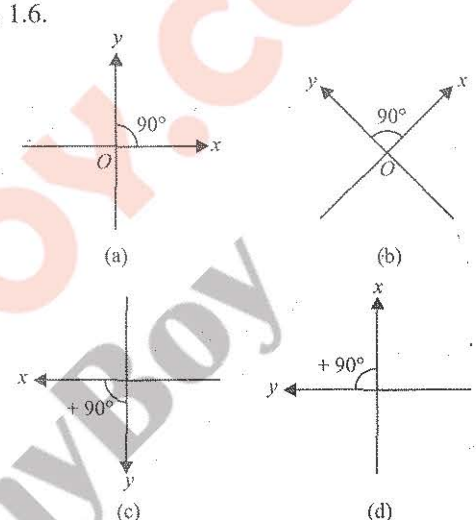


Fig. 1.6

5. Once origin, x- and y-axes are fixed, z-axis becomes automatically fixed. Convenience of the observer goes away. z-axis is the fixed direction passing through the origin and perpendicular to both x- and y-axes.

Position of a Point

As you already know it well, in case of plane coordinate geometry, i.e., when the position of a point always remains contained in a plane (called x–y plane), the position of a point is specified by its distances from the origin along (or parallel to) x and y-axes, as shown in Fig. 1.7.

You can easily observe that the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) in Fig. 1.7 are (4, 2), (−4, 3), (−5, −4), and (2, −2), respectively.

Distance Formulae

1. The distance between two points (x_1, y_1) and (x_2, y_2) $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. The distance of the point (x_1, y_1) from the origin $= \sqrt{(x_1^2 + y_1^2)}$.

The coordinates of the mid-point of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

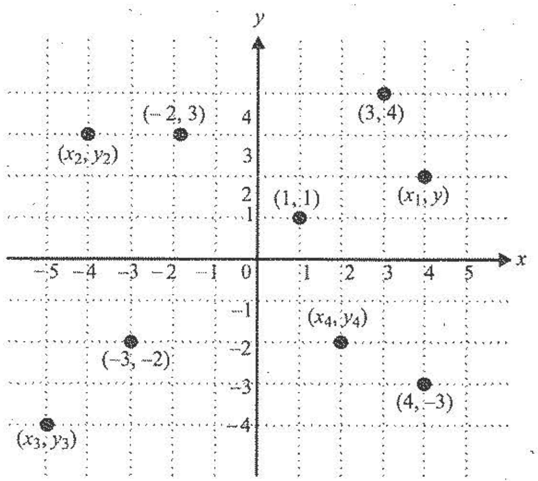


Fig. 1.7

Slope of a Line

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by $m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$, where θ is the angle which the line makes with the positive direction of x-axis (Fig. 1.8).

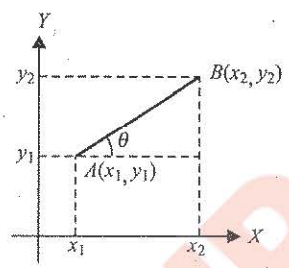


Fig. 1.8

Straight Line Equations

1. $Ax + By + C = 0$, is the general form of the equation of a straight line.
2. Equation of x-axis is $y = 0$.
3. Equation of y-axis is $x = 0$.
4. Equation of a straight line parallel to y-axis and at a distance a from it is given by $x = a$.
5. Equation of a straight line parallel to x-axis and at a distance b from it is given by $y = b$.
   a. Constant function, $x = a$ (Fig. 1.9).

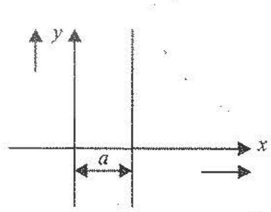


Fig. 1.9

   b. Constant function, $y = b$ (Fig. 1.10).
6. $y = mx + c$ is a line which cuts off an intercept c on y-axis and makes an angle θ with the +ve direction of x-axis in

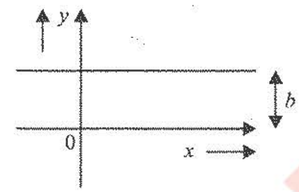


Fig. 1.10

anticlockwise direction; and $m = \tan\theta$ is called its slope or gradient.

7. $y = mx$, is any line through the origin and having slope m.
   a. When $c = 0$, $y = mx$
   The graph between x and y will be a straight line as x bears the direct dependence on y (Fig. 1.11).

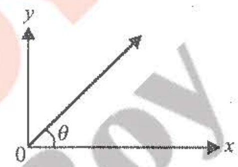


Fig. 1.11

   Here, m represents the slope of line.
   $\therefore \dfrac{dy}{dx} = m = \tan\theta$
   b. When $c \neq 0$, $y = mx + c$
   Graph for this equation is also a straight line but with a +ve intercept on y-axis. As when x goes to zero, y accordingly takes value c. So, the straight line will start from $y = c$ instead of origin (Fig. 1.12).

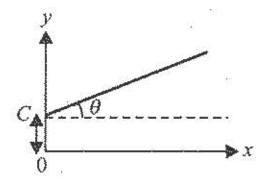


Fig. 1.12

   $m = \tan\theta$ is the slope of the straight line here also.
   c. When $c = 0$, $m < 0$, $y = mx$
   For $m < 0$, $\theta > 90°$ (Fig. 1.13).

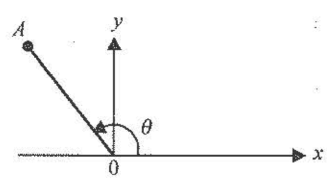


Fig. 1.13

   d. When $c \neq 0$, $m < 0$, $y = mx + c$
   For $m < 0$, $\theta > 90°$ (Fig. 1.14).
8. $\dfrac{x}{a} + \dfrac{y}{b} = 1$, is a line in intercept form where a and b are the intercepts on the axes of x and y, respectively (Fig. 1.15).
9. $y - y_1 = m(x - x_1)$, is the equation of a line through a given point (x_1, y_1) and having slope m.

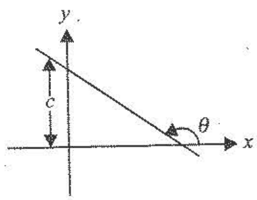


Fig. 1.14

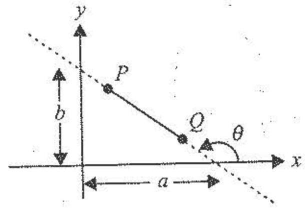


Fig. 1.15

Slope

The slope m of the line $Ax + By + C = 0$ is given by

$$m = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-A}{B}$$

Illustration 1.6 **Consider two points $P_1(2, 7)$ and $P_2(6, 15)$. Write the equation and draw the straight line through these points.**

Sol.

Step 1. Obtain the gradient which is m.

Step 2. c can be found by using the (x, y) values of any given point.

Step 1. Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{15 - 7}{6 - 2} = \dfrac{\text{Height}}{\text{Distance}} = \dfrac{8}{4} = 2.$

So, $\qquad y = 2x + c. \qquad (1)$

Step 2. To find c, put $(x = 2, y = 7)$ or $(x = 6, y = 15)$ in equation (1).

$$7 = 2 \times 2 + c \Rightarrow c = 3$$

So, $m = 2$, $c = 3$.

$\therefore$ the equation becomes $y = 2x + 3$.

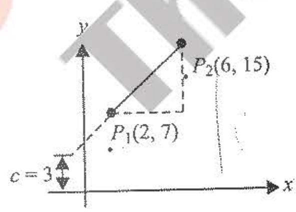


Fig. 1.16

Illustration 1.7 **Plot the line $2x - 3y = 12$.**

Sol. Method 1:

Given $3y = 2x - 12 \Rightarrow y = \dfrac{2}{3}x - 4$

$\therefore$ using $y = mx + c$, $m = \dfrac{2}{3}$ and $c = -4$

Here, the positive slope (m) means that the angle made by the line with x-axis should be less than 90° and negative c means the line will intercept with negative y-axis (Fig. 1.17).

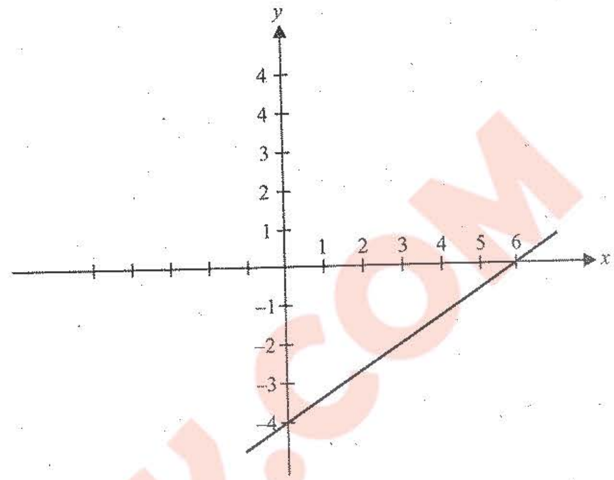


Fig. 1.17

If $y = 0$, then $\dfrac{2}{3}x = 4 \quad \Rightarrow \quad x = \dfrac{4 \times 3}{2} = 6$

Method 2: Using $\dfrac{x}{a} + \dfrac{y}{b} = 1 \quad \Rightarrow \quad \dfrac{2x}{12} - \dfrac{3y}{12} = 1$

$\Rightarrow \quad \dfrac{x}{6} + \dfrac{y}{-4} = 1$

Illustration 1.8 **Plot the line $-3x - 5y = 15$.**

Sol. Method 1:

Given $5y = -3x - 15$

$\therefore$ using $y = mx + c$, $y = -\dfrac{3}{5}x - 3$

Here, the slope is negative, i.e., the line makes an angle greater than 90° with x-axis. As intercept is negative, it indicates that the line will cut y-axis at negative side of it (Fig. 1.18).

When $x = 0$, $\quad y = -3$.

When $y = 0$, $\quad x = -5$.

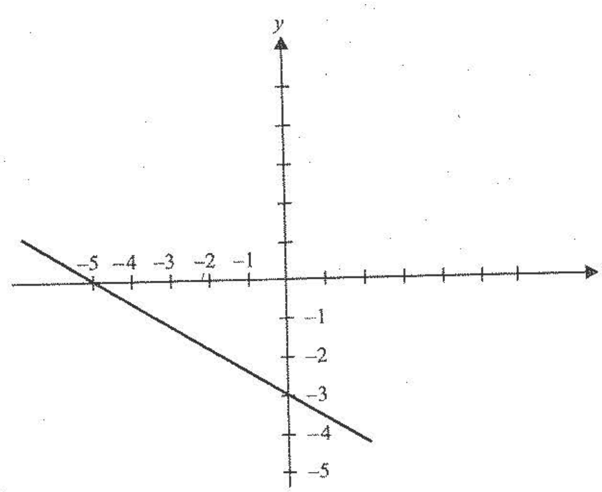


Fig. 1.18

Method 2: Using $\dfrac{x}{a} + \dfrac{y}{b} = 1$; $\dfrac{-3x}{15} - \dfrac{5y}{15} = 1$

$$\Rightarrow \quad \frac{x}{(-5)} + \frac{y}{(-3)} = 1$$

Concept Application Exercise 1.2

1. Plot the lines: (i) $3x + 2y = 0$, (ii) $x - 3y + 6 = 0$.
2. If a particle starts moving with initial velocity $u = 1$ m/s and with acceleration $a = 2$ m/s^2, the velocity of the particle at any time is given by $v = u + at = 1 + 2t = 1 + 2t$; plot the velocity–time graph of the particle.
3. A particle starts moving with initial velocity $u = 25$ m/s with retardation $a = -2$ m/s^2. Draw the velocity–time graph.

Parabola: The Quadratic Equations

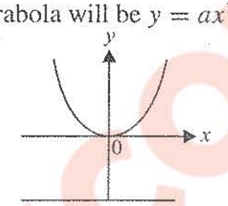
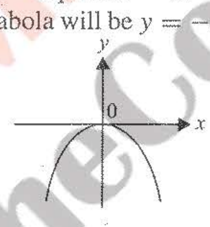
Let us now discuss graphs of quadratic equations.

For equation $y = ax^2 + bx + c$ (where a, b, and c are constants), the graph between x and y will be an asymmetric parabola. As long as $a \neq 0$, this equation represents a quadratic function. So, what is the simplest quadratic equation? It is $y = ax^2$ (obtained by putting $b = 0$, $c = 0$), which is the equation of parabola.

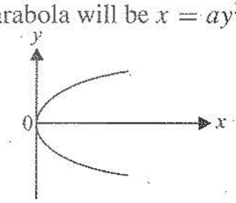
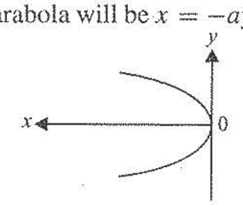
Conclusion

Equation of parabola is a quadratic equation in its simplest form. This parabola has its vertex at origin (0, 0) because when we put $x = 0$, it gives $y = 0$.

- The graph for $y = ax^2$ will be a symmetric parabola about y-axis. The orientation of parabola will be decided by the sign of a.

When a is Positive	**When a is Negative**
The equation of the parabola will be $y = ax^2$. 	The equation of the parabola will be $y = -ax^2$. 

- If we exchange x and y in this equation, i.e., $x = ay^2$, then the axis of symmetry changes and becomes x-axis. As we know this orientation changes as per the sign of a, so the orientation will be opposite when a is negative.

When a is Positive	**When a is Negative**
The equation of the parabola will be $x = ay^2$. 	The equation of the parabola will be $x = -ay^2$. 

- For equation with $c = 0$, $y = ax^2 + bx$. The graph between x and y is an asymmetric parabola, but the orientation of the graph varies with the signs of a and b. Let us take the special case when both a and b are positive.

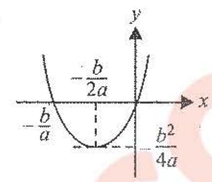


Fig. 1.19

When $y = 0$, then $x = 0$ or $x = -b/a$. At $x = -b/2a$, y is min and $y_{\min} = -b^2/4a$. It is known as vertex (see Fig. 1.19).

Plotting Quadratic Equations

1. General quadratic equation is $y = ax^2 + bx + c$.
2. The graph of a quadratic equation is always a parabola.
3. Orientation of graph depends upon sign of a.
   a. When a is +ve, the graph will open up.
   b. When a is –ve, the graph will open down.
4. The x-coordinate of the vertex is equal to $-b/2a$, i.e., $x = -b/2a$.
5. Put this value back in given equation and find y. Point (x, y) so obtained represents the vertex.
6. Choose two values of x which are to the right or left of the x-coordinate of the vertex.
7. Substitution of these values will give values of y.
   Using these values of (x, y), the graph can be plotted successfully.

Note: ***Since a parabola is symmetric about the line passing through its vertex, the mirror image of points taken with the same value will give other side of parabola.***

Illustration 1.9 **Plot the graph for the equation $y = -x^2 + 4x - 1$.**

Sol. $a = -1$, $b = 4$, $c = -1$. As a is negative, so parabola should open down.

Vertex: $x = \dfrac{-b}{2a} = 2$. Put this value of x to get $y = 3$.

Hence, the vertex of the parabola is (2, 3).

Assume two values of x as follows and find corresponding values of y.

x	1	-1
y	2	-6

Points obtained: (1, 2), (−1, −6)

All points obtained : (2, 3) (Vertex), [1, 2], [−1, −6]

Symmetry of parabola: Mirror image points of (1, 2) and (−1, −6) are (3, 2) and (5, −6).

Now, sketch the parabola as shown in Fig. 1.20.

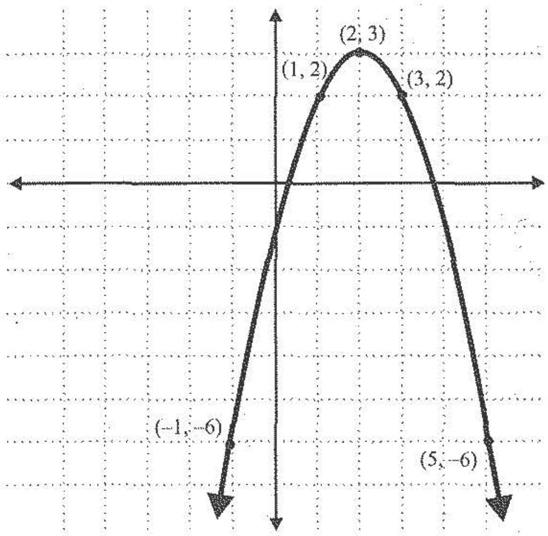


Fig. 1.20

Illustration 1.10 **Plot the graph for the equation $y = x^2 - 4x$.**

Sol. $a = 1$, $b = -4$. As a is positive, so parabola should open up. As $c = 0$, so parabola will pass through origin.

Vertex: $x = \dfrac{-b}{2a} = 2$, so $y = -4 \Rightarrow$ Vertex $= (2, -4)$

Assuming two values of x,

x	1	0
y	−3	0

All points obtained: (2, −4), (1, −3), (0, 0)

Symmetry of parabola: Mirror image points of (1, −3) and (0, 0) are (3, −3) and (4, 0).

Now, sketch the parabola as shown in Fig. 1.21.

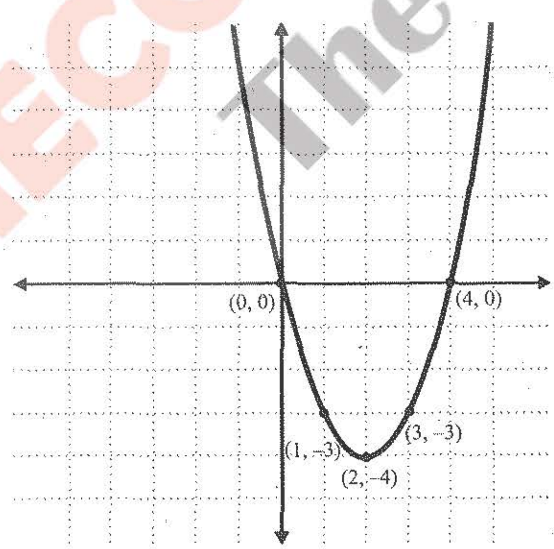


Fig. 1.21

Concept Application Exercise 1.3

1. Find the vertex of the following quadratic equations and plot the graph:
   a. $y = x^2 - 8x$
   b. $y = -2x^2 + 3$
   c. $y = x^2 - 6x + 4$
2. If a particle starts moving along x-axis from origin with initial velocity $u = 1$ m/s and acceleration $a = 2$ m/s^2, the relationship between displacement and time will be
   $x = ut + \frac{1}{2}at^2 = 1 \times t + \frac{1}{2} \times 2 \times t^2 = t + t^2$.
   Draw the displacement (x)–time (t) graph.

DIFFERENTIATION

The purpose of differential calculus is to study the nature (i.e., increase or decrease) and the amount of variation in a quantity when another quantity (on which first quantity depends) varies independently. In our day-to-day life, we often face such types of situations, e.g., growth of plants, expansion of solids on heating, variation in the velocity of a uniformly accelerated object, growth in the population of a country.

Quantity: Anything which can be measured is called a quantity.

Constants and Variables: A quantity whose value remains constant throughout the mathematical operation is called a constant, e.g., integers, fractions, π, e, etc. On the other hand, a quantity which can have any numerical value within certain specific limits is called a variable. A variable is usually represented by u, v, w, x, y, z, etc.

Dependent and Independent Variables: A variable which can have any arbitrary value within specific limits is called as independent variable whereas one whose value depends upon the numerical values assigned to the independent variable is defined as dependant variable.

Differential Coefficient or Derivative of a Function

Suppose y be a function of x, i.e., $y = f(x)$. (i)

The value of the function or the dependent variable y depends on the value of independent variable x. If we change the value of independent variable x to $x + \Delta x$, then value of the function will also change. Let it becomes $y + \Delta y$. Hence

$$y + \Delta y = f(x + \Delta x) \quad \text{(ii)}$$

Subtracting equation (i) from (ii), we get

$$y + \Delta y - y = f(x + \Delta x) - f(x)$$

or $$\Delta y = f(x + \Delta x) - f(x) \quad \text{(iii)}$$

Above equation provides the change in the value of function y, when the value of variable x is changed from x to $x + \Delta x$.

Dividing both sides of the equation (iii) by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{(iv)}$$

Maximum value of $v = 6 \times \frac{4}{9} - 6 \times \frac{8}{27} = \frac{8}{3} - \frac{16}{9}$

$= \frac{8}{9}\ \text{ms}^{-1}$ (by putting $t = \frac{2}{3}$ in v)

Example 1.25 **On a certain planet, the instantaneous velocity of a ball thrown up is given by $v = 2t - 6$ (v is in ms^{-1} and t is in sec).**

1. **Find the expression for the displacement of the particle, given that the particle started its journey at $x = 1$ m.**
2. **What is the value of g on the surface of this planet?**

Sol. 1. To find the displacements, we need to integrate the velocity, i.e.,

$$x = \int v dt = \int (2t-6)dt = \frac{2t^2}{2} - 6t + c = t^2 - 6t + c$$

where c is the constant of integration. Its value can be calculated from the given boundary condition that $x = 1$, $t = 0$.

$\Rightarrow \quad c = 1\ \text{m} \Rightarrow x = t^2 - 6t + 1$

2. g is acceleration due to gravity. So, to find acceleration we need to differentiate velocity.

As $\quad v = 2t - 6 \Rightarrow \quad a = \frac{dv}{dt} = 2\ \text{ms}^{-2}$

So, the acceleration due to gravity on the planet under consideration is $2\ \text{m/s}^{-2}$.

Example 1.26 **Let the instantaneous velocity of a rocket, just after landing, is given by the expression, $v = 2t + 3t^2$ (where v is in ms^{-1} and t is in seconds). Find out the distance traveled by the rocket from $t = 2$ s to $t = 3$ s.**

Sol. To find distance traveled we need to integrate v. [The limits of integration will be from 2 to 3 as we have to find the distance traveled between $t = 2$ s and $t = 3$ s.

$$x = \int_2^3 v dt = \int_2^3 (2t + 3t^2)dt = \left|\frac{2t^2}{2} + \frac{3t^3}{3}\right|_2^3$$

$$= |t^2 + t^3|_2^3 = 24\ \text{m}$$

Example 1.27 **A particle moves with a constant acceleration $a = 2\ \text{ms}^{-2}$ along a straight line. If it moves with an initial velocity of $5\ \text{ms}^{-1}$, then obtain an expression for its instantaneous velocity.**

Sol. We know that acceleration is time rate of change of velocity, i.e., $a = \frac{dv}{dt}$ and differentiation is the inverse operation of integration. So, by integrating acceleration we can obtain the expression of velocity.

So, $\quad v = \int a dt = 2\int dt = 2t + c$

where c is the constant of integration and its value can be obtained from the initial conditions.

At $t = 0$, $v = 5\ \text{ms}^{-1}$. We have $5 = 2\times 0 + c$

$\Rightarrow \quad c = 5\ \text{ms}^{-1}$

Therefore, $v = 2t + 5$ is the required expression for the instantaneous velocity.

Example 1.28 **In the previous problem, if the particle occupies a position $x = 7$ m at $t = 1$ s, then obtain an expression for the instantaneous displacement of the particle.**

Sol. Again, we can use the idea that displacement is the integration of velocity w.r.t. time.

So, $x = \int v dt = \int (2t+5)dt = \frac{2t^2}{2} + 5t + c = t^2 + 5t + c$

where c is the constant of integration. Its value can be determined by using the given condition. (As particular details have been given about the particle.)

At $t = 1$ s, $x = 7$ m $\Rightarrow 7 = 1^2 + 5\times 1 + c \Rightarrow c = 1$ m

Hence, the expression becomes $x = t^2 + 5t + 1$

Concept Application Exercise 1.6

1. Displacement of a particle is given by $y = (6t^2 + 3t + 4)$ m, where t is in second. Calculate the instantaneous speed of the particle.
2. The velocity of a particle is given by $v = 12 + 3(t + 7t^2)$. What is the acceleration of the particle?
3. A particle starts from origin with uniform acceleration. Its displacement after t seconds is given in meter by the relation $x = 2 + 5t + 7t^2$. Calculate the magnitude of its
   a. initial velocity,
   b. velocity at $t = 4$ s,
   c. uniform acceleration, and
   d. displacement at $t = 5$ s.
4. The acceleration of a particle is given by $a = t^3 - 3t^2 + 5$, where a is in ms^{-2} and t in sec. At $t = 1$ s, the displacement and velocity are 8.30 m and 6.25 ms^{-1}, respectively. Calculate the displacement and velocity at $t = 2$ sec.
5. A particle starts moving along x-axis from $t = 0$, its position varying with time as $x = 2t^3 - 3t^2 + 1$.
   a. At which time instants is its velocity zero?
   b. What is the velocity when it passes through origin?
6. A particle moves along x-axis obeying the equation $x = t(t-1)(t-2)$, where x is in meter and t is in second
   a. Find the initial velocity of the particle.
   b. Find the initial acceleration of the particle.
   c. Find the time when the displacement of the particle is zero.
   d. Find the displacement when the velocity of the particle is zero.
   e. Find the acceleration of the particle when its velocity is zero.
7. The speed of a car increases uniformly from zero to 10 ms^{-1} in 2 s and then remains constant (Fig. 1.29).
   a. Find the distance traveled by the car in the first two seconds.

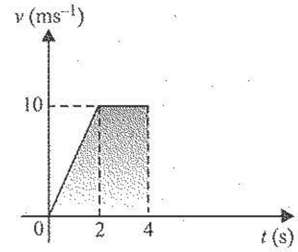


Fig. 1.29

b. Find the distance traveled by the car in the next two seconds.

c. Find the total distance traveled in 4 s.

8. A car accelerates from rest with 2 ms^{-2} for 2 s and then decelerates constantly with 4 ms^{-2} for t_0 second to come to rest. The graph for the motion is shown in Fig. 1.30.

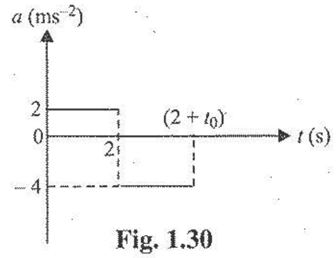


Fig. 1.30

a. Find the maximum speed attained by the car.

b. Find the value of t_0.

9. A stationary particle of mass $m = 1.5$ kg is acted upon by a variable force. The variation of force with respect to displacement is plotted in the following Fig. 1.31.

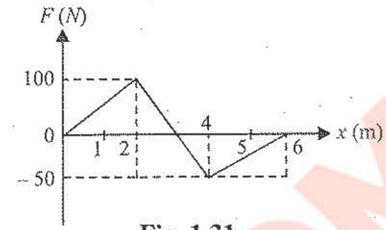


Fig. 1.31

a. Calculate the velocity acquired by the particle after getting displaced through 6 m.

b. What is the maximum speed attained by the particle and at what time is it attained?

10. A body moving along x-axis has, at any instant, its x-coordinate to be $x = a + bt + ct^2$. What is the acceleration of the particle?

11. The displacement of a body at any time t after starting is given by $s = 15t - 0.4t^2$. Find the time when the velocity of the body will be 7 ms^{-1}.

12. A particle moves along a straight line such that its displacement at any time t is given by $s = t^3 - 6t^2 + 3t + 4$ m. Find the velocity when the acceleration is 0.

13. The coordinates of a moving particle at time t are given by $x = ct^2$ and $y = bt^2$. Find the speed of the particle at any time t.

14. The displacement x of a particle moving in one dimension under the action of a constant force is related to time t by the equation $t = \sqrt{x} + 3$, where x is in meter and t is in seconds. Find the displacement of the particle when its velocity is zero.

CHAPTER

2

Vectors

In physics, various quantities are broadly classified into two categories:

1. Scalars, and **2.** Vectors.

SCALARS

Scalars are those physical quantities which have magnitude only but no direction. For example: mass, length, time, work, etc.

VECTORS

Vectors are those physical quantities which have both magnitude and direction. For example: velocity, acceleration, momentum, force, etc.

Further, vectors can be of two types:

1. **Polar vectors:** These are vectors which have a starting point or a point of application.
   For example: velocity, force, linear momentum, etc.
2. **Axial vectors:** These are vectors whose directions are along the axis of rotation.
   For example: angular velocity, angular momentum, torque, etc.

Note: ***A vector is meaningful only if we know both magnitude and direction of a vector. Without knowing direction, the description of a vector quantity is incomplete.***

Use of vectors: Many laws of physics can be expressed in compact form by the use of vectors. In this way, the complicated laws are greatly simplified and then they become easier to apply. For example: $\vec{F} = m\vec{a}$, $\vec{\tau} = I\vec{\alpha}$, $\vec{F} = q\vec{v} \times \vec{B}$, etc.

REPRESENTATION OF VECTOR

A vector is represented by a directed line segment, with an arrow head. For example, a vector $\vec{F}$ is represented by a directed line PQ (see Fig. 2.1).

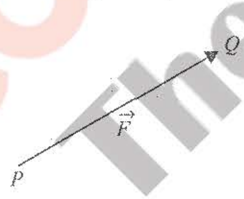


Fig. 2.1

Point P is called *tail* or *origin* of the vector. Point Q, the end point of vector, is called *tip*, *head* or *terminal point* of the vector.

1. The length of the line represents the magnitude of the vector.
2. The arrow head represents the direction of the vector.

NOTATION OF VECTOR

1. By single letter with an arrow overhead.
   For example: Force can be represented by $\vec{F}$ and its magnitude is represented as $\left|\vec{F}\right|$.
2. By bold letters.
   For example: Force can be represented by **F**. In this case, magnitude is represented in the same way as above such as $\left|\vec{F}\right|$ or by same symbol, but *Italic* F.

INTRODUCTION TO DIFFERENT TYPES OF VECTORS

Collinear Vectors

The vectors which either act along the same line or along the parallel lines are called collinear vectors. These vectors may act either in the same direction or in the opposite direction.

Parallel Vectors

Two collinear vectors having the same direction are called parallel or like vectors (see Fig. 2.2). Angle between them is 0°.

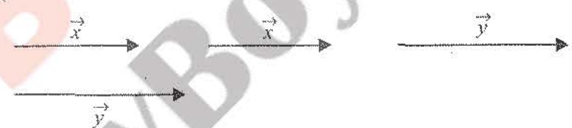


Fig. 2.2

Antiparallel Vectors

Two collinear vectors acting in opposite directions are called antiparallel or opposite vectors (see Fig. 2.3). The angle between them is 180° or π-radian.

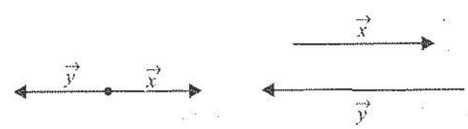


Fig. 2.3

Equal Vectors

Two vectors are said to be equal vectors if they have equal magnitude and same direction. Two equal vectors $\vec{A}$ and $\vec{B}$ are represented by two equal and parallel lines having arrow heads in the same direction (see Fig. 2.4).

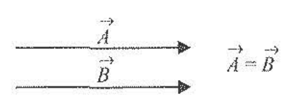


Fig. 2.4

Negative of a Vector

A negative vector of a given vector is a vector having same magnitude with the direction opposite to that of given vector (see Fig. 2.5). The negative vector of $\vec{A}$ is represented by $-\vec{A}$.

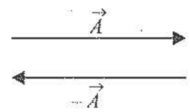


Fig. 2.5

Coplanar Vectors

These are vectors which lie in the same plane. In Fig. 2.6, $\vec{A}$, $\vec{B}$, and $\vec{C}$ are acting in the same plane, i.e., XY plane, so they are coplanar vectors.

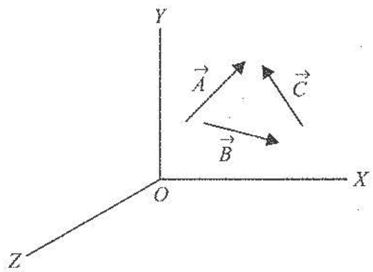


Fig. 2.6

Unit Vector

It is a vector whose magnitude is equal to unity (one). A unit vector in a given direction can be obtained by dividing a vector in that direction by its magnitude. Unit vector of $\vec{A}$ is written as $\hat{A}$ and is read as A cap.

So, $\hat{A} = \dfrac{\vec{A}}{A}$, where $\hat{A}$ is the unit vector along the direction of $\vec{A}$.

The direction of unit vector will be same as that of the vector from which it is obtained. It means $\hat{A}$ is parallel to $\vec{A}$.

Also, from above: $\vec{A} = A\hat{A}$ or

vector = magnitude × direction

In every direction, we can obtain a unit vector. In x, y and z directions, unit vectors are predefined (Fig. 2.7).

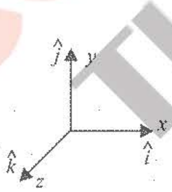


Fig. 2.7

$\hat{i}$ is the unit vector along x-axis,
$\hat{j}$ is the unit vector along y-axis, and
$\hat{k}$ is the unit vector along z-axis.

Now, a unit vector in any other direction can be obtained in terms of $\hat{i}$, $\hat{j}$, and $\hat{k}$ as shown in the next section.

Note: ***A unit vector is a dimensionless quantity, so it has no units. It represents only direction.***

Expression of a Vector in terms of $\hat{i}$, $\hat{j}$, and $\hat{k}$ and Magnitude of a Vector

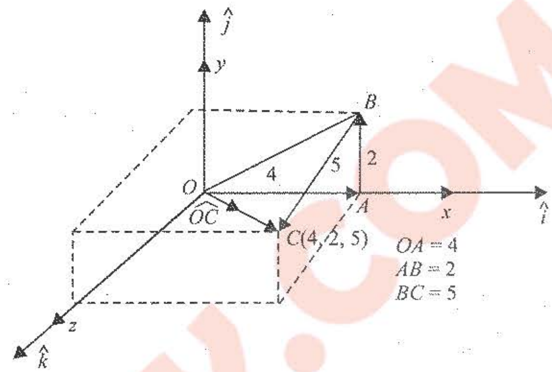


Fig. 2.8

Suppose a particle goes from O to A, 4 m along x-axis; A to B, 2 m along y-axis; and then B to C, 5 m along z-axis (Fig. 2.8). Finally, the particle reaches at C.

Coordinates of C: (4, 2, 5) m

We can write: $\overrightarrow{OA} = 4\hat{i}$, $\overrightarrow{AB} = 2\hat{j}$, and $\overrightarrow{BC} = 5\hat{k}$.

O is the initial position and C is the final position. $\overrightarrow{OC}$ is the net displacement, which is the resultant of $\overrightarrow{OA}$, $\overrightarrow{AB}$, and $\overrightarrow{BC}$.

So, we can write: $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

i.e., $\overrightarrow{OC} = 4\hat{i} + 2\hat{j} + 5\hat{k}$

(Hence, here we represent $\overrightarrow{OC}$ in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$.)

Magnitude (modulus) of $\overrightarrow{OC}$**:** It is written as $\left|\overrightarrow{OC}\right|$ or OC.

In ΔOAB: $OB^2 = OA^2 + AB^2$.

In ΔOBC: $OC^2 = OB^2 + BC^2$ $\quad [\because \angle OBC = 90°]$

$$\Rightarrow \quad OC^2 = OA^2 + AB^2 + BC^2$$

$$\Rightarrow \quad OC = \sqrt{OA^2 + AB^2 + BC^2}$$

$$= \sqrt{4^2 + 2^2 + 5^2} = \sqrt{45}$$

$\Rightarrow \quad OC = |\overrightarrow{OC}| = \sqrt{45}$. This is the magnitude of $\overrightarrow{OC}$.

Note: ***Magnitude of a vector or modulus of a vector is equal to length of the vector.***

Unit vector: $\widehat{OC} = \dfrac{\overrightarrow{OC}}{OC} = \dfrac{4\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{45}}$

$$= \frac{4}{\sqrt{45}}\hat{i} + \frac{2}{\sqrt{45}}\hat{j} + \frac{3}{\sqrt{45}}\hat{k}$$

(Hence, we express a unit vector in the direction of $\overrightarrow{OC}$ in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$.)

Important: Direction of $\widehat{OC}$ and $\overrightarrow{OC}$ will be same.

Now, in general, if we have a point P in space whose coordinates are x, y and z, then a vector from origin O to P is known as position vector of P w.r.t. origin O (Fig. 2.9). We can write:

$$\vec{r} = \overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

Magnitude: $r = OP = \sqrt{x^2 + y^2 + z^2}$,

Unit vector: $\hat{r} = \dfrac{\vec{r}}{r} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

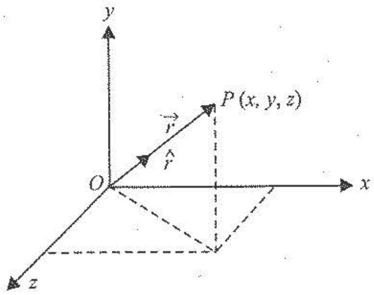


Fig. 2.9

Position Vector and Displacement Vector

Consider two points A and B in space whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively (Fig. 2.10).

Position vector of A: $\overrightarrow{OA} = \vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$,

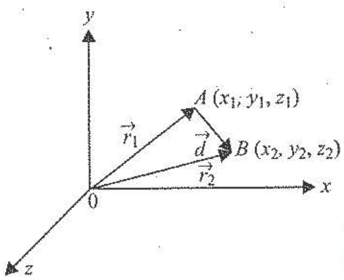


Fig. 2.10

Position vector of B: $\overrightarrow{OB} = \vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Position vector provides us some important information: It gives an idea about the direction and the distance of the point from origin in space.

Now consider a particle which goes from point A to B. Then, $\overrightarrow{AB} = \vec{d}$ is displacement vector of the particle. *Displacement vector is that vector which tells us how much and in which direction an object has changed its position in a given interval of time.*

Here, $\vec{r_1}$ is initial position vector of the particle and $\vec{r_2}$ is final position vector of the particle.

$\therefore$ displacement vector = final position vector – initial position vector

$$\Rightarrow \quad \vec{d} = \vec{r_2} - \vec{r_1}$$

$$\Rightarrow \quad \vec{d} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Note: ***Position vector gives us information about position of a point in space, displacement vector about displacement. Similarly, a velocity vector or a force vector will tell us about velocity or force and their magnitudes and directions. So, a vector gives us information about the same physical quantity by which it is known.***

- Position vector is also known as radius vector.
- If A and B are two points whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively, then a vector from A to B is given by (Fig. 2.11):

  $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

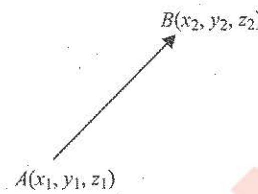


Fig. 2.11

- A vector can be completely zero, if all of its individual components are zero. For example: let a vector $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is given. If $\vec{A} = 0$, then it is possible only if $a_1 = 0$, $a_2 = 0$ and $a_3 = 0$.
- If two vectors are equal, then their individual components are also equal separately.
  For example: Let two vectors $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are given. If $\vec{A} = \vec{B}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$ (but if $A = B$, then it means $\sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{b_1^2 + b_2^2 + b_3^2}$.

Illustration 2.1 **A particle initially at point $A(2, 4, 6)$ m moves finally to the point $B(3, 2, -3)$ m. Write the initial position vector, final position vector and displacement vector of the particle.**

Sol. Initial position vector: $\vec{r}_1 = 2\hat{i} + 4\hat{j} + 6\hat{k}$

Final position vector: $\vec{r}_2 = 3\hat{i} + 2\hat{j} - 3\hat{k}$

Displacement: $\vec{d} = \vec{r}_2 - \vec{r}_1 = (3-2)\hat{i} + (2-4)\hat{j} + (-3-6)\hat{k}$
$= \hat{i} - 2\hat{j} - 9\hat{k}$

Illustration 2.2 **A particle has the following displacements in succession: (i) 12 m towards east, (ii) 5 m towards north, and (iii) 6 m vertically upwards. Find the magnitude of the resultant displacement.**

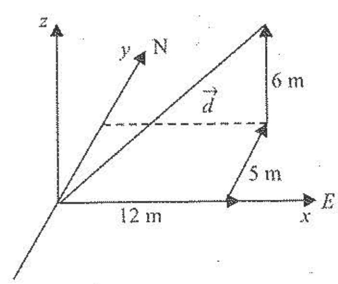


Fig. 2.12

Sol. From Fig. 2.12: $\vec{d} = 12\hat{i} + 5\hat{j} + 6\hat{k}$,

Magnitude: $d = \sqrt{12^2 + 5^2 + 6^2} = \sqrt{205}$ m

Illustration 2.3 **Determine a vector which when added to the resultant of $\vec{A} = 2\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{B} = 3\hat{i} - 5\hat{j} - \hat{k}$ gives unit vector along negative y-direction.**

Sol. Let $\vec{C}$ be the vector which we have to find.

Then, given: $(\vec{A} + \vec{B}) + \vec{C} = -\hat{j}$

$$\Rightarrow \quad (2\hat{i} + 5\hat{j} - \hat{k}) + (3\hat{i} - 5\hat{j} - \hat{k}) + \vec{C} = -\hat{j}$$

$$\Rightarrow \quad \vec{C} = -5\hat{j} - \hat{j} + 2\hat{k}$$

Resultant Vector

The resultant vector of two or more vectors is defined as that single vector which produces the same effect both in magnitude and direction as produced by individual vectors taken together.

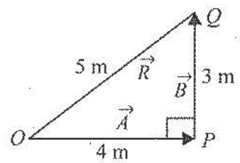


Fig. 2.13

Example: Let a person goes from O to P traveling 4 m and then P to Q traveling 3 m perpendicular to OP. The person could also go directly from O to Q, traveling 5 m (Fig. 2.13). In both the cases, displacement of the person would be same. Here, $\overrightarrow{OQ}$ produces same effect as produced by $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ taken together.

So, $\overrightarrow{OQ}$ is called the resultant of $\overrightarrow{OP}$ and $\overrightarrow{PQ}$. We can also write $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$.

Note: ***Resultant means addition.***

We cannot write $OQ = OP + PQ$, because $OQ = 5$ m and $OP + PQ = 4 + 3 = 7$ m.

Points to Remember

1. If two vectors $\vec{A}$ and $\vec{B}$ are parallel, then we can write $\vec{B} = m\vec{A}$, where m is a number. If vectors are parallel, then m is +ve and if vectors are antiparallel, then m is –ve (Fig. 2.14).

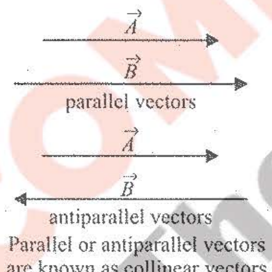


Fig. 2.14

2. If two vectors are parallel, then their unit vectors are equal, i.e., $\hat{A} = \hat{B}$ and if two vectors are antiparallel, then we have $\hat{A} = -\hat{B}$.
3. If three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar, then we can write: $\vec{A} = m\vec{B} + n\vec{C}$, where m and n are some numbers. Minimum number of unequal vectors whose sum can be zero is three and these three vectors must be coplanar.
4. If four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ are in any arbitrary directions or in different planes, then we can write: $\vec{A} = m\vec{B} + n\vec{C} + p\vec{D}$, where m, n and p are some numbers. The resultant of three non-coplanar vectors can never be zero. *The minimum number of non-coplanar vectors whose sum can be zero is four.*

Illustration 2.4 **Two vectors $\vec{P} = 2\hat{i} - b\hat{j} + 2\hat{k}$ and $\vec{Q} = \hat{i} + \hat{j} + \hat{k}$ are parallel. Find the value of b.**

Sol. If $\vec{P}$ and $\vec{Q}$ are parallel, then we have $\vec{P} = m\vec{Q}$, where m is some number.

$\Rightarrow \quad 2\hat{i} - b\hat{j} + 2\hat{k} = m(\hat{i} + \hat{j} + \hat{k})$

$\Rightarrow \quad 2\hat{i} - b\hat{j} + 2\hat{k} = m\hat{i} + m\hat{j} + m\hat{k}$

Equating the components of $\hat{i}$ and $\hat{j}$ from both sides: $m = 2$ and $-b = m \Rightarrow \quad b = -2$

Illustration 2.5 **If vectors $2\hat{i} + 2\hat{j} - 2\hat{k}$, $5\hat{i} + y\hat{j} + \hat{k}$, and $\hat{i} + 2\hat{j} + 2\hat{k}$ are coplanar, then find the value of y.**

Sol. If $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar, then we have: $\vec{A} = m\vec{B} + n\vec{C}$

$\Rightarrow 2\hat{i} + 2\hat{j} - 2\hat{k} = m(5\hat{i} + y\hat{j} + \hat{k}) + n(\hat{i} + 2\hat{j} + 2\hat{k})$

$\Rightarrow 2\hat{i} + 2\hat{j} - 2\hat{k} = (5m + n)\hat{i} + (my + 2n)\hat{j} + (m + 2n)\hat{k}$

Equating components of $\hat{i}$, $\hat{j}$, and $\hat{k}$ from both sides:

$$5m + n = 2,\ my + 2n = 2,\ m + 2n = -2.$$

Solving them, we get: $m = 2/3$, $n = -4/3$ and $y = 7$.

ADDITION OF VECTORS

How to Add Two Vectors Graphically (Tip to Tail Method)

1. Draw the two vectors by arrow head lines using the same suitable scale.
2. Put the second vector such that its tail coincides with the head or tip of the first vector.
3. Now, draw a single vector from the tail of the first vector to the head of the second vector. This single vector represents the resultant of the two vectors.

Let us discuss some cases:

1. When two vectors are acting in the same direction (Fig. 2.15). Let the two vectors $\vec{x}$ and $\vec{y}$ be acting in the same direction.

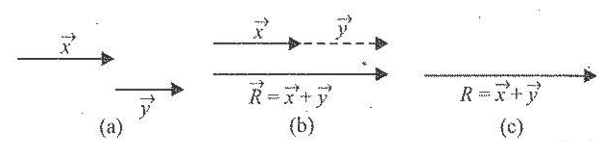


Fig. 2.15

2. When two vectors are acting in opposite directions: To find the resultant, in this case, coincide the head of $\vec{x}$ on the tail of $\vec{y}$ and then draw a single vectors $\vec{R}$ from the tail of $\vec{x}$ to the head of $\vec{y}$. The vector $\vec{R}$ gives the resultant of vectors $\vec{x}$ and $\vec{y}$ (see Fig. 2.16).

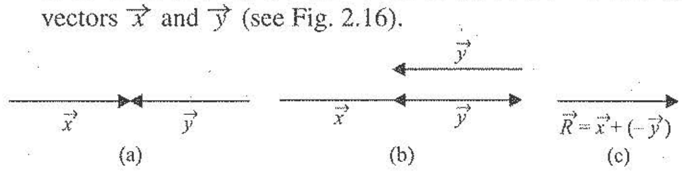


Fig. 2.16

The direction of the resultant vector is the same as that of bigger vector.

3. When two vectors are acting at some angle:
   First join the tail of $\vec{y}$ with the head of $\vec{x}$ and then, to find the resultant in this case, draw a vector $\vec{R}$ from the tail of $\vec{x}$ to the head of $\vec{y}$. This single vector $\vec{R}$ drawn is the resultant vector (Fig. 2.17).

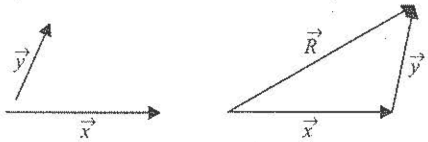


Fig. 2.17

$\vec{R}$ represents the resultant of $\vec{x}$ and $\vec{y}$ both in magnitude and direction. So, $\vec{R} = \vec{x} + \vec{y}$.

TRIANGLE LAW OF VECTOR ADDITION

If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented by the third side of the triangle (both in magnitude and direction) taken in reverse direction.

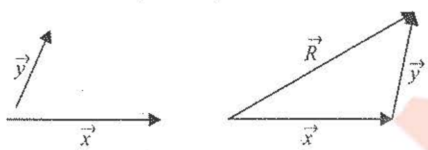


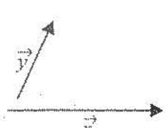
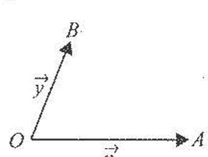
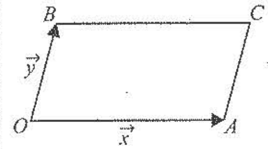
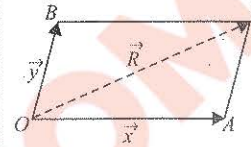
Fig. 2.18

Suppose $\vec{x}$ and $\vec{y}$ are two vectors acting on a particle at the same time. They are represented as two sides of a triangle and $\vec{R}$ represents the third side (Fig. 2.18).

Thus, $\vec{R}$ represents the resultant of $\vec{x}$ and $\vec{y}$ both in magnitude and direction. So, we can write: $\vec{R} = \vec{x} + \vec{y}$.

PARALLELOGRAM LAW OF VECTOR ADDITION

It states that if two vectors can be represented both in magnitude and direction by two adjacent sides of a parallelogram, then the resultant is represented completely both in magnitude and direction by the corresponding diagonal of the parallelogram.

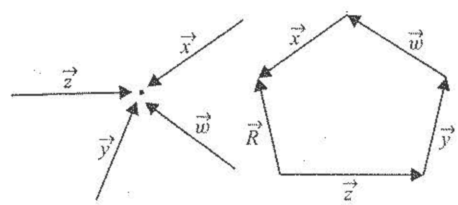
(i) Consider two vectors $\vec{x}$ and $\vec{y}$ as shown.  (a)	(ii) Draw these two vectors $\vec{x}$ and $\vec{y}$ from a common point O.  (b)
(iii) Now complete the ‖ gm as shown.  (c)	(iv) Draw the diagonal $\overrightarrow{OC}$ which represents resultant of $\vec{x}$ and $\vec{y}$. $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$ So, $\vec{R} = \vec{x} + \vec{y}$  (d)

ADDITION OF MORE THAN TWO VECTORS

For this, we can use following steps:

1. Represent these vectors by arrow head lines using the same suitable scale.
2. Put these vectors in such a way that the head of one coincides with the tail of second and so on to the last vector.
3. Then, draw a single vector from the tail of first vector to the head of the last vector.
4. This single vector represents the resultant of all the vectors.

The above mentioned process may be known as *polygon law of vector addition*. It states that if any number of vectors acting on a particle at the same time are represented in magnitude and direction by the sides of an open polygon taken in order, then their resultant is represented both in magnitude and direction by the closing side of the polygon. Consider four vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, $\vec{z}$ as shown in Fig. 2.19(a).



(a) (b)

Fig. 2.19

Then, $\vec{R} = \vec{x} + \vec{w} + \vec{y} + \vec{z}$, as shown in Fig. 2.19(b), represents the resultant of $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$.

VECTOR ADDITION BY ANALYTICAL METHOD

Here, we will treat both triangle law and the parallelogram law of vector addition analytically to find the resultant of two vectors.

1. Analytical treatment of triangle law of vector addition: Let us consider two vectors $\vec{P}$ and $\vec{Q}$ acting simultaneously on a particle and inclined at an angle θ. Let these vectors be represented both in magnitude and direction by the two sides $\overrightarrow{OA}$ and $\overrightarrow{AC}$ of ΔOAC taken in same order. Then, the third side

$\overrightarrow{OC}$ represents the resultant (taken in opposite order) (Fig. 2.20).

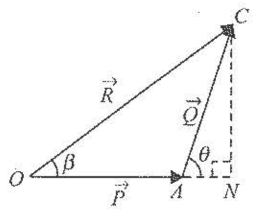


Fig. 2.20

Draw $CN \perp ON$ (extended).

From the right angled ΔCNO,

$$OC^2 = R^2 = ON^2 + NC^2 = (OA + AN)^2 + NC^2$$

$$\Rightarrow \quad R^2 = (P + AN)^2 + NC^2 \qquad \text{(i)}$$

In right-angled ΔANC,

$$\sin\theta = \frac{NC}{Q} \quad \Rightarrow \quad NC = Q\sin\theta \text{ and } \cos\theta = \frac{AN}{Q}$$

$$\Rightarrow \quad AN = Q\cos\theta$$

From equation (i): $R^2 = (P + Q\cos\theta)^2 + Q^2\sin^2\theta$

$$\Rightarrow \quad R^2 = P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$

$$\Rightarrow \quad R^2 = P^2 + Q^2(\cos^2\theta + \sin^2\theta) + 2PQ\cos\theta$$

$$\Rightarrow \quad R^2 = P^2 + Q^2 + 2PQ\cos\theta \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$\Rightarrow \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} \qquad \text{(ii)}$$

The equation of the magnitude of the resultant vector can be written in either of the following two ways.

1. $\left|\vec{R}\right| = \sqrt{\left|\vec{P}\right|^2 + \left|\vec{Q}\right|^2 + 2\left|\vec{P}\right|\left|\vec{Q}\right|\cos\theta}$

2. $\left|\vec{P} + \vec{Q}\right| = \sqrt{\left|\vec{P}\right|^2 + \left|\vec{Q}\right|^2 + 2\left|\vec{P}\right|\left|\vec{Q}\right|\cos\theta}$

Let β be the angle which $\vec{R}$ makes with $\vec{P}$. Then,

$$\tan\beta = \frac{NC}{ON} = \frac{NC}{OA + AN} = \frac{Q\sin\theta}{P + Q\cos\theta}$$

$$\therefore \quad \beta = \tan^{-1}\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right) = \tan^{-1}\left(\frac{|\vec{Q}|\sin\theta}{|\vec{P}| + |\vec{Q}|\cos\theta}\right)$$

which gives the direction of the resultant vector.

2. Analytical treatment of parallelogram law of vector addition: Let us consider two vectors $\vec{P}$ and $\vec{Q}$ acting simultaneously at a point and let us further assume that they can be represented both in magnitude and direction by the two adjacent sides of a parallelogram $\overrightarrow{OA}$ and $\overrightarrow{OB}$ inclined at an angle θ w.r.t. each other. Draw $CN \perp ON$ (extended) (Fig. 2.21).

Remaining procedure is same as above in triangle law.

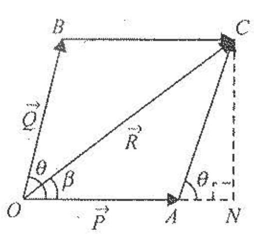


Fig. 2.21

Special Cases

Case 1. When the given vectors act in the same direction ($\theta = 0°$).

So, $R = \sqrt{P^2 + Q^2 + 2PQ\cos 0°}$ [From equation (ii)]

$$= \sqrt{P^2 + Q^2 + 2PQ} = \sqrt{(P+Q)^2} = P + Q$$

$$[\because \cos 0° = 1]$$

or $\left|\vec{R}\right| = \left|\vec{P}\right| + \left|\vec{Q}\right|$ which represents the magnitude of the resultant.

Case 2. When the given vectors act in opposite directions ($\theta = 180°$).

So, $R = \sqrt{P^2 + Q^2 + 2PQ\cos 180°} = \sqrt{P^2 + Q^2 - 2PQ}$

$$= \sqrt{(P-Q)^2} \qquad [\because \cos 180° = -1]$$

$$R = \pm(P - Q) = P - Q \quad \text{or} \quad Q - P$$

or $\left|\vec{R}\right| = \left|\left|\vec{P}\right| - \left|\vec{Q}\right|\right|$ which represents the magnitude of the resultant.

Case 3. When the given vectors $\vec{P}$ and $\vec{Q}$ act at right angle to each other ($\theta = 90°$).

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos 90°} = \sqrt{P^2 + Q^2}$$

$$[\because \cos 90° = 0]$$

or $\left|\vec{R}\right| = \sqrt{\left|\vec{P}\right|^2 + \left|\vec{Q}\right|^2}$ and $\tan\beta = \dfrac{Q\sin 90°}{P + Q\cos 90°}$

or $\tan\beta = \dfrac{Q}{P}$ $\qquad [\because \sin = 90° = 1]$

Note: ***Subtraction of two vectors can be followed from addition***

Suppose we have to subtract a vector $\vec{B}$ from $\vec{A}$. So, we have to find $\vec{A} - \vec{B}$. It can also be written as $\vec{A} + (-\vec{B})$. Hence, subtraction of a vector $\vec{B}$ from $\vec{A}$ becomes as the addition of vectors $\vec{A}$ and $-\vec{B}$ (Fig. 2.22).

$$S = \sqrt{A^2 + B^2 + 2AB\cos(180 - \theta)}$$

$$\Rightarrow \quad S = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

$$\tan\beta = \frac{B\sin(180-\theta)}{A + B\cos(180-\theta)} \quad \Rightarrow \tan\beta = \frac{B\sin\theta}{A - B\cos\theta}$$

Vector addition is commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, Vector subtraction is anti-commutative: $\vec{A} - \vec{B} = -(\vec{B} - \vec{A})$ (Fig. 2.23).

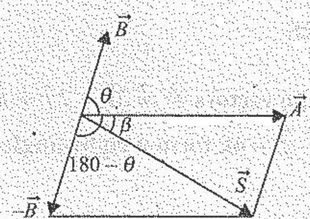


Fig. 2.22

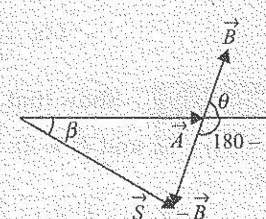


Fig. 2.23

Illustration 2.6 **Two forces of 10 N and 15 N are acting at a point at an angle of 45° with each other. Find out the magnitude and the direction of their resultant force.**

Sol. Given $A = 10$ N, $B = 15$ N, $\theta = 45°$

Magnitude of resultant force: $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$= \sqrt{10^2 + 15^2 + 2 \times 10 \times 15 \times \cos 45°} = 23.76$ N

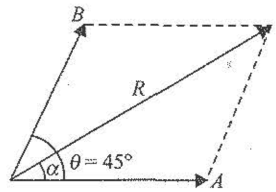


Fig. 2.24

Direction: $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta} = \dfrac{15\sin 45°}{10 + 15\cos 45°} = 0.5147$

$\Rightarrow \quad \alpha = \tan^{-1}(0.5147) = 27°$

Illustration 2.7 **Two forces of equal magnitudes are acting at a point. The magnitude of their resultant is equal to magnitude of the either. Find the angle between the force vectors.**

Sol. Given $R = A = B$. Using $R^2 = A^2 + B^2 + 2AB\cos\theta$.

$A^2 = A^2 + A^2 + 2AA\cos\theta$

$\Rightarrow \cos\theta = -\dfrac{1}{2} \quad \Rightarrow \theta = 120°$

Condition for Zero Resultant Vectors

1. The resultant of two vectors can only be zero if they are equal in magnitude and opposite in direction (Fig. 2.25).

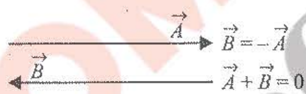


Fig. 2.25

2. The resultant of three or more vectors can be zero if they constitute a close figure when taken in same order (Fig. 2.26).

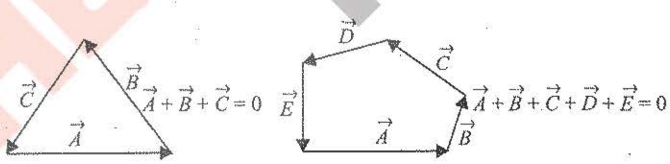


Fig. 2.26

Illustration 2.8 **Show that the vectors $\vec{P} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{Q} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{R} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right-angled triangle.**

Sol. $\quad P = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$,

$Q = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35}$,

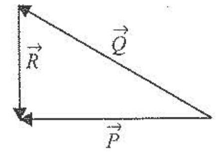


Fig. 2.27

and

$$R = \sqrt{2^2 + 1^2 + (-4)^2} = \sqrt{21}$$

We see that $\vec{P} = \vec{Q} + \vec{R}$ and $Q^2 = P^2 + R^2$ is satisfied. Hence, they will form a right-angled triangle, with $\vec{Q}$ as hypotenuse and $\vec{P}$ and $\vec{R}$ the other two sides as shown in Fig. 2.27.

Lami's Theorem

It states that if the resultant of three vectors is zero, then magnitude of a vector is directly proportional to the sine of angle between the other two vectors (see Fig. 2.28). Or it can be stated as *if the resultant of three vectors is zero, then the ratio of magnitude of a vector to the sine of angle between the other two vectors is constant, i.e.,*

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$$

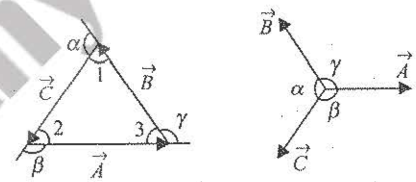


Fig. 2.28

Illustration 2.9 **Given that $\vec{A} + \vec{B} + \vec{C} = \vec{0}$. Out of three vectors two are equal in magnitude and the magnitude of the third vector is $\sqrt{2}$ times that of either of the two having equal magnitude. Find the angles between the vectors.**

Sol. Given $A = B$, $C = \sqrt{2}A = \sqrt{2}B = B$. From Fig. 2.29:

$\alpha = \beta$ and $\alpha + \beta + \gamma = 180° \quad \Rightarrow \gamma = 180° - 2\alpha$

Apply Lami's theorem: $\dfrac{A}{\sin\alpha} = \dfrac{C}{\sin y}$

$\Rightarrow \quad \dfrac{A}{\sin\alpha} = \dfrac{\sqrt{2}A}{\sin(180 - 2\alpha)}$

$\Rightarrow \quad \dfrac{1}{\sin\alpha} = \dfrac{\sqrt{2}}{\sin 2\alpha}$

$\Rightarrow \quad \dfrac{1}{\sin\alpha} = \dfrac{\sqrt{2}}{2\sin\alpha\cos\alpha} \Rightarrow \cos\alpha = \dfrac{1}{\sqrt{2}} \Rightarrow \alpha = 45°$

$\Rightarrow \quad \beta = 45°$ and $\gamma = 180° - 2\alpha = 90°$

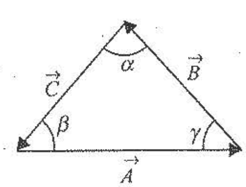


Fig. 2.29

Angle between $\vec{A}$ and $\vec{B} = 180° - \gamma = 90°$, angle between $\vec{B}$ and $\vec{C} = 180 - \alpha = 135°$, angle between $\vec{C}$ and $\vec{A} = 180° - \beta = 135°$.

Concept Application Exercise 2.1

1. A force of $(2\hat{i} + 3\hat{j} + \hat{k})$ N and another force of $(\hat{i} + \hat{j} + \hat{k})$ N are acting on a body. What is the magnitude of total force acting on the body?
2. If $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = 7\hat{i} + 24\hat{j}$, then find the vector having the same magnitude as that of $\vec{b}$ and parallel to $\vec{a}$.
3. In the vector diagram given below (Fig. 2.30), what is the angle between $\vec{A}$ and $\vec{B}$? (Given: $C = \dfrac{B}{2}$)

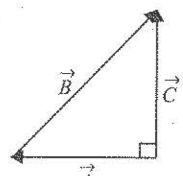


Fig. 2.30

4. What is the angle made by $3\hat{i} + 4\hat{j}$ with x-axis?
5. Three forces $\vec{A} = (\hat{i} + \hat{j} + \hat{k})$, $\vec{B} = (2\hat{i} - \hat{j} + 3\hat{k})$, and $\vec{C}$ are acting on a body which is kept at equilibrium. Find $\vec{C}$.
6. At what angle should the two force vectors $2F$ and $\sqrt{2}F$ act so that the resultant force is $\sqrt{10}F$?
7. Two forces while acting on a particle in opposite directions, have the resultant of 10 N. If they act at right angles to each other, the resultant is found to be 50 N. Find the two forces.
8. Two forces each equal to $F/2$ act at right angle. Their effect may be neutralized by a third force acting along their bisector in the opposite direction. What is the magnitude of that third force?
9. The resultant of two forces has magnitude 20 N. One of the forces is of magnitude $20\sqrt{3}$ N and makes an angle of 30° with the resultant. What is the magnitude of the other force?
10. The sum of the magnitudes of two vectors is 18. The magnitude of their resultant is 12. If the resultant is perpendicular to one of the vectors, then find the magnitudes of the two vectors.

RECTANGULAR COMPONENTS OF A VECTOR IN TWO DIMENSIONS

When a vector is split into two mutually perpendicular directions in a plane, the component vectors are called rectangular components of the given vector in a plane.

Fig. 2.31 shows vector $\vec{A}$ represented by $\overrightarrow{OP}$.

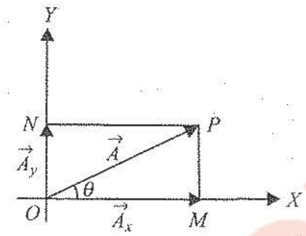


Fig. 2.31

Component along x-axis $A_x = A\cos\theta$ (Horizontal component) (1)

Component along y-axis $A_y = A\sin\theta$ (Vertical component) (2)

So, $\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{i} + A_y\hat{j}$

Squaring and adding equations (1) and (2), we get

$$A_x^2 + A_y^2 = A^2(\sin^2\theta + \cos^2\theta) = A^2$$

$$\Rightarrow \quad A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \tan\theta = \frac{PM}{OM} = \frac{A_y}{A_x}$$

So, knowledge of components of a vector gives information about angle which the vector makes with different axes.

Illustration 2.10 **A force of 10 N is inclined at an angle of 30° to the horizontal. Find the horizontal and vertical components of the force.**

Sol. Let $R = 10$ N. Horizontal component:

$$R_x = R\cos 30° = 10\sqrt{3}/2 = 5\sqrt{3}\ \text{N}$$

Vertical component: $R_y = R\sin 30° = 10/2 = 5$ N

Illustration 2.11 **The x and y components of vector $\vec{A}$ are 4 and 6 m, respectively. The x and y components of vector $\vec{A} + \vec{B}$ are 10 and 9 m, respectively. Calculate for the vector $\vec{B}$ the followings: (i) its x and y components; (ii) its length, assuming that $\vec{A}$ and $\vec{B}$ lie in $x - y$ plane; and (iii) the angle it makes with the x-axis.**

Sol. Given $\vec{A} = 4\hat{i} + 6\hat{j}$ (1)

and $\vec{A} + \vec{B} = 10\hat{i} + 9\hat{j}$ (2)

Subtract equation (1) from (2), we get $\vec{B} = 6\hat{i} + 3\hat{j}$

1. Hence, x and y components of $\vec{B}$ are 6 and 3 m, respectively.
2. Length of $\vec{B}$ = magnitude of $\vec{B} = \sqrt{6^2 + 3^2} = 3\sqrt{5}$ m.
3. Let $\vec{B}$ makes an angle α with x-axis, then $\tan\alpha = 3/6$.
   $\Rightarrow \alpha = \tan^{-1}(1/2) = 26.6°$

RECTANGULAR COMPONENTS OF A VECTOR IN THREE DIMENSIONS

When a vector is split into mutually perpendicular directions in 3-D space, the component vectors obtained are called rectangular components of the given vector in 3-D space.

Fig. 2.32 shows vector $\vec{A}$ represented by $\overrightarrow{OP}$.

Here, $\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

The magnitude of $\vec{A}$ is given by, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

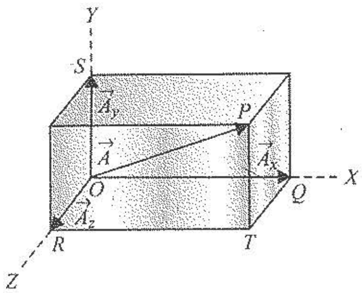


Fig. 2.32

Direction Cosines

Let A is a point in space whose coordinates are (x, y, z), then its position vector w.r.t. the origin of coordinate system is given by: $\vec{r} = \overrightarrow{OA} = x\hat{i} + y\hat{j} + z\hat{k}$ (see Fig. 2.33).

And $r = OA = \sqrt{x^2 + y^2 + z^2}$

Angles of $\vec{r}$ with x-, y- and z-axis, respectively, are given by:

$$\cos\alpha = \frac{x}{r} = l, \cos\beta = \frac{y}{r} = m, \cos\gamma = \frac{z}{r} = n$$

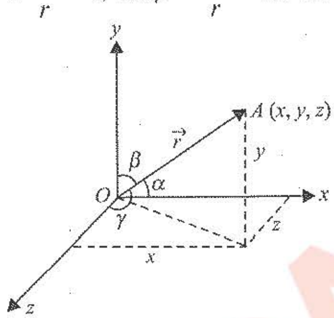


Fig. 2.33

The direction cosines l, m, and n of a vector are the cosines of the angles α, β and γ which a given vector makes with x-, y-, and z-axis, respectively.

Now, squaring and adding l, m, and n

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2 + y^2 + z^2}{r^2}$$

or $$l^2 + m^2 + n^2 = \frac{r^2}{r^2} = 1$$

It means the sum of squares of the direction cosines of a vector is always unity.

Illustration 2.12 **Given $\vec{A} = 5\hat{i} + 2\hat{j} + 4\hat{k}$. Find: (i) $|\vec{A}|$ and (ii) the direction cosines of vector $\vec{A}$.**

Sol.

(i) As $\vec{A} = 5\hat{i} + 2\hat{j} + 4\hat{k} \Rightarrow \quad |\vec{A}| = \sqrt{25 + 4 + 16} = \sqrt{45}$

(ii) $\cos\alpha = l = \frac{x}{r} = \frac{5}{\sqrt{45}}, \cos\beta = m = \frac{y}{r} = \frac{2}{\sqrt{45}},$

$\cos\gamma = n = \frac{z}{r} = \frac{4}{\sqrt{45}}$

PRODUCT OF TWO VECTORS

There are two ways of vector multiplication.

1. Scalar or dot product
2. Vector or cross product.

Scalar or Dot Product

The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of two vectors and the cosine of the smaller angle between them (Fig. 2.34). It is given by $\vec{A} \cdot \vec{B} = AB\cos\theta$.

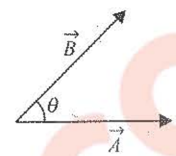


Fig. 2.34

Special Cases

1. If $\theta = 0°$, $\vec{A} \cdot \vec{B} = AB$ (maximum value) $[\because \cos 0° = 1]$
2. If $\theta = 180°$, $\vec{A} \cdot \vec{B} = -AB$ (negative maximum value) $[\because \cos 180° = -1]$
3. If $\theta = 90°$, $\vec{A} \cdot \vec{B} = 0$ (minimum value) $[\because \cos 90° = 0]$

   So, if two vectors are perpendicular, then their dot product is zero.
4. If θ is acute, then $\vec{A} \cdot \vec{B}$ is +ve. $[\because \cos\theta$ is +ve when θ is acute$]$
5. If θ is obtuse, then $\vec{A} \cdot \vec{B}$ is −ve. $[\because \cos\theta$ is −ve when θ is obtuse$]$

Note: ***The dot product of two vectors is always a scalar quantity.***

Dot Product of Unit Vectors Along x-, y-, and z-directions

Dot product of a unit vector with itself is unity and with other perpendicular unit vectors is zero (Fig. 2.35).

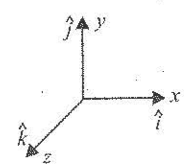


Fig. 2.35

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \text{ and}$$
$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = \hat{k} \cdot \hat{i} = \hat{i} \cdot \hat{k} = 0$$

In component form, the product is expressed as:

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$. Then

$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_x\hat{i}(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

So, $\vec{A} \cdot \vec{B} = A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y(\hat{i}\cdot\hat{j}) + A_xB_z(\hat{i}\cdot\hat{k}) + A_yB_x(\hat{j}\cdot\hat{i}) + A_yB_y(\hat{j}\cdot\hat{j}) + A_yB_z(\hat{j}\cdot\hat{k}) + A_zB_x(\hat{k}\cdot\hat{i}) + A_zB_y(\hat{k}\cdot\hat{j}) + A_zB_z(\hat{k}\cdot\hat{k}) = A_xB_x + A_yB_y + A_zB_z$

or $\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$

Illustration 2.13 **Find the dot product of two vectors $\vec{A} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ and $\vec{B} = 2\hat{i} - 3\hat{j} - 6\hat{k}$.**

Sol. $\vec{A}\vec{B} = 3 \times 2 + 2 \times (-3) + (-4) \times (-6) = 24$

Dot product of a vector with itself

A vector is parallel to itself. So, the angle of a vector with itself is zero.

$\therefore \quad \vec{A} \cdot \vec{A} = AA\cos 0° = A^2 \qquad [\because \cos 0° = 1]$

Hence, the dot product of a vector with itself is square of its magnitude.

We can also write: $\vec{A} \cdot \vec{A} = \left|\vec{A}\right|^2 \Rightarrow \quad \left|\vec{A}\right| = \sqrt{\vec{A}\cdot\vec{A}}$

For example: $\left|\vec{P} + \vec{Q}\right| = \sqrt{(\vec{P} + \vec{Q})\cdot(\vec{P} + \vec{Q})}$

(Taking $\vec{A} = \vec{P} + \vec{Q}$)

$\Rightarrow \quad \left|\vec{P} + \vec{Q}\right|^2 = (\vec{P} + \vec{Q})\cdot(\vec{P} + \vec{Q})$

$\Rightarrow \quad \left|\vec{P} + \vec{Q}\right|^2 = P^2 + Q^2 + 2\vec{P}\cdot\vec{Q} = P^2 + Q^2 + 2PQ\cos\theta$

Similarly, $\left|\vec{P} - \vec{Q}\right|^2 = P^2 + Q^2 - 2\vec{P}\cdot\vec{Q} = P^2 + Q^2 - 2PQ\cos\theta$

Important Points

1. Angle between two vectors can be calculated from:
$$\cos\theta = \frac{\vec{A}\vec{B}}{AB} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$
2. If $\left|\vec{A} + \vec{B}\right| = \left|\vec{A} - \vec{B}\right|$, then angle between $\vec{A}$ and $\vec{B}$ is $90°$.

   Proof: Given $\left|\vec{A} + \vec{B}\right| = \left|\vec{A} - \vec{B}\right|$

   Squaring both sides, we get $\left|\vec{A} + \vec{B}\right|^2 = \left|\vec{A} - \vec{B}\right|^2$

   $\Rightarrow \quad A^2 + B^2 + 2\vec{A}\cdot\vec{B} = A^2 + B^2 - 2\vec{A}\cdot\vec{B}$

   $\Rightarrow \quad 4\vec{A}\cdot\vec{B} = 0$

   $\Rightarrow \quad \vec{A}\cdot\vec{B} = 0$. Hence, $\vec{A}$ is perpendicular to $\vec{B}$.
3. If $\left(\vec{A} + \vec{B}\right)\cdot\left(\vec{A} - \vec{B}\right) = 0$, then $\vec{A}$ and $\vec{B}$ are equal in magnitude, i.e., $A = B$.

   Proof: Given $\left(\vec{A} + \vec{B}\right)\cdot\left(\vec{A} - \vec{B}\right) = 0$

   $\Rightarrow \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} - \vec{B}\cdot\vec{B} = 0 \Rightarrow A^2 - B^2 = 0$

   $\Rightarrow \quad A^2 = B^2 \quad \Rightarrow \quad A = B$. Hence proved.
4. We can find addition of two vectors using dot product: In Fig. 2.36, $\vec{R} = \vec{A} + \vec{B}$.

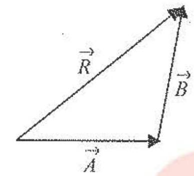


Fig. 2.36

$\Rightarrow \quad \left|\vec{R}\right| = \left|\vec{A} + \vec{B}\right|$

$\Rightarrow \quad \left|\vec{R}\right|^2 = \left|\vec{A} + \vec{B}\right|^2$

$\Rightarrow \quad R^2 = (\vec{A} + \vec{B})\cdot(\vec{A} + \vec{B}) = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$

$\Rightarrow R = \sqrt{A^2 + B^2 + 2\vec{A}\cdot\vec{B}} = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

Similarly, we can find subtraction of two vectors also.

Illustration 2.14 **If the sum of two unit vectors is a unit vector, then find the magnitude of their difference.**

Sol. Let $\hat{n}_1$ and $\hat{n}_2$ are the two unit vectors, then their sum is

$\vec{n}_S = \hat{n}_1 + \hat{n}_2 \quad \Rightarrow \quad n_S^2 = n_1^2 + n_2^2 + 2n_1n_2\cos\theta = 1 + 1 + 2\cos\theta$

Since it is given that $\vec{n}_S$ is a unit vector, so $n_S = 1$. Therefore,

$1 = 1 + 1 + 2\cos\theta \quad \Rightarrow \cos\theta = -\dfrac{1}{2}$

$\Rightarrow \quad \theta = 120°$

Now, the difference vector is $\vec{n}_d = \vec{n}_1 - \vec{n}_2$

$\Rightarrow \quad n_d^2 = n_1^2 + n_2^2 - 2n_1n_2\cos\theta = 1 + 1 - 2\cos(120°) = 3$

$\Rightarrow \quad n_d = \sqrt{3}$

Illustration 2.15 **Find the value of m so that the vector $3\hat{i} - 2\hat{j} + \hat{k}$ may be perpendicular to the vector $2\hat{i} + 6\hat{j} + m\hat{k}$.**

Sol. For the vectors to be perpendicular their dot product has to be zero.

$\therefore \quad (3\hat{i} - 2\hat{j} + \hat{k})\cdot(2\hat{i} + 6\hat{j} + m\hat{k}) = 0$

$\Rightarrow \quad 6 - 12 + m = 0 \quad \Rightarrow \quad m - 6 = 0 \Rightarrow \quad m = 6$

Vector or Cross Product

Cross product of two vectors $\vec{A}$ and $\vec{B}$ is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and sine of the shortest angle between them, i.e., $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector which represents the direction of $\vec{A} \times \vec{B}$ and it is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. It is given by right handed screw rule depicted by Fig. 2.37. Note that $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

Magnitude of $\vec{A} \times \vec{B}$: $\left|\vec{A} \times \vec{B}\right| = AB\sin\theta$

From here, we can write: $\vec{A} \times \vec{B} = \left|\vec{A} \times \vec{B}\right|\hat{n}$

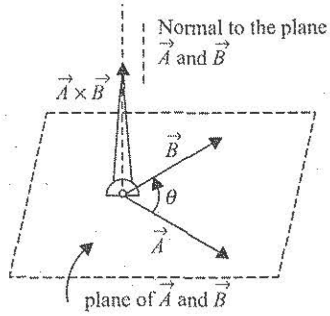


Fig. 2.37

$$\Rightarrow \quad \hat{n} = \frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}$$

Unit Vectors and Their Cross Product

$\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors along x-, y-, and z-axis, respectively. The magnitude of each vector is 1 and the angle between any of two unit vectors is 90°. So, $\hat{i}\times\hat{j} = (1)(1)\sin 90°\hat{n} = \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the plane containing vector $\hat{i}$ and $\hat{j}$.

To find out the resultant of any two unit vectors in a cross product use the following rules (see Fig. 2.38).

1. Multiplication of any two unit vectors in anticlockwise direction gives third unit vector with positive sign.
2. Multiplication of any two unit vectors in clockwise direction gives third unit vector with negative sign.

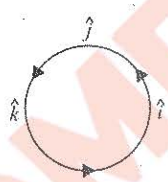


Fig. 2.38

From these rules, we obtain the following results.

From Rule 1:

1. $\hat{i}\times\hat{j} = \hat{k}$ **2.** $\hat{j}\times\hat{k} = \hat{i}$ **3.** $\hat{k}\times\hat{i} = \hat{j}$

From Rule 2:

1. $\hat{j}\times\hat{i} = -\hat{k}$ **2.** $\hat{k}\times\hat{j} = -\hat{i}$ **3.** $\hat{i}\times\hat{k} = -\hat{j}$

CROSS PRODUCT METHOD 1: USING COMPONENT FORM

$$\begin{aligned}\vec{A}\times\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\\ &= A_xB_x(\hat{i}\times\hat{i}) + A_yB_x(\hat{j}\times\hat{i}) + A_zB_x(\hat{k}\times\hat{i})\\ &\quad + A_xB_y(\hat{i}\times\hat{j}) + A_yB_y(\hat{j}\times\hat{j}) + A_zB_y(\hat{k}\times\hat{j})\\ &\quad + A_xB_z(\hat{i}\times\hat{k}) + A_yB_z(\hat{j}\times\hat{k}) + A_zB_z(\hat{k}\times\hat{k})\end{aligned}$$

[As $\hat{i}\times\hat{i} = 0, \hat{j}\times\hat{j} = 0, \hat{k}\times\hat{k} = 0$ and $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{i} = \hat{j}, \hat{i}\times\hat{k} = -\hat{j}, \hat{k}\times\hat{j} = -\hat{i}$]

So, we have $\vec{A}\times\vec{B} = A_yB_x(-\hat{k}) + A_zB_x\hat{j} + A_xB_y\hat{k}$

$$+ A_zB_y(-\hat{i}) + A_xB_z(-\hat{j}) + A_yB_z(\hat{i})$$

$$= (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

CROSS PRODUCT METHOD 2: DETERMINANT METHOD

Cross product of two vectors $\vec{A}$ and $\vec{B}$ can be obtained easily by using the following method.

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

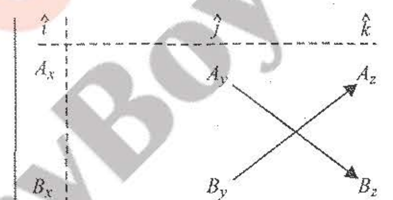


Fig. 2.39

Here, we will use $\hat{i}, \hat{j}, \hat{k}$ one by one. When $\hat{i}$ is chosen, its corresponding row and column become bound and remaining elements are subtracted after cross multiplication (Fig. 2.39). So, $\hat{i}(A_yB_z - B_yA_z)$, is the component along $\hat{i}$.

Similarly, in the case of $\hat{j}$, the row and column in which it is present become bound and remaining elements are subtracted after cross multiplication (Fig. 2.40).

So, $\hat{i}(A_yB_z - B_yA_z) - \hat{j}(A_xB_z - B_xA_z)$ is the component along $\hat{i}$ and $\hat{j}$.

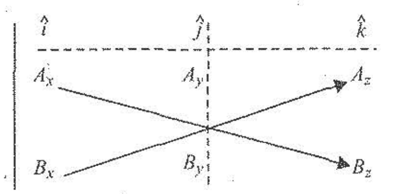


Fig. 2.40

Same argument will follow for $\hat{k}$ as is for $\hat{i}$ and $\hat{j}$ (Fig. 2.41).

$$\therefore \vec{A}\times\vec{B} = \hat{i}(A_yB_z - B_yA_z) - \hat{j}(A_xB_z - B_xA_z) + \hat{k}(A_xB_y - B_xA_y)$$

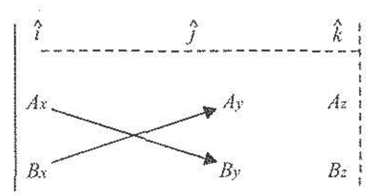


Fig. 2.41

Properties of Cross Product

1. **Anticommutative property**
   The vector product of two vectors is anticommutative.
   $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$ and $\vec{B} \times \vec{A} = BA \sin\theta(-\hat{n})$
   $= -AB \sin\theta \hat{n} = -(\vec{A} \times \vec{B})$
   So, $\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$. It means $\vec{B} \times \vec{A} \neq \vec{A} \times \vec{B}$.
2. **Distributive property**
   Vector product is distributive, i.e.,
   $$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$
3. **Associative property**
   $$(\vec{A} + \vec{B}) \times (\vec{C} + \vec{D}) = \vec{A} \times \vec{C} + \vec{A} \times \vec{D} + \vec{B} \times \vec{C} + \vec{B} \times \vec{D}$$
4. **Cross product of two parallel vectors**
   Cross product of the parallel vectors is zero.
   As $\theta = 0°$ (for parallel vectors), so
   $(\vec{A} \times \vec{B}) = AB \sin 0° \hat{n} = 0$

Important Points

1. If two vectors represent the two adjacent sides of a parallelogram, then the magnitude of the cross product will give the area of the parallelogram. Mathematically:
   Two vectors $\vec{A}$ and $\vec{B}$ are represented by the two adjacent sides PQ and PS, respectively, of the parallelogram as shown in Fig. 2.42.

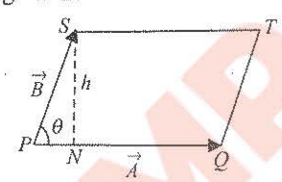


Fig. 2.42

   Now, from the magnitude of the cross product:
   $|\vec{A} \times \vec{B}| = AB \sin\theta = Ah =$ base $\times$ height of parallelogram = area of parallelogram.
2. If two vectors represent the two sides of a triangle, then half the magnitude of their cross product will give the area of the triangle.
   Consider two vectors $\vec{A}$ and $\vec{B}$ represented by the two sides PQ and PS of triangle PQS (Fig. 2.43).
   Using cross product: $\vec{A} \times \vec{B} = AB \sin\theta \hat{n}$
   Taking magnitude, $|\vec{A} \times \vec{B}| = AB \sin\theta$
   $= A(B \sin\theta) = A \times h =$ base $\times$ height

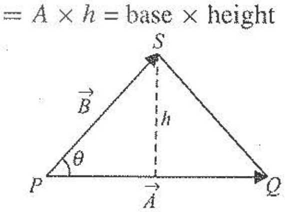


Fig. 2.43

   Multiplying by $1/2$ on both sides,
   $\frac{1}{2}|\vec{A} \times \vec{B}| = \frac{1}{2} \times$ base $\times$ height = area of triangle.

Illustration 2.16 **Calculate the area of the triangle determined by the two vectors $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = -3\hat{i} + 7\hat{j}$.**

Sol. We know that the half of magnitude of the cross product of two vectors gives the area of the triangle.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -3 & 7 & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(21+12) = 33\hat{k}$$

Taking magnitude $|\vec{A} \times \vec{B}| = \sqrt{33^2} = 33$. So, area of triangle $= \frac{1}{2}|\vec{A} \times \vec{B}| = \frac{33}{2}$ sq. unit.

Illustration 2.17 **Calculate the area of the parallelogram when adjacent sides are given by the vectors $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - 3\hat{j} + \hat{k}$.**

Sol. We know that the area of the parallelogram is equal to the magnitude of the cross product of given vectors.

Now, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -3 & 1 \end{vmatrix}$

$= \hat{i}(2+9) + \hat{j}(6-1) + \hat{k}(-3-4)$
$= 11\hat{i} + 5\hat{j} - 7\hat{k}$

So, area of parallelogram:

$|\vec{A} \times \vec{B}| = \sqrt{11^2 + 5^2 + (-7)^2} = \sqrt{195}$ sq. unit.

Concept Application Exercise 2.2

1. What is the area of parallelogram whose adjacent sides are given by vectors $\vec{A} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{B} = 4\hat{i} + 5\hat{j}$?
2. If the vectors $4\hat{i} + \hat{j} - 3\hat{k}$ and $2m\hat{i} + 6m\hat{j} + \hat{k}$ are perpendicular to each other, then find the value of m.
3. The magnitude of the vector product of two vectors is $\sqrt{3}$ times their scalar product. What is the angle between the two vectors?
4. What is the angle between $\hat{i} + \hat{j} + \hat{k}$ and $\hat{i}$?
5. The linear velocity of a rotating body is given by $\vec{v} = \vec{\omega} \times \vec{r}$. If $\vec{\omega} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{r} = 4\hat{j} - 3\hat{k}$, then what is the magnitude of $\vec{v}$?
6. Find the magnitude of component of $3\hat{i} - 2\hat{j} + \hat{k}$ along the vector $12\hat{i} + 3\hat{j} - 4\hat{k}$.

Solved Examples

Example 2.1 **A car is moving around a circular track with a constant speed v of 20 ms^{-1} (as shown in Fig. 2.44). At different times, the car is at A, B and C, respectively. Find the change in velocity:**
1. from A to C,

2. from A to B.

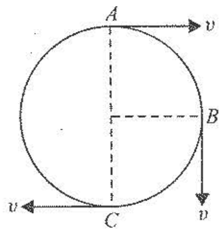


Fig. 2.44

Sol.

1. Change in velocity, as the car moves from A to C,

$$\Delta\vec{v}_{CA} = \vec{v}_C - \vec{v}_A = 20 - (-20) = 40 \text{ ms}^{-1}$$

$$\Delta\vec{v}_{CA} = 40 \text{ ms}^{-1} \text{ in the direction of } \vec{v}_C$$

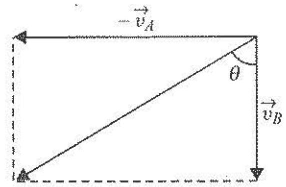


Fig. 2.45

2. Change in velocity, as the car moves from A to B,

$$\Delta\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$$

$$\Delta v_{BA} = \sqrt{20^2 + 20^2} = \sqrt{800} \text{ ms}^{-1} = 28.28 \text{ ms}^{-1}$$

Also, $\tan\theta = \dfrac{20}{20} = 1 \quad \Rightarrow \quad \theta = 45^\circ$

This is the required direction of change in velocity.

Example 2.2 **Given $\vec{A} = 0.3\hat{i} + 0.4\hat{j} + c\hat{k}$. Calculate the value of c if A is a unit vector.**

Sol. If $\vec{A}$ is a unit vector, then its magnitude must be unity.

$\Rightarrow \quad \sqrt{a_x^2 + a_y^2 + a_z^2} = 1$, i.e., $\sqrt{(0.3)^2 + (0.4)^2 + c^2} = 1$

or $\;0.09 + 0.16 + c^2 = 1$ or $c^2 = 1 - 0.25 = 0.75$ or $c = 0.87$

Example 2.3 **Determine that vector which when added to the resultant of $\vec{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$ gives a unit vector along y-direction.**

Sol. Here, $\vec{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$

Resultant $\vec{R} = \vec{A} + \vec{B} = (3\hat{i} - 5\hat{j} + 7\hat{k}) + (2i + 4\hat{j} - 3\hat{k})$

$$= 5\hat{i} - \hat{j} + 4\hat{k}$$

Let the vector to be added is $\vec{X}$. So, the unit vector along y-direction $\hat{j} = 5\hat{i} - \hat{j} + 4\hat{k} + \vec{X}$

So the required vector $\vec{X} = \hat{j} - (5\hat{i} - \hat{j} + 4\hat{k})$

$$= -5\hat{i} + 2\hat{j} - 4\hat{k}.$$

Example 2.4 **Find the unit vector of $\vec{A} = 2\hat{i} + 3\hat{j} + 2\hat{k}$.**

Sol. $\vec{A} = 2\hat{i} + 3\hat{j} + 2\hat{k}$. We know that $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$,

where $\;|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$$= \sqrt{2^2 + 3^2 + 2^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$$

$$\therefore \quad \hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{17}}$$

Example 2.5 **Find the unit vector of $(\vec{A} + \vec{B})$, where $\vec{A} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{B} = 3\hat{i} - 2\hat{j} - 2\hat{k}$.**

Sol. $(\vec{A} + \vec{B}) = (2\hat{i} - \hat{j} + 3\hat{k}) + (3\hat{i} - 2\hat{j} - 2\hat{k}) = 5\hat{i} - 3\hat{j} + \hat{k} = \vec{C}$ (say)

Magnitude of $\vec{C} = C = [5^2 + (-3)^2 + 1^2]^{1/2} = \sqrt{35}$

So, $\hat{C} = \dfrac{\vec{C}}{C} = \dfrac{5\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{35}}$.

Example 2.6 **The greatest and least resultant of two forces acting at a point are 10 and 6 N, respectively. If each force is increased by 3 N, find the resultant of new forces when acting at a point at an angle of 90° with each other.**

Sol. Let A and B be the two forces

Greatest resultant $= A + B = 10$ (i)

Least resultant $= A - B = 6$ (ii)

Solving equations (i) and (ii), we get, $A = 8$ N; $B = 2$ N

When each force is increased by 3 N, then

$A' = A + 3 = 8 + 3 = 11$ N, $B' = B + 3 = 2 + 3 = 5$ N.

As the new forces are acting at an angle of 90° (i.e., $\theta = 90^\circ$), so $R' = \sqrt{A'^2 + B'^2} = \sqrt{(11)^2 + (5)^2} = \sqrt{146}$ N.

Example 2.7 **Two forces whose magnitudes are in the ratio 3 : 5 give a resultant of 28 N. If the angle of their inclination is 60°, find the magnitude of each force.**

Sol. Let A and B be the two forces.

Then $A = 3x$; $B = 5x$; $R = 28$ N and $\theta = 60^\circ$

Dividing A by B, $A/B = 3/5$

We know that $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow \quad 28 = \sqrt{(3x)^2 + (5x)^2 + 2(3x)(5x)\cos 60^\circ}$

$\Rightarrow \quad 28 = \sqrt{9x^2 + 25x^2 + 15x^2} = 7x \quad \Rightarrow x = \dfrac{28}{7} = 4$

Hence, the forces are: $A = 3 \times 4 = 12$ N, $B = 5 \times 4 = 20$ N.

Example 2.8 **One of the rectangular components of a velocity of 100 ms^{-1} is 50 ms^{-1}. Find the other component.**

Sol. Here, $v = 100 \text{ ms}^{-1}$, Let $v_x = 50 \text{ ms}^{-1}$. $v_x = v\cos\theta$

$\Rightarrow \quad 50 = 100\cos\theta$

$\Rightarrow \quad \cos\theta = \dfrac{1}{2}$ or $\theta = 60^\circ$

$\therefore \quad v_y = v\sin\theta = 100\sin 60^\circ = 100 \times \dfrac{\sqrt{3}}{2} = 50\sqrt{3} \text{ ms}^{-1}$

Example 2.9 **An aeroplane takes off at an angle of 60° to the horizontal. If muzzle velocity of the plane is 200 kmh^{-1}, calculate its horizontal and vertical components.**

Sol. Here, $v = 200 \text{ kmh}^{-1}$, $\theta = 60^\circ$.

$\therefore$ Horizontal component $v_x = v\cos\theta = 200\cos 60^\circ$

$$= 200 \times \frac{1}{2} = 100 \text{ kmh}^{-1}$$

Vertical component $v_y = v\sin\theta = 200\sin 60^\circ$

$$= 200 \times \frac{\sqrt{3}}{2} = 100\sqrt{3} \text{ kmh}^{-1}$$

Example 2.10 **Prove that $(\vec{A} + 2\vec{B}) \cdot (2\vec{A} - 3\vec{B}) = 2A^2 + AB\cos\theta - 6B^2$.**

Sol. $(\vec{A} + 2\vec{B}) \cdot (2\vec{A} - 3\vec{B}) = 2\vec{A}\cdot\vec{A} - 3\vec{A}\cdot\vec{B} + 4\vec{B}\cdot\vec{A} - 6(\vec{B}\cdot\vec{B})$

$= 2(\vec{A}\cdot\vec{A}) - 3\vec{A}\vec{B}\cos\theta + 4\vec{A}\vec{B}\cos\theta - 6(\vec{B}\cdot\vec{B})$

$= 2A^2 + AB\cos\theta - 6B^2$

Example 2.11 **A body constrained to move along the z-axis of a coordinate system is subjected to a constant force $\vec{F}$ given by $\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$ N, where $\hat{i}$, $\hat{j}$, and $\hat{k}$ represent unit vectors along x-, y-, and z-axis of the system, respectively. Calculate the work done by this force in displacing the body through a distance of 4 m along the z-axis.**

Sol. Displacement $= 4\hat{k}$, Force: $\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$

Since work W is the scalar product of force and displacement,

$\therefore W = (-\hat{i} + 2\hat{j} + 3\hat{k}) \cdot 4\hat{k} = -4(\hat{i}\cdot\hat{k}) + 8(\hat{j}\cdot\hat{k}) + 12(\hat{k}\cdot\hat{k})$

$W = 12$ joule, because $\hat{i}\cdot\hat{k} = 0 = \hat{j}\cdot\hat{k}$ and $\hat{k}\cdot\hat{k} = 1$

Example 2.12

1. **Prove that the vectors $\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} + 5\hat{k}$, and $\vec{C} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right-angled triangle.**
2. **Determine the unit vector parallel to the cross product of the vectors $\vec{A} = 3\hat{i} - 5\hat{j} + 10\hat{k}$ and $\vec{B} = 6\hat{i} + 5\hat{j} + 2\hat{k}$.**

Sol.

1. The given vectors will constitute a triangle only if one of the given vectors is equal to vector sum of the remaining two vectors. In the given problem, $\vec{B} + \vec{C} = \vec{A}$. So, the given vectors do form a triangle. This triangle will be right angled only if the dot product of two vectors (out of the given three) is zero.

$\vec{A}\cdot\vec{B} = (3\hat{i} - 2\hat{j} + \hat{k})\cdot(\hat{i} - 3\hat{j} + 5\hat{k}) = 3(\hat{i}\cdot\hat{i}) + 6(\hat{j}\cdot\hat{j}) + 5(\hat{k}\cdot\hat{k}) = 3 + 6 + 5 = 14$

$\vec{B}\cdot\vec{C} = (\hat{i} - 3\hat{j} + 5\hat{k})\cdot(2\hat{i} + \hat{j} - 4\hat{k}) = 2(\hat{i}\cdot\hat{i}) - 3(\hat{j}\cdot\hat{j}) - 20(\hat{k}\cdot\hat{k}) = 2 - 3 - 20 = -21$

$\vec{C}\cdot\vec{A} = (2\hat{i} + j - 4\hat{k})\cdot(3\hat{i} - 2\hat{j} + \hat{k}) = 6(\hat{i}\cdot\hat{i}) - 2(\hat{j}\cdot\hat{j}) - 4(\hat{k}\cdot\hat{k}) = 6 - 2 - 4 = 0$

Since the dot product of $\vec{C}$ and $\vec{A}$ is zero, therefore it implies that $\vec{C}$ is perpendicular to $\vec{A}$.

2. The unit vector parallel to $(\vec{A} \times \vec{B})$ is given by $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

So, let us first determine $\vec{A} \times \vec{B}$.

Now, $\vec{A} \times \vec{B} = (3\hat{i} - 5\hat{j} + 10\hat{k}) \times (6\hat{i} + 5\hat{j} + 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -5 & 10 \\ 6 & 5 & 2 \end{vmatrix} = \hat{i}(-10 - 50) + \hat{j}(60 - 6) + \hat{k}(15 + 30)$$

$= -60\hat{i} + 54\hat{j} + 45\hat{k}$

Magnitude: $|\vec{A} \times \vec{B}| = \sqrt{(-60)^2 + (54)^2 + (45)^2} = \sqrt{8541}$

So, required unit vector: $\hat{n} = \dfrac{-60\hat{i} + 54\hat{j} + 45\hat{k}}{\sqrt{8541}}$.

Example 2.13 **A man rows a boat with a speed of 18 kmh^{-1} in the north-west direction. The shoreline makes an angle of 15° south of west. Obtain the components of the velocity of the boat along the shoreline and perpendicular to the shoreline.**

Sol. The north-west direction of the boat makes an angle of 60° with the shoreline (Fig. 2.46).

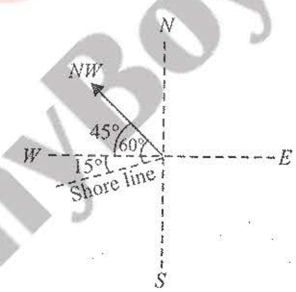


Fig. 2.46

Component of the velocity of boat along the shoreline

$$= 18\cos 60^\circ \text{ kmh}^{-1} = 9 \text{ kmh}^{-1}$$

Component of the boat velocity along a line normal to the shoreline

$$= 18\sin 60^\circ \text{ kmh}^{-1} = 18 \times \frac{\sqrt{3}}{2} \text{ kmh}^{-1} = 15.59 \text{ kmh}^{-1}$$

Example 2.14 **A point P lies in the xy plane. Its position can be specified by its x, y coordinates or by a radially directed vector $\vec{r} = (x\hat{i} + y\hat{j})$ making an angle θ with the x-axis. Find a vector $\hat{i}_r$ of unit magnitude in the direction of vector $\vec{r}$ and a vector $\hat{i}_\theta$ of unit magnitude normal to the vector $\hat{i}_r$ and lying in the xy plane (Fig. 2.47).**

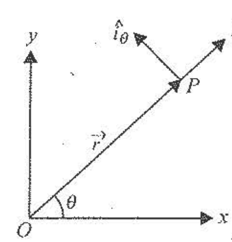


Fig. 2.47

Sol. When a vector is divided by its magnitude, we get a unit vector (Fig. 2.48).

$$\hat{i}_r = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j}}{r} = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r}; \quad \cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

Again, $\sin\theta = \dfrac{y}{r}$ or $y = r\sin\theta$.

So, we get: $\hat{i}_r = \hat{i}\cos\theta + \hat{j}\sin\theta$

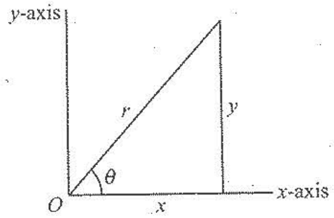


Fig. 2.48

Now, let $\hat{i}_\theta = \hat{i}\alpha + \hat{i}\beta$, where α and β are coefficients to be determined.

Making use of scalar product:

$$\hat{i}_r \cdot \hat{i}_\theta = (\hat{i}\cos\theta + \hat{j}\sin\theta)\cdot(\hat{i}\alpha + \hat{j}\beta) \quad \text{(i)}$$

Since dot product of perpendicular vectors is zero,

$$\hat{i}_r \cdot \hat{i}_\theta = 0 \quad \Rightarrow \quad \alpha\cos\theta + \beta\sin\theta = 0$$

$$\Rightarrow \quad \alpha = -\beta\frac{\sin\theta}{\cos\theta}$$

Again, since $\hat{i}_\theta$ is a vector of unit magnitude: $\alpha^2 + \beta^2 = 1$.

Put the value of α and get: $\beta = \pm\cos\theta$ and $\alpha = \mp\sin\theta$.

And we get $\hat{i}_\theta = \mp\hat{i}\sin\theta \pm \hat{j}\cos\theta$. Since $\hat{i}_\theta$ should have –ve x component and +ve y component, so finally we have: $\hat{i}_\theta = -\hat{i}\sin\theta + \hat{j}\cos\theta$

EXERCISES

Subjective Type

Solutions on page 2.22

1. **a.** What is the essential condition for the addition of two vectors?
   b. Is addition of any two scalar quantities meaningful?
   c. Component of a vector can be a scalar. State true or false.
   d. Can two vectors of same magnitude add to give zero resultant vector? If yes, under what conditions?
   e. Can two vectors of different magnitudes add to give zero resultant vector? Can three vectors give the zero resultant vector on addition. If yes, under what conditions?
   f. Can a rectangular component of a vector have magnitude greater than the vector itself?
   g. Can a vector be zero if one of its component is not zero?
   h. Can scalar product of two vectors be a negative quantity?
   i. State the condition (regarding the value of dot or cross product) for which two vectors are:
      i. parallel to each other,
      ii. perpendicular to each other.
   j. Is possession of magnitude and direction sufficient for calling a quantity a vector quantity?
   k. Is it necessary that sum of two unit vectors is also a unit vector?
   l. What will be the difference in the product of
      i. a real number with a vector, and
      ii. a scalar with a vector.
      iii. Explain the difference between the following data?
         - 4 (5 kmh^{-1}, east)
         - 4 h (5 kmh^{-1}, east)

2. Two equal forces have a resultant equal to one and a half times the either force. Find the angle between the forces.

3. At what angle two forces $(P + Q)$ and $(P - Q)$ act so that their resultant is
   a. $\sqrt{3P^2 + Q^2}$ **b.** $\sqrt{2(P^2 + Q^2)}$

4. Two forces 7 and 3 N simultaneously act on a body. What is the value of their (i) maximum resultant, (ii) minimum resultant, and (iii) what will be the resultant if the forces act at right angle to each other?

5. Find the resultant force of the following forces which are acting simultaneously upon a particle.
   a. 30 N due East **b.** 20 N due North
   c. 50 N due West **d.** 40 N due South

6. For the vectors $\vec{A}$ and $\vec{B}$ in Fig. 2.49, use a scale drawing to find the magnitude and direction of

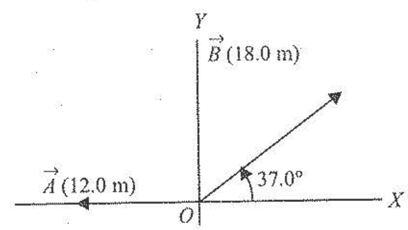


Fig. 2.49

   a. the vector sum $\vec{A} + \vec{B}$.
   b. the vector difference $\vec{A} - \vec{B}$.
   c. From your answers to parts (a) and (b), find the magnitude and direction of
      i. $-\vec{A} - \vec{B}$ **ii.** $\vec{B} - \vec{A}$

7. If two vectors $\vec{a} = 4$ ms^{-1} and $\vec{b} = 7$ ms^{-1} be inclined at an angle of 60° to each other, then determine the direction and magnitude of their resultant.

8. Find a unit vector along and opposite to the vector $3\hat{i} - 4\hat{j} + 12\hat{k}$.

9. If $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = 6\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = \hat{i} + \hat{k}$, then find

a. $3\vec{a}+2\vec{b}+\vec{c}$
b. $\vec{a}-6\vec{b}+2\vec{c}$

10. Given two vectors $\vec{A}=4.00\hat{i}+3.00\hat{j}$ and $\vec{B}=5.00\hat{i}-2.00\hat{j}$.
   a. Find the magnitude of each vector.
   b. Write an expression for the vector difference $\vec{A}-\vec{B}$ using unit vectors.
   c. Find the magnitude and direction of the vector difference $\vec{A}-\vec{B}$.
   d. In a vector diagram show $\vec{A}$, $\vec{B}$, and $\vec{A}-\vec{B}$, and also show that your diagram agrees qualitatively with your answer in part (c).

11. **a.** Is the vector $(\hat{i}+\hat{j}+\hat{k})$ a unit vector? Justify your answer.
   b. Can a unit vector have any rectangular component with a magnitude greater than unity? Justify your answer.
   c. If $\vec{A}=a(3\hat{i}+4\hat{j})$, where a is a constant, determine the value of a that makes $\vec{A}$ a unit vector.

12. Resolve the vector $\vec{R}=2\hat{i}+3\hat{j}$ along the directions of $\hat{i}+2\hat{j}$ and $\hat{i}-\hat{j}$ and write down the resolved components.

13. Find the rectangular component of vector $\vec{R}=2\hat{i}+3\hat{j}$ along $\vec{A}=\hat{i}+\hat{j}$.

14. Find the angle between each of the following pairs of vectors.
   a. $\vec{A}=-2.00\hat{i}+6.00\hat{j}$ and $\vec{B}=2.00\hat{i}-3.00\hat{j}$
   b. $\vec{A}=3.00\hat{i}+5.00\hat{j}$ and $\vec{B}=10.00\hat{i}+6.00\hat{j}$
   c. $\vec{A}=-4.00\hat{i}+2.00\hat{j}$ and $\vec{B}=7.00\hat{i}+14.00\hat{j}$

15. A cube is placed so that one corner is at the origin and three edges are along the x-, y-, and z-axis of a coordinate system (Fig. 2.50). Use vectors to compute

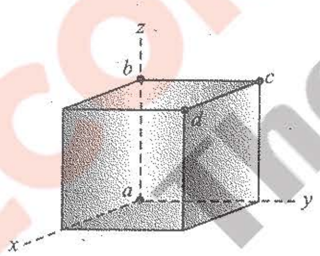


Fig. 2.50

   a. the angle between the edge along the z-axis (line ab) and the diagonal from the origin to the opposite corner (line ad).
   b. the angle between line ac (the diagonal of a face) and line ad.

16. You are given vectors $\vec{A}=5\hat{i}-6.5\hat{j}$ and $\vec{B}=10\hat{i}+7\hat{j}$. A third vector $\vec{C}$ lies in the $x-y$ plane. Vector $\vec{C}$ is perpendicular to vector $\vec{A}$ and the scalar product of $\vec{C}$ with $\vec{B}$ is 15. From this information, find the components of vector $\vec{C}$.

17. Two vectors $\vec{A}$ and $\vec{B}$ have magnitudes $A=3.00$ and $B=3.00$. Their vector product is $\vec{A}\times\vec{B}=-5.00\hat{k}+2.00\hat{i}$. What is the angle between $\vec{A}$ and $\vec{B}$?

18. Two vectors have magnitudes 5 units and 12 units, respectively. Find their cross product if the angle between them is 30°.

19. Given two vectors $\vec{A}=3\hat{i}+\hat{j}+\hat{k}$ and $\vec{B}=\hat{i}-\hat{j}-\hat{k}$. Find the
   a. area of the triangle whose two sides are represented by the vectors $\vec{A}$ and $\vec{B}$.
   b. area of the parallelogram whose two adjacent sides are represented by the vectors $\vec{A}$ and $\vec{B}$.
   c. area of the parallelogram whose diagonals are represented by the vectors $\vec{A}$ and $\vec{B}$.

20. On a horizontal flat ground, a person is standing at a point A. At this point, he installs a 5 m long pole vertically. Now, he moves 5 m towards east and then 2 m towards north and reaches at a point B. There he installs another 3 m long vertical pole. A bird flies from the top of first pole to the top of second pole. Find the displacement and magnitude of the displacement of the bird.

21. Find the vector sum of N coplanar forces, each of magnitude F, when each force is making an angle of $\frac{2\pi}{N}$ with that preceding it.

22. Can you find at least one vector perpendicular to $3\hat{i}-4\hat{j}+7\hat{k}$?

23. Establish the following inequalities:
   a. $|\vec{A}+\vec{B}|\le|\vec{A}|+|\vec{B}|$
   b. $|\vec{A}+\vec{B}|\ge\big||\vec{A}|-|\vec{B}|\big|$
   c. $|\vec{A}-\vec{B}|\le|\vec{A}|+|\vec{B}|$
   d. $|\vec{A}-\vec{B}|\ge\big||\vec{A}|-|\vec{B}|\big|$

24. Two forces P and Q acting at a point are such that if P is reversed, the direction of the resultant is turned through 90°, then prove that magnitudes of the forces are equal.

25. Unit vectors $\hat{P}$ and $\hat{Q}$ are inclined at an angle θ, then prove that $|\hat{P}-\hat{Q}|=2\sin(\theta/2)$.

Objective Type

Solutions on page 2.25

1. The sum and difference of two perpendicular vectors of equal lengths are
   a. also perpendicular and of equal length
   b. also perpendicular and of different lengths
   c. of equal length and have an obtuse angle between them
   d. of equal length and have an acute angle between them

2. The minimum number of vectors having different planes which can be added to give zero resultant is
   a. 2 **b.** 3 **c.** 4 **d.** 5

3. A vector perpendicular to $\hat{i} + \hat{j} + \hat{k}$ is
   a. $\hat{i} - \hat{j} + \hat{k}$
   b. $\hat{i} - \hat{j} - \hat{k}$
   c. $-\hat{i} - \hat{j} - \hat{k}$
   d. $3\hat{i} + 2\hat{j} - 5\hat{k}$

4. From Fig. 2.51, the correct relation is

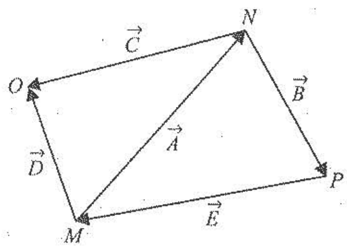


Fig. 2.51

   a. $\vec{A} + \vec{B} + \vec{E} = \vec{0}$
   b. $\vec{C} - \vec{D} = -\vec{A}$
   c. $\vec{B} + \vec{E} - \vec{C} = -\vec{D}$
   d. All of the above

5. Out of the following set of forces, the resultant of which cannot be zero
   a. 10, 10, 10
   b. 10, 10, 20
   c. 10, 20, 20
   d. 10, 20, 40

6. The resultant of two vectors $\vec{A}$ and $\vec{B}$ is perpendicular to the vector $\vec{A}$ and its magnitude is equal to half of the magnitude of vector $\vec{B}$. The angle between $\vec{A}$ and $\vec{B}$ is

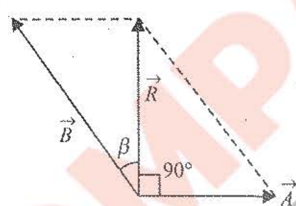


Fig. 2.52

   a. 120°
   b. 150°
   c. 135°
   d. None of these

7. The ratio of maximum and minimum magnitudes of the resultant of two vectors $\vec{a}$ and $\vec{b}$ is 3 : 1. Now, $|\vec{a}|$ is equal to
   a. $|\vec{b}|$
   b. $2|\vec{b}|$
   c. $3|\vec{b}|$
   d. $4|\vec{b}|$

8. Two forces, each equal to F, act as shown in Fig. 2.53. Their resultant is

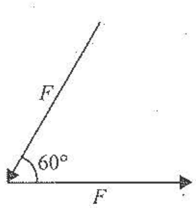


Fig. 2.53

   a. $F/2$
   b. F
   c. $\sqrt{3}F$
   d. $\sqrt{5}F$

9. Vector $\vec{A}$ is 2 cm long and is 60° above the x-axis in the first quadrant. Vector $\vec{B}$ is 2 cm long and is 60° below the x-axis in the fourth quadrant. The sum $\vec{A} + \vec{B}$ is a vector of magnitude
   a. 2 cm along + y-axis
   b. 2 cm along + x-axis
   c. 2 cm along – x-axis
   d. 2 cm along – x-axis

10. What is the angle between two vector forces of equal magnitude such that the resultant is one-third as much as either of the original forces?
   a. $\cos^{-1}\left(-\frac{17}{18}\right)$
   b. $\cos^{-1}\left(\frac{1}{3}\right)$
   c. 45°
   d. 120°

11. The angle between $\vec{A} + \vec{B}$ and $\vec{A} \times \vec{B}$ is
   a. 0
   b. $\pi/4$
   c. $\pi/2$
   d. π

12. The projection of a vector $\vec{r} = 3\hat{i} + \hat{j} + 2\hat{k}$ on the x–y plane has magnitude
   a. 3
   b. 4
   c. $\sqrt{14}$
   d. $\sqrt{10}$

13. If $\left|\vec{A} + \vec{B}\right| = \left|\vec{A}\right| = \left|\vec{B}\right|$, then the angle between $\vec{A}$ and $\vec{B}$ is
   a. 120°
   b. 60°
   c. 90°
   d. 0°

14. If vectors $\vec{A} = \hat{i} + 2\hat{j} + 4\hat{k}$ and $\vec{B} = 5\hat{i}$ represent the two sides of a triangle, then the third side of the triangle can have length equal to
   a. 6
   b. $\sqrt{56}$
   c. Both of the above
   d. None of the above

15. Given $\left|\vec{A}_1\right| = 2$, $\left|\vec{A}_2\right| = 3$ and $\left|\vec{A}_1 + \vec{A}_2\right| = 3$. Find the value of $\left(\vec{A}_1 + 2\vec{A}_2\right) \cdot \left(3\vec{A}_1 - 4\vec{A}_2\right)$
   a. 64
   b. 60
   c. 62
   d. 61

16. Three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ satisfy the relation $\vec{A} \cdot \vec{B} = 0$ and $\vec{A} \cdot \vec{C} = 0$. The vector $\vec{A}$ is parallel to
   a. $\vec{B}$
   b. $\vec{C}$
   c. $\vec{B} \cdot \vec{C}$
   d. $\vec{B} \times \vec{C}$

17. If $\vec{A} = \vec{B} + \vec{C}$, and the magnitudes of $\vec{A}$, $\vec{B}$, $\vec{C}$ are 5, 4, and 3 units, then angle between $\vec{A}$ and $\vec{C}$ is
   a. $\cos^{-1}\left(\frac{3}{5}\right)$
   b. $\cos^{-1}\left(\frac{4}{5}\right)$
   c. $\sin^{-1}\left(\frac{3}{4}\right)$
   d. $\frac{\pi}{2}$

18. Given: $\vec{A} = A\cos\theta\hat{i} + A\sin\theta\hat{j}$. A vector $\vec{B}$ which is perpendicular to $\vec{A}$ is given by
   a. $B\cos\theta\hat{i} - B\sin\theta\hat{j}$

b. $B\sin\theta\hat{i} - B\cos\theta\hat{j}$
c. $B\cos\theta\hat{i} + B\sin\theta\hat{j}$
d. $B\sin\theta\hat{i} + B\cos\theta\hat{j}$

19. The angle which the vector $\vec{A} = 2\hat{i} + 3\hat{j}$ makes with y-axis, where $\hat{i}$ and $\hat{j}$ are unit vectors along x- and y-axes, respectively, is

a. $\cos^{-1}(3/5)$
b. $\cos^{-1}(2/3)$
c. $\tan^{-1}(2/3)$
d. $\sin^{-1}(2/3)$

20. Given: $\vec{P} = 3\hat{i} - 4\hat{j}$. Which of the following is perpendicular to $\vec{P}$?

a. $3\hat{i}$
b. $4\hat{j}$
c. $4\hat{i} + 3\hat{j}$
d. $4\hat{i} - 3\hat{j}$

21. In going from one city to another, a car travels 75 km north, 60 km north-west and 20 km east. The magnitude of displacement between the two cities is (Take $1/\sqrt{2} = 0.7$)

a. 170 km
b. 137 km
c. 119 km
d. 140 km

22. What is the angle between $\vec{A}$ and $\vec{B}$, if $\vec{A}$ and $\vec{B}$ are the adjacent sides of a parallelogram drawn from a common point and the area of the parallelogram is $AB/2$?

a. 15°
b. 30°
c. 45°
d. 60°

23. Two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$. What is the angle between $\vec{a}$ and $\vec{b}$?

a. 0°
b. 90°
c. 60°
d. 180°

24. Given: $\vec{A} = 4\hat{i} + 6\hat{j}$ and $\vec{B} = 2\hat{i} + 3\hat{j}$. Which of the following is correct?

a. $\vec{A} \times \vec{B} = \vec{0}$
b. $\vec{A} \cdot \vec{B} = 24$
c. $\dfrac{|\vec{A}|}{|\vec{B}|} = \dfrac{1}{2}$
d. $\vec{A}$ and $\vec{B}$ are antiparallel

25. Given: $\vec{A} = 2\hat{i} + p\hat{j} + q\hat{k}$ and $\vec{B} = 5\hat{i} + 7\hat{j} + 3\hat{k}$. If $\vec{A} \parallel \vec{B}$, then the values of p and q are, respectively,

a. $\dfrac{14}{5}$ and $\dfrac{6}{5}$
b. $\dfrac{14}{3}$ and $\dfrac{6}{5}$
c. $\dfrac{6}{5}$ and $\dfrac{1}{3}$
d. $\dfrac{3}{4}$ and $\dfrac{1}{4}$

26. If $\vec{A}$ is perpendicular to $\vec{B}$, then

a. $\vec{A} \times \vec{B} = 0$
b. $\vec{A} \cdot \left[\vec{A} + \vec{B}\right] = A^2$
c. $\vec{A} \cdot \vec{B} = AB$
d. $\vec{A} \cdot \left[\vec{A} + \vec{B}\right] = A^2 + AB$

27. If the angle between vectors $\vec{a}$ and $\vec{b}$ is an acute angle, then the difference $\vec{a} - \vec{b}$ is

a. the major diagonal of the parallelogram
b. the minor diagonal of the parallelogram
c. any of the above
d. none of the above

28. Given that $\vec{A} + \vec{B} = \vec{C}$. If $\left|\vec{A}\right| = 4$, $\left|\vec{B}\right| = 5$ and $\left|\vec{C}\right| = \sqrt{61}$. The angle between $\vec{A}$ and $\vec{B}$ is

a. 30°
b. 60°
c. 90°
d. 120°

29. Given vector $\vec{A} = 2\hat{i} + 3\hat{j}$, the angle between $\vec{A}$ and y-axis is

a. $\tan^{-1}(3/2)$
b. $\tan^{-1}(2/3)$
c. $\sin^{-1}(2/3)$
d. $\cos^{-1}(2/3)$

30. If $\vec{b} = 3\hat{i} + 4\hat{j}$ and $\vec{a} = \hat{i} - \hat{j}$, the vector having the same magnitude as that of $\vec{b}$ and parallel to $\vec{a}$ is

a. $\dfrac{5}{\sqrt{2}}(\hat{i} - \hat{j})$
b. $\dfrac{5}{\sqrt{2}}(\hat{i} + \hat{j})$
c. $5(\hat{i} - \hat{j})$
d. $5(\hat{i} + \hat{j})$

31. Choose the wrong statement

a. Three vectors of different magnitudes may be combined to give zero resultant.
b. Two vectors of different magnitudes can be combined to give a zero resultant.
c. The product of a scalar and a vector is a vector quantity.
d. All of the above are wrong statements.

32. What displacement at an angle 60° to the x-axis has an x-component of 5 m? $\hat{i}$ and $\hat{j}$ are unit vectors in x and y directions, respectively.

a. $5\hat{i}$
b. $5\hat{i} + 5\hat{j}$
c. $5\hat{i} + 5\sqrt{3}\hat{j}$
d. All of the above

33. Mark the correct statement

a. $\left|\vec{a} + \vec{b}\right| \geq |\vec{a}| + \left|\vec{b}\right|$
b. $\left|\vec{a} + \vec{b}\right| \leq |\vec{a}| + \left|\vec{b}\right|$
c. $\left|\vec{a} - \vec{b}\right| \geq |\vec{a}| + \left|\vec{b}\right|$
d. All of the above

34. Out of the following forces, the resultant of which cannot be 10 N?

a. 15 N and 20 N
b. 10 N and 10 N
c. 5 N and 12 N
d. 12 N and 1 N

35. Which of the following pairs of forces cannot be added to give a resultant force of 4 N?

a. 2 N and 8 N
b. 2 N and 2 N
c. 2 N and 6 N
d. 2 N and 4 N

36. In an equilateral triangle ABC, AL, BM, and CN are medians. Forces along BC and BA represented by them will have a resultant represented by

a. $2AL$
b. $2BM$
c. $2CN$
d. AC

37. The vector sum of two forces is perpendicular to their vector difference. The forces are

a. equal to each other
b. equal to each other in magnitude
c. not equal to each other in magnitude
d. cannot be predicted

38. If a parallelogram is formed with two sides represented by vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}+\vec{b}$ represents the

a. major diagonal when the angle between vectors is acute
b. minor diagonal when the angle between vectors is obtuse
c. both of the above
d. none of the above

39. The resultant $\vec{C}$ of $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$. Also, $|\vec{A}| = |\vec{C}|$. The angle between $\vec{A}$ and $\vec{B}$ is

a. $\frac{\pi}{4}$ rad **b.** $\frac{3\pi}{4}$ rad
c. $\frac{5\pi}{4}$ rad **d.** $\frac{7\pi}{4}$ rad

40. Two forces of $\vec{F_1} = 500$ N due east and $\vec{F_2} = 250$ N due north have their common initial point. $\vec{F_2} - \vec{F_1}$ is

a. $250\sqrt{5}$ N, $\tan^{-1}(2)$W of N
b. 250 N, $\tan^{-1}(2)$W of N
c. zero
d. 750 N, $\tan^{-1}(3/4)$ N of W

41. The resultant of the three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ shown in Fig. 2.54 is

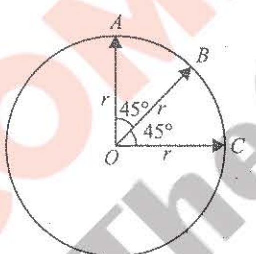


Fig. 2.54

a. r **b.** $2r$
c. $r(1+\sqrt{2})$ **d.** $r(\sqrt{2}-1)$

42. Two vectors $\vec{a}$ and $\vec{b}$ are at an angle of 60° with each other. Their resultant makes an angle of 45° with $\vec{a}$. If $|\vec{b}| = 2$ units, then $|\vec{a}|$ is

a. $\sqrt{3}$ **b.** $\sqrt{3}-1$
c. $\sqrt{3}+1$ **d.** $\sqrt{3}/2$

43. The resultant of two vectors $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. If the magnitude of $\vec{Q}$ is doubled, the new resultant vector becomes perpendicular to $\vec{P}$. Then, the magnitude of $\vec{R}$ is equal to

a. $P+Q$ **b.** P
c. $P-Q$ **d.** Q

44. A vector $\vec{A}$ when added to the vector $\vec{B} = 3\hat{i} + 4\hat{j}$ yields a resultant vector that is in the positive y-direction and has a magnitude equal to that of $\vec{B}$. Find the magnitude of $\vec{A}$.

a. $\sqrt{10}$ **b.** 10 **c.** 5 **d.** $\sqrt{15}$

45. $ABCDEF$ is a regular hexagon with point O as centre. The value of $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$ is

a. $2\overrightarrow{AO}$ **b.** $4\overrightarrow{AO}$ **c.** $6\overrightarrow{AO}$ **d.** 0

46. In a two dimensional motion of a particle, the particle moves from point A, position vector $\vec{r}_1$, to point B, position vector $\vec{r}_2$. If the magnitudes of these vectors are, respectively, $r_1 = 3$ and $r_2 = 4$ and the angles they make with the x-axis are $\theta_1 = 75°$ and $\theta_2 = 15°$, respectively, then find the magnitude of the displacement vector.

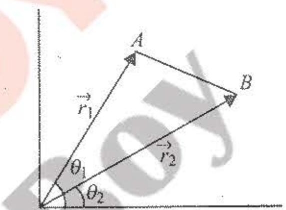


Fig. 2.55

a. 15 **b.** $\sqrt{13}$ **c.** 17 **d.** $\sqrt{15}$

47. The sum of the magnitudes of two forces acting at a point is 16 N. The resultant of these forces is perpendicular to the smaller force and has a magnitude of 8 N. If the smaller force is of magnitude x, then the value of x is

a. 2 N **b.** 4 N **c.** 6 N **d.** 7 N

48. The angle between two vectors $\vec{A}$ and $\vec{B}$ is θ. Resultant of these vectors $\vec{R}$ makes an angle $\theta/2$ with $\vec{A}$. Which of the following is true?

a. $A = 2B$ **b.** $A = B/2$
c. $A = B$ **d.** $AB = 1$

49. The resultant of three vectors 1, 2, and 3 units whose directions are those of the sides of an equilateral triangle is at an angle of

a. 30° with the first vector
b. 15° with the first vector
c. 100° with the first vector
d. 150° with the first vector

50. A particle moves in the xy plane with only an x-component of acceleration of 2 ms^{-2}. The particle starts from the origin at $t = 0$ with an initial velocity having an x-component of 8 ms^{-1} and y-component of -15 ms^{-1}. The total velocity vector at any time t is

a. $[(8+2t)\hat{i} - 15\hat{j}]$ ms^{-1}
b. zero
c. $2t\hat{i} + 15\hat{j}$
d. directed along z-axis

51. A unit vector along incident ray of light is $\hat{i}$. The unit vector for the corresponding refracted ray of light is $\hat{r}$. $\hat{n}$ is a unit vector normal to the boundary of the medium and directed towards the incident medium. If m be the refractive index of the medium, then Snell's law (2[nd]) of refraction is

a. $\hat{i} \times \hat{n} = \mu(\hat{n} + \vec{r})$
b. $\hat{i} \cdot \hat{n} = \mu(\hat{r} \cdot \hat{n})$
c. $\hat{i} \times \hat{n} = \mu(\hat{r} \times \hat{n})$
d. $\mu(\hat{i} \times \hat{n}) = \hat{r} \times \hat{n}$

52. The simple sum of two co-initial vectors is 16 units. Their vector sum is 8 units. The resultant of the vectors is perpendicular to the smaller vector. The magnitudes of the two vectors are

a. 2 units and 14 units
b. 4 units and 12 units
c. 6 units and 10 units
d. 8 units and 8 units

53. The components of a vector along x- and y-directions are $(n + 1)$ and 1, respectively. If the coordinate system is rotated by an angle $\theta = 60°$, then the components change to n and 3. The value of n is

a. 2 **b.** $\cos 60°$ **c.** $\sin 60°$ **d.** 3.5

54. Two point masses 1 and 2 move with uniform velocities $\vec{v_1}$ and $\vec{v_2}$, respectively. Their initial position vectors are $\vec{r_1}$ and $\vec{r_2}$, respectively. Which of the following should be satisfied for the collision of the point masses?

a. $\dfrac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} + \vec{r_1}|} = \dfrac{\vec{v_2} + \vec{v_1}}{|\vec{v_2} - \vec{v_1}|}$

b. $\dfrac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|} = \dfrac{\vec{v_2} - \vec{v_1}}{|\vec{v_2} - \vec{v_1}|}$

c. $\dfrac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} + \vec{r_1}|} = \dfrac{\vec{v_2} - \vec{v_1}}{|\vec{v_2} + \vec{v_1}|}$

d. $\dfrac{\vec{r_2} + \vec{r_1}}{|\vec{r_2} + \vec{r_1}|} = \dfrac{\vec{v_2} - \vec{v_1}}{|\vec{v_2} + \vec{v_1}|}$

55. What is the resultant of three coplanar forces: 300 N at 0°, 400 N at 30°, and 400 N at 150°?

a. 500 N **b.** 700 N
c. 1,100 N **d.** 300 N

Multiple Correct Answers Type

Solutions on page 2.29

1. Which of the following statement is/are correct (see Fig. 2.56)?

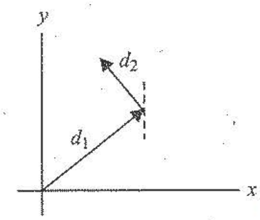


Fig. 2.56

a. The signs of x-component of $\vec{d_1}$ is positive and that of $\vec{d_2}$ is negative.

b. The signs of the y-component of $\vec{d_1}$ and $\vec{d_2}$ are positive and negative, respectively.

c. The signs of x- and y-components of $\vec{d_1} + \vec{d_2}$ are positive.

d. None of these.

2. Given two vectors $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = \hat{i} + \hat{j}$. θ is the angle between $\vec{A}$ and $\vec{B}$. Which of the following statements is/are correct?

a. $\left|\vec{A}\right| \cos\theta \left(\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}\right)$ is the component of $\vec{A}$ along $\vec{B}$.

b. $\left|\vec{A}\right| \sin\theta \left(\dfrac{\hat{i} - \hat{j}}{\sqrt{2}}\right)$ is the component of $\vec{A}$ perpendicular to $\vec{B}$.

c. $\left|\vec{A}\right| \cos\theta \left(\dfrac{\hat{i} - \hat{j}}{\sqrt{2}}\right)$ is the component of $\vec{A}$ along $\vec{B}$.

d. $\left|\vec{A}\right| \sin\theta \left(\dfrac{\hat{i} - \hat{j}}{2}\right)$ is the component of $\vec{A}$ perpendicular to $\vec{B}$.

3. If $\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ are two vectors, then the unit vector

a. perpendicular to $\vec{A}$ is $\left(\dfrac{-\hat{j} + \hat{k}}{\sqrt{2}}\right)$

b. parallel to $\vec{A}$ is $\dfrac{(2\hat{i} + \hat{j} + \hat{k})}{\sqrt{6}}$

c. perpendicular to $\vec{B}$ is $\left(\dfrac{-\hat{j} + \hat{k}}{\sqrt{2}}\right)$

d. parallel to $\vec{A}$ is $\dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

4. If $\left(\vec{v_1} + \vec{v_2}\right)$ is perpendicular to $\left(\vec{v_1} - \vec{v_2}\right)$, then

a. $\vec{v_1}$ is perpendicular to $\vec{v_2}$
b. $\left|\vec{v_1}\right| = \left|\vec{v_2}\right|$
c. $\vec{v_1}$ is null vector
d. the angle between $\vec{v_1}$ and $\vec{v_2}$ can have any value

5. Two vectors $\vec{A}$ and $\vec{B}$ lie in one plane. Vector $\vec{C}$ lies in a different plane. Then, $\vec{A} + \vec{B} + \vec{C}$

a. cannot be zero
b. can be zero
c. lies in the plane of $\vec{A}$ or $\vec{B}$
d. lies in a plane different from that of any of the three vectors

ANSWERS AND SOLUTIONS

Subjective Type

1.
a. The two vectors should be of the same nature, i.e., a force vector can be added only into another force vector, not in a velocity vector, say.

b. No, they should have same nature, i.e., mass can be added into mass, not in length, say.

c. False, component of a vector is also a vector.

d. Yes, if they are equal and opposite.

e. Two vectors of different magnitudes cannot add to give zero resultant. Three vectors of different magnitude can add to give zero resultant if they are coplanar.

f. No

g. No, a vector can be zero if all components are zero.

h. Yes

i.
i. cross product is zero
ii. dot product is zero

j. No, it should follow the vector rules of addition multiplication etc. For example, electric current has both magnitude and direction but still it is a vector quantity.

k. No

l.
i. nature of vector remains same, only magnitude may change
ii. magnitude of the vector changes
iii. • 4 (5 kmh^{-1}, east) = 20 kmh^{-1}, east
Here, we are multiplying a velocity 5 kmh^{-1}, east with a real number 4. The final result obtained is 20 kmh^{-1}, east which is also a velocity. So, nature of vector remains same if it is multiplied with a real number, only magnitude may change.
• 4 h (5 kmh^{-1}, east) = 20 km, east
Here, we multiply velocity with scalar quantity, time 4 h. The result obtained is 20 km, east which is a displacement vector. Here, nature of vector changes.

2. $R^2 = A^2 + B^2 + 2AB\cos\theta$ Given $A = B, R = 3A/2$

$$\Rightarrow \left(\frac{3}{2}A\right)^2 = A^2 + A^2 + 2A^2\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{1}{8} \Rightarrow \theta = 83^0$$

3. **a.** $R = \sqrt{3P^2 + Q^2}, \quad A = P + Q, B = P - Q$

Apply $R^2 = A^2 + B^2 + 2AB\cos\theta$

$$\Rightarrow \quad 3P^2 + Q^2 = (P + Q)^2 + (P-Q)^2 + 2(p + Q)(P - Q)\cos\theta$$

$$\Rightarrow \quad \cos = 1/2 \Rightarrow \theta = 60°$$

b. Here, $R = \sqrt{2(P^2 + Q^2)}$; $A = P + Q, B = P - Q$

Apply $R^2 = A^2 + B^2 + 2AB\cos\theta$

$$\Rightarrow \quad 2(P^2 + Q^2) = (P + Q)^2 + (P-Q)^2 + 2(P + Q)(P - Q)\cos\theta$$

$$\Rightarrow \quad \cos\theta = 0 \Rightarrow \theta = 90^0$$

4. **a.** Resultant is maximum when both vectors act in same direction. $R_{max} = A + B = 7 + 3 = 10$ N

b. Resultant is minimum when both vectors act in opposite direction: $R_{min} = A - B = 7 - 3 = 4$ N

c. If both vectors act at right angle, then
$R = \sqrt{A^2 + B^2} = \sqrt{7^2 + 3^2} = \sqrt{58}$ N

5. Resultant force: $\vec{F} = 30\hat{i} + 20\hat{j} - 50\hat{i} - 40\hat{j}$
$= 20\hat{i} - 20\hat{j}$ $F = \sqrt{20^2 + 20^2} = 20\sqrt{2}$ S – W

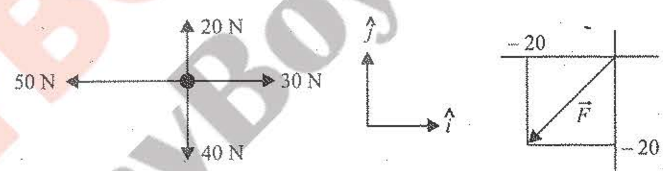


Fig. 2.57

6. **a.** $\vec{R} = \vec{A} + \vec{B}$

$R = \sqrt{12^2 + 18^2 + 2 \times 12 \times 18\cos(180 - 37°)}$
$= 11.1$ m

$$\tan(\alpha - 37°) = \frac{12\sin(180 - 37°)}{18 + 12\cos(180 - 37°)}$$

$\Rightarrow \alpha - 37° = 40.6 \Rightarrow \alpha = 77.6°$ with x-axis (Fig. 2.58)

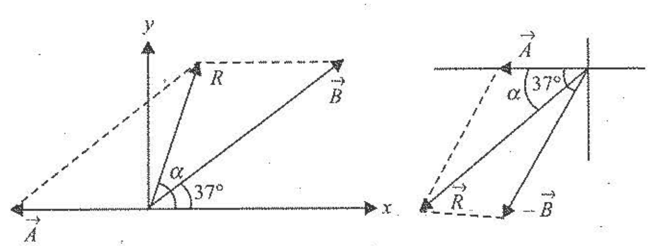


Fig. 2.58

b. $\vec{R} = \vec{A} - \vec{B}$

$R = \sqrt{12^2 + 18^2 + 2 \times 12 \times 18\cos 37°} = 28.5$

$$\tan\alpha = \frac{18\sin 37°}{12 + 18\cos 37°} \Rightarrow \alpha = 22°$$

Angle with x-axis = 180° + 22° = 202°

c. $-\vec{A} - \vec{B}$ is opposite to $\vec{A} + \vec{B}$ and having same magnitude as that of $\vec{A} + \vec{B}$.

d. $\vec{B} - \vec{A}$ is opposite to $\vec{A} - \vec{B}$ and having same magnitude as that of $\vec{A} - \vec{B}$.

7. $R = \sqrt{4^2 + 7^2 + 2 \times 4 \times 7 \cos 60^\circ} = \sqrt{93}$ m/s

$$\tan\alpha = \frac{7\sin 60^\circ}{4 + 7\cos 60^\circ} = \frac{7\sqrt{3}}{15}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{7\sqrt{3}}{15}\right)$$

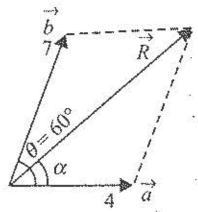


Fig. 2.59

8. Let $\vec{A} = 3\hat{i} - 4\hat{j} + 12\hat{k}$, then $A = \sqrt{3^2 + 4^2 + 12^2} = 13$

Unit vector along $\vec{A}$ is $\hat{A} = \dfrac{\vec{A}}{A} = \dfrac{3\hat{i} - 4\hat{j} + 12\hat{k}}{13}$

Unit vector opposite to $\vec{A}$ is $-\hat{A} = -\dfrac{\vec{A}}{A}$

$$= -\left(\frac{3\hat{i} - 4\hat{j} + 12\hat{k}}{13}\right)$$

9. **a.** $3\vec{a} + 2\vec{b} + \vec{c} = 3(2\hat{i} - 3\hat{j}) + 2(6\hat{i} + 2\hat{j} - 3\hat{k}) + \hat{i} + \hat{k} = 19\hat{i} - 5\hat{j} - 5\hat{k}$

b. $\vec{a} - 6\vec{b} + 2\vec{c} = 2\hat{i} - 3\hat{j} - 6(6\hat{i} + 2\hat{j} - 3\hat{k}) + 2(\hat{i} + \hat{k}) = -32\hat{i} - 15\hat{j} + 20\hat{k}$

10. **a.** $A = \sqrt{4^2 + 3^2} = 5$, $B = \sqrt{5^2 + 2^2} = \sqrt{29}$

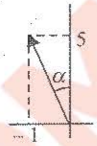


Fig. 2.60

b. $\vec{A} - \vec{B} = -\hat{i} + 5\hat{j}$

c. Magnitude: $|\vec{A} - \vec{B}| = \sqrt{1^2 + 5^2} = \sqrt{26}$

$\tan\alpha = \dfrac{1}{5} \Rightarrow \alpha = \tan^{-1}\left(\dfrac{1}{5}\right)$

d. See Fig. 2.61

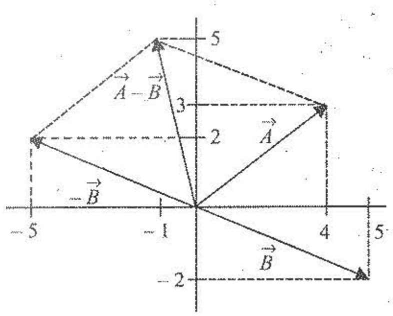


Fig. 2.61

11. **a.** Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

Magnitude: $|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Since magnitude is not 1, so $\vec{a}$ is not a unit vector.

b. No, a rectangular component cannot be greater than a vector itself. Since rectangular component, say, $R_x = R\cos\theta$, $\cos\theta$ never becomes greater than 1. So R_x never becomes greater than R.

12. $\vec{R} = 2\hat{i} + 3\hat{j}$, Let $\vec{A} = \hat{i} + 2\hat{j}$, $\vec{B} = \hat{i} - \hat{j}$

Then, we can write $\vec{R} = m\vec{A} + n\vec{B}$,

where $m\vec{A}$ is the component of $\vec{R}$ along $\vec{A}$ and

$n\vec{B}$ is the component of $\vec{R}$ along $\vec{B}$

$$\Rightarrow \quad 2\hat{i} + 3\hat{j} = m\hat{i} + 2m\hat{j} + n\hat{i} - n\hat{j}$$

$$\Rightarrow \quad m + n = 2,\ 2m - n = 3$$

From these $m = 5/3$, $n = 1/3$

$$m\vec{A} = \frac{5}{3}(\hat{i} + 2\hat{j}),$$

$$n\vec{B} = \frac{1}{3}(\hat{i} - \hat{j})$$

13. Rectangular component of $\vec{R}$ along $\vec{A}$ is $= \left(\vec{R}\cdot\hat{A}\right)\hat{A}$

$$= \frac{\left(\vec{R}\cdot\vec{A}\right)\vec{A}}{A^2} = \frac{(2\times 1 + 3\times 1)\left(\hat{i} + \hat{j}\right)}{\left(\sqrt{2}\right)^2} = \frac{5}{2}\left(\hat{i} + \hat{j}\right)$$

14. **a.** $\vec{A}\cdot\vec{B} = -2 \times 2 + 6 \times (-3) = -22$,

$A = \sqrt{2^2 + 6^2} = \sqrt{40}$, $B = \sqrt{2^2 + 3^2} = \sqrt{13}$,

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{-22}{\sqrt{40}\sqrt{13}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{-22}{\sqrt{40}\sqrt{13}}\right)$$

b. In the same way as above: $\theta = \cos^{-1}\left[\dfrac{-22}{\sqrt{40}\sqrt{13}}\right]$

c. $\vec{A}\cdot\vec{B} = -4 \times 7 + 2 \times 14 = 0$

So, angle between $\vec{A}$ and $\vec{B}$ is 90°

15. **a.** Let side of the cube is d, then $\vec{ab} = d\hat{k}$ $\vec{ad} = d\hat{i} + d\hat{j} + d\hat{k}$; $\cos\theta = \dfrac{\vec{ab}\cdot\vec{ad}}{\left|\vec{ab}\right|\left|\vec{ad}\right|} = \dfrac{d^2}{d\sqrt{3}d} = \dfrac{1}{\sqrt{3}}$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

b. $\vec{ac} = d\hat{j} + d\hat{k}$, $\vec{ad} = d\hat{i} + d\hat{j} + d\hat{k}$

$$\cos\theta = \frac{\vec{ac}\cdot\vec{ad}}{|\vec{ac}|\left|\vec{ad}\right|} = \frac{2d^2}{\sqrt{2}d\sqrt{3}d} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$$

16. Since $\vec{C}$ lies at x-y plane, so let us assume $\vec{C} = x\hat{i} + y\hat{j}$,
Now, $\vec{C} \cdot \vec{A} = 0 \Rightarrow 5x - 6.5y = 0$ (i)
and $\vec{C} \cdot \vec{B} = 15 \quad \Rightarrow 10x + 7y = 15$ (ii)
From equations (i) and (ii), $y = \dfrac{3}{4}, x = \dfrac{39}{40}$

17. $|\vec{A} \times \vec{B}| = \sqrt{5^2 + 2^2} = \sqrt{29}$

$$\sin\theta = \frac{|\vec{A} \times \vec{B}|}{AB} = \frac{\sqrt{29}}{3 \times 3}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left[\frac{\sqrt{29}}{9}\right]$$

18. $|\vec{A} \times \vec{B}| = AB\sin\theta = 5 \times 12 \times \sin 30^\circ = 30$ units

19. **a.** Area $= \dfrac{1}{2}|\vec{A} \times \vec{B}|$

b. Area $= |\vec{A} \times \vec{B}|$

c. Area $= \dfrac{1}{2}|\vec{A} \times \vec{B}|$

20. $\overrightarrow{AP} = 5\hat{k}$, $\overrightarrow{AQ} = 5\hat{i} + 2\hat{j} + 3\hat{k}$
Displacement of the bird $= \overrightarrow{PQ} = \overrightarrow{AQ} - \overrightarrow{AP}$
$= 5\hat{i} + 2\hat{j} - 2\hat{k}$

$$PQ = \sqrt{5^2 + 2^2 + 2^2} = \sqrt{33} \text{ m}$$

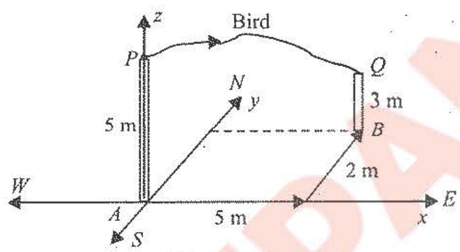


Fig. 2.62

21. Consider a regular polygon of N sides. Each side of the polygon is same. C is the centre of polygon (Fig. 2.63).
$\theta = \dfrac{2\pi}{N} \to \theta$ is the angle subtended at the centre of polygon by any one side.

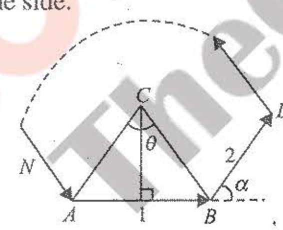


Fig. 2.63

We can prove that $\alpha = \theta$ as: $\angle ABC + \angle CBD + \alpha = 180^\circ$ (i)
But $\angle ABC = \angle CBD = 90^\circ - \dfrac{\theta}{2}$ (ii)
From equations (i) and (ii), we get $\alpha = \theta$

$$\Rightarrow \quad \alpha = \frac{2\pi}{N}$$

So, if we have N vectors of equal magnitude and they are arranged in such a way that each vector makes an angle $\dfrac{2\pi}{N}$ $(= \alpha)$ with its preceding vector, we find that head of the last vector will coincide with the tail of first vector. Hence, resultant of all these vectors becomes zero.

22. Let $x\hat{i} + y\hat{j} + z\hat{k}$ is perpendicular to $3\hat{i} - 4\hat{j} + 7\hat{k}$, then their dot product should be zero.
$\Rightarrow \quad 3x - 4y + 7z = 0$. Take $x = 1$, $y = 2$, $z = 5/7$
So, $\hat{i} + 2\hat{j} + \dfrac{5}{7}\hat{k}$ is perpendicular to $3\hat{i} - 4\hat{j} + 7\hat{k}$.

23. **a.** Sum of any two sides of a triangle is greater than or equal to third side.

$$\Rightarrow \quad PR \le PQ + QR$$

$$\Rightarrow \quad \left|\vec{A} + \vec{B}\right| \le \left|\vec{A}\right| + \left|\vec{B}\right|$$

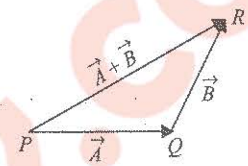


Fig. 2.64

b. $PR + QR \ge PQ \quad \Rightarrow \quad \left|\vec{A} + \vec{B}\right| + \left|\vec{B}\right| \ge \left|\vec{A}\right|$
$\Rightarrow \quad \left|\vec{A} + \vec{B}\right| \ge \left|\vec{A}\right| - \left|\vec{B}\right| \Rightarrow \quad \left|\vec{A} + \vec{B}\right| \ge \left|\left|\vec{A}\right| - \left|\vec{B}\right|\right|$ [(c) and (d) parts, do as (a) and (b)]

24. $\vec{R} = \vec{P} + \vec{Q}$, $\vec{S} = -\vec{P} + \vec{Q}$, Given that $\vec{S}$ is $\perp$ $\vec{R}$
So, $\vec{R} \cdot \vec{S} = 0 \Rightarrow \quad \left(\vec{P} + \vec{Q}\right) \cdot \left(-\vec{P} + \vec{Q}\right) = 0$
$\Rightarrow \quad -P^2 + Q^2 = 0$
$\Rightarrow \quad P^2 = Q^2$
$\Rightarrow \quad P = Q$
Hence proved.

Alternatively: $\beta = 90 - \alpha$, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$ (i)

$$\tan\beta = \frac{Q\sin(180 - \theta)}{P + Q\cos(180 - \theta)}$$

$$\Rightarrow \quad \tan(90 - \alpha) = \frac{Q\sin\theta}{P - Q\cos\theta}$$

$$\Rightarrow \quad \cot\alpha = \frac{Q\sin\theta}{P - Q\cos\theta} \quad \text{(ii)}$$

Multiply equations (i) and (ii): $\quad 1 = \dfrac{Q^2\sin^2\theta}{P^2 - Q^2\cos^2\theta}$

Simplify to get: $P = Q$

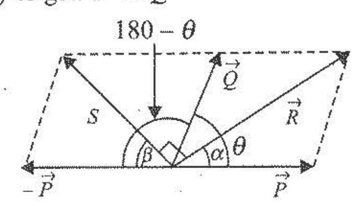


Fig. 2.65

25. $|\hat{P}| = |\hat{Q}| = 1$, because they are unit vectors.

L.H.S. $= |\hat{P} - \hat{Q}| = \sqrt{(\hat{P} - \hat{Q}) \cdot (\hat{P} - \hat{Q})}$

$= \sqrt{P^2 + Q^2 - 2\hat{P} \cdot \hat{Q}} = \sqrt{1^2 + 1^2 - 2PQ\cos\theta}$

$= \sqrt{2 - 2 \times 1 \times 1\cos\theta} = \sqrt{2(1 - \cos\theta)}$

$= \sqrt{2\left[2\sin^2(\theta/2)\right]} = 2\sin(\theta/2) =$ R.H.S.

Objective Type

1. a. Given $\left|\vec{A}\right| = \left|\vec{B}\right|$ or $A = B$.

Sum: $\vec{R} = \vec{A} + \vec{B}$

$$\Rightarrow \quad \left|\vec{R}\right| = \sqrt{A^2 + B^2} = \sqrt{2}A$$

Difference: $\vec{S} = \vec{A} - \vec{B}$

$$\Rightarrow \quad \left|\vec{S}\right| = \sqrt{A^2 + B^2} = \sqrt{2}A$$

$$\alpha_1 = 45^\circ, \alpha_2 = 45^\circ$$

Hence, $\vec{R}$ and $\vec{S}$ will be perpendicular and also of equal lengths.

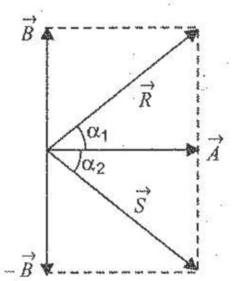


Fig. 2.66

2. c. The minimum number of vectors having different planes which can be added to give zero resultant is 4.

3. d. We see that dot product of $\hat{i} + \hat{j} + \hat{k}$ with $3\hat{i} + 2\hat{j} - 5\hat{k}$ is zero.

4. d. In ΔMNO: $\vec{A} + \vec{C} - \vec{D} = 0 \Rightarrow \vec{C} - \vec{D} = -\vec{A}$

Hence (b) is correct.

In ΔMNP: $\vec{A} + \vec{B} + \vec{E} = 0$

Hence (a) is correct.

In $\square MPNO$: $-\vec{E} - \vec{B} + \vec{C} - \vec{D} = 0$

$\Rightarrow \vec{B} + \vec{E} - \vec{C} = -\vec{D}$

Hence, (c) is correct.

5. d. For the resultant of some vectors to be zero, they should form a closed figure taken in same order.

6. b. $\cos\beta = \dfrac{R}{B} = \dfrac{1}{2} \Rightarrow \beta = 60^\circ$

Angle between $\vec{A}$ and $\vec{B} = 90^\circ + \beta = 150^\circ$

7. b. $\dfrac{a+b}{a-b} = \dfrac{3}{1}$ or $3a - 3b = a + b$

or $2a = 4b$ or $a = 2b$

8. b. Note that the angle between two forces is 120° and not 60°.

$$R^2 = F^2 + F^2 + 2F^2 \cos 120^\circ$$

$$\text{or } R^2 = 2F^2 + 2F^2\left(-\frac{1}{2}\right) = F^2 \text{ or } R = F$$

9. b. Here, the angle between two vectors of equal magnitude is 120°. So, resultant has the same magnitude as either of the given vectors. Moreover, it is mid-way between the two vectors, i.e., it is along x-axis.

10. a. $\left(\dfrac{1}{3}\right)^2 = 1^2 + 1^2 + 2 \times 1 \times 1 \cos\theta$

or $\dfrac{1}{9} = 2(1 + \cos\theta)$ or $1 + \cos\theta = \dfrac{1}{18}$

or $\cos\theta = \dfrac{1}{18} - 1 = -\dfrac{17}{18}$ or $\theta = \cos^{-1}\left(-\dfrac{17}{18}\right)$

11. c. $\vec{A} + \vec{B}$ will be in the plane containing $\vec{A}$ and $\vec{B}$, whereas $\vec{A} \times \vec{B}$ will be $\perp$ to that plane.

12. d. Consider only x and y components: $\sqrt{3^2 + 1^2} = \sqrt{10}$

13. a.

14. c. Let third side is $\vec{C}$, then $\left|\vec{C}\right| = \left|\vec{A} + \vec{B}\right|$ or

$\left|\vec{C}\right| = \left|\vec{A} - \vec{B}\right|$

15. a. $A_1 = 2$, $A_2 = 3$, $\left|\vec{A}_1 + \vec{A}_2\right| = 3$

$\Rightarrow \left|\vec{A}_1 + \vec{A}_2\right|^2 = 9 \Rightarrow A_1^2 + A_2^2 + 2\vec{A}_1 \cdot \vec{A}_2 = 9$

$\Rightarrow \quad 2^2 + 3^2 + 2\vec{A}_1 \cdot \vec{A}_2 = 9 \quad \Rightarrow \vec{A}_1 \cdot \vec{A}_2 = -2$

Now, $\left(\vec{A}_1 + 2\vec{A}_2\right) \cdot \left(3\vec{A}_1 + 4\vec{A}_2\right) = 3A_1^2 - 8A_2^2$

$+2\vec{A}_1 \cdot \vec{A}_2 = 3(2)^2 - 8(3)^2 + 2(-2) = -64$

16. d. $\vec{A} \cdot \vec{B} = 0$ (given) $\quad \Rightarrow \quad \vec{A} \perp \vec{B}$

$\vec{A} \cdot \vec{C} = 0$ (given) $\quad \Rightarrow \quad \vec{A} \perp \vec{C}$

$\vec{A}$ is perpendicular to both $\vec{B}$ and $\vec{C}$.

We know, from the definition of cross product, that $\vec{B} \times \vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$.

So, $\vec{A}$ is parallel to $\vec{B} \times \vec{C}$.

17. a. $\cos\theta = \dfrac{C}{A} = \dfrac{3}{5}$ or $\theta = \cos^{-1}\left(\dfrac{3}{5}\right)$. See Fig. 2.67.

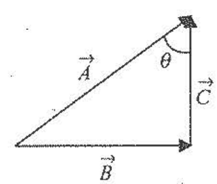


Fig. 2.67

18. b. Clearly, $\vec{B}$ should be either in second quadrant or fourth quadrant. In none of the given options, we have '$-\hat{i}$ term'. So, second quadrant is ruled out. Also, $\vec{B}$ should make an angle of $90^\circ - \theta$ with x-axis (Fig. 2.68). So, $\vec{B}$ should be $B\cos(90^\circ - \theta)\hat{i} - B\sin(90^\circ - \theta)\hat{j} = B\sin\theta\hat{i} - B\cos\theta\hat{j}$.

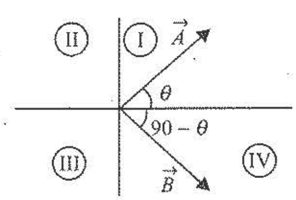


Fig. 2.68

19. c. See Fig. 2.69

$\tan\beta = \dfrac{2}{3}$ or $\beta = \tan^{-1}\left(\dfrac{2}{3}\right)$

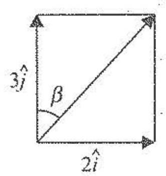


Fig. 2.69

20. c. $\vec{P}$ is in fourth quadrant.

$4\hat{i} + 3\hat{j}$ is in the first quadrant.

Clearly, $4\hat{i} + 3\hat{j}$ can be perpendicular to $\vec{P}$. For confirmation, let us check whether their dot product is zero.

$(3\hat{i} - 4\hat{j}) \cdot (4\hat{i} + 3\hat{j}) = 12 - 12 = 0$

This shows that $4\hat{i} + 3\hat{j}$ is perpendicular to $3\hat{i} - 4\hat{j}$.

21. c. $\vec{S} = 75\hat{j} + [60\cos 45°\,\hat{j} - 60\sin 45°\hat{i}] + 20\hat{i}$

$\vec{S} = (20 - 60 \times 0.7)\hat{i} + (60 \times 0.7 + 75)\hat{j}$

or $\vec{S} = -22\hat{i} + 117\hat{j}$

$S = \sqrt{22^2 + 117^2} = \sqrt{484 + 13689} = \sqrt{14173}$

$= 119$ km

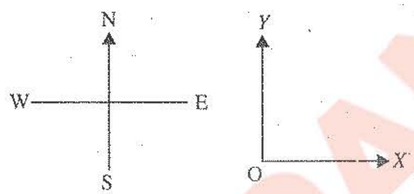


Fig. 2.70

22. b. Area of parallelogram: $\left|\vec{A} \times \vec{B}\right| = AB/2$ (given)

$\Rightarrow AB\sin\theta = AB/2 \quad \Rightarrow \sin\theta = 1/2$

$\Rightarrow \quad \theta = 30°$

23. b. $a^2 + b^2 + 2ab\cos\theta = a^2 + b^2 - 2ab\cos\theta$

or $\quad 4ab\cos\theta = 0$

But $4ab \neq 0 \Rightarrow \cos\theta = 0$ or $\theta = 90°$

Aliter

$(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are the diagonals of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

Since $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, therefore the two diagonals of a parallelogram are equal. So, think of square. This leads to $\theta = 90°$.

24. a. $\vec{A} \times \vec{B} = (4\hat{i} + 6\hat{j}) \times (2\hat{i} + 3\hat{j})$

$= 12(\hat{i} \times \hat{j}) + 12(\hat{j} \times \hat{i}) = 12(\hat{i} \times \hat{j}) - 12(\hat{i} \times \hat{j}) = 0$

Again, $\vec{A} \cdot \vec{B} = (4\hat{i} + 6\hat{j}) \cdot (2\hat{i} + 3\hat{j}) = 8 + 18 = 26$

Again, $\dfrac{|\vec{A}|}{|\vec{B}|} = \dfrac{\sqrt{16 + 36}}{\sqrt{4 + 9}} \neq \dfrac{1}{2}$.

Also, $\vec{B} = \dfrac{1}{2}\vec{A}$

$\Rightarrow \quad \vec{A}$ and $\vec{B}$ are parallel and not antiparallel.

25. a. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & p & q \\ 5 & 7 & 3 \end{vmatrix} = 0$

or $\hat{i}(3p - 7q) + \hat{j}(5q - 6) + \hat{k}(14 - 5p) = 0$

$3p = 7q,\ 5q - 6 = 0$ or $q = \dfrac{6}{5}$

$14 - 5p = 0$ or $5p = 14$ or $p = \dfrac{14}{5}$

26. b. If $\vec{A}$ is perpendicular to $\vec{B}$, then $\vec{A} \cdot \vec{B} = 0$ and $\vec{A} \times \vec{B} \neq 0$

27. b. $\overrightarrow{PR} = \vec{a} + \vec{b} \quad \Rightarrow$ major diagonal

$\overrightarrow{SQ} = \vec{a} - \vec{b} \quad \Rightarrow$ minor diagonal

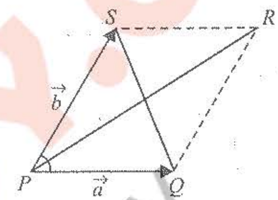


Fig. 2.71

28. b. $\vec{A} + \vec{B} = \vec{C}$

$(\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{C} \cdot \vec{C}$

$\Rightarrow \quad A^2 + B^2 + 2AB\cos\theta = C^2$

$\Rightarrow \quad 4^2 + 5^2 = 2 \times 4 \times 5\cos\theta = 61$

$\Rightarrow \cos\theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta = 60°$

29. b. $\tan\theta = \dfrac{2}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{2}{3}\right)$

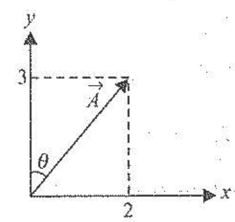


Fig. 2.72

30. a. Let that vector is $\vec{C}$. Then

$\vec{C} = C\hat{C} = b\hat{a} \quad \Rightarrow \quad \vec{C} = \dfrac{b\vec{a}}{a} = \dfrac{5}{\sqrt{2}}(\hat{i} - \hat{j})$

31. b. For the resultant of two vectors to be zero, they should be equal and opposite.

32. c. In first option (a), vector is along x-axis (Fig. 2.73).

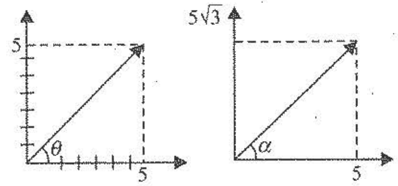


Fig. 2.73

In (b), angle of vector with x-axis

$\tan\theta = \dfrac{5}{5} = 1 \quad \Rightarrow \quad \theta = 45°$

In (c), angle of vector with x-axis

$$\tan\alpha = \frac{5\sqrt{3}}{5} = \sqrt{3} \quad \Rightarrow \quad \alpha = 60^\circ$$

33. b. See Fig. 2.74

$$AC \le AB + BC \quad \Rightarrow \quad |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$$

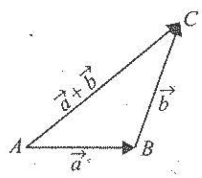


Fig. 2.74

34. d. The resultant of two forces can lie between $A - B$ and $A + B$, i.e., $12 - 1 = 11$ N and $12 + 1 = 13$ N.

35. a. Find min $(A - B)$ and maximum $(A + B)$ value of each case, then check if 4 N lies between them.

36. b. $\vec{BA} + \vec{BC} = \vec{BD}$

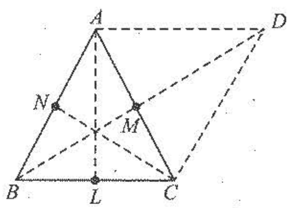


Fig. 2.75

$\vec{BA} + \vec{BC} = 2\,\vec{BM}$. Hence, the answer is $2BM$.

37. b. $(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B}) = 0$

$$A^2 - B^2 = 0 \quad \Rightarrow \quad A^2 = B^2$$

$$\Rightarrow \quad A = B$$

38. c.

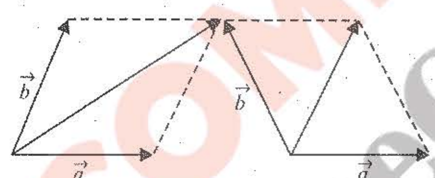


Fig. 2.76

$\vec{a} + \vec{b} \rightarrow$ major diagonal, $\vec{a} + \vec{b} \rightarrow$ minor diagonal

39. b. $\tan\theta' = \dfrac{C}{A} = 1$

$$\Rightarrow \quad \theta' = 45^\circ = \frac{\pi}{4}$$

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

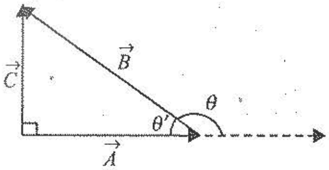


Fig. 2.77

40. a. $\vec{F_2} - \vec{F_1} = \vec{F_2} + (-\vec{F_1})$

$= 250$ N due north $+ 500$ N due west

$$\tan\theta = \frac{500}{250} = 2$$

$$|\vec{F_2} - \vec{F_1}| = \sqrt{(500)^2 + (250)^2} = 250\sqrt{5}\ \text{N}$$

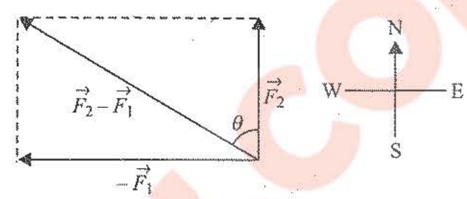


Fig. 2.78

41. c. $\vec{OC}$ and $\vec{OA}$ are equal in magnitude and inclined to each other at an angle of 90°. So, their resultant is $\sqrt{2}r$. It acts mid-way between $\vec{OC}$ and $\vec{OA}$, i.e., along OB.

Now, both r and $\sqrt{2}r$ are along the same line and in the same direction.

$\therefore$ resultant $= r + \sqrt{2}r = r(1 + \sqrt{2})$

42. b. $\tan 45^\circ = \dfrac{2\sin 60^\circ}{a + 2\cos 60^\circ} = \dfrac{\sqrt{3}}{a+1}$

or $1 = \dfrac{\sqrt{3}}{a+1}$ or $a + 1 = \sqrt{3}$ or $a = \sqrt{3} - 1$

43. d. $\tan 90^\circ = \dfrac{2Q\sin\theta}{P + 2Q\cos\theta} \Rightarrow \infty = \dfrac{2Q\sin\theta}{P + 2Q\cos\theta}$

$$\Rightarrow \quad P + 2Q\cos\theta = 0$$

Now, $R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$$\Rightarrow \quad R^2 = Q^2 + P[P + 2Q\cos\theta]$$

$$\Rightarrow \quad R^2 = Q^2 \quad \Rightarrow R = Q$$

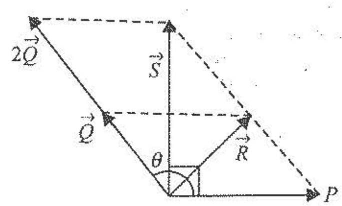


Fig. 2.79

Alternate method:

$$\vec{R} = \vec{P} + \vec{Q} \Rightarrow \vec{P} = \vec{R} - \vec{Q} \text{ and } \vec{S} = \vec{P} + 2\vec{Q}$$

$$= \vec{R} - \vec{Q} + 2\vec{Q}$$

Now, $\vec{S}$ and $\vec{P}$ are perpendicular (Fig. 2.78), so

$$\vec{S} \cdot \vec{P} = 0 \quad \Rightarrow (\vec{R} + \vec{Q}) \cdot (\vec{R} - \vec{Q}) = 0$$

$$\Rightarrow R^2 = Q^2 \quad \Rightarrow R = Q$$

44. a. Given $\vec{C} = \left|\vec{B}\right|\hat{j} \Rightarrow \vec{C} = 5\hat{j}$

Let $\vec{C} = \vec{A} + \vec{B} = \vec{A} + 3\hat{i} + 4\hat{j}$

$$\Rightarrow 5\hat{j} = \vec{A} + 3\hat{i} + 4\hat{j}$$

$$\Rightarrow \vec{A} = -3\hat{i} + \hat{j} \Rightarrow \left|\vec{A}\right| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

45. c. $\overrightarrow{AB} + \overrightarrow{AF} = \overrightarrow{AO} \quad \Rightarrow \overrightarrow{AB} = \overrightarrow{AO} - \overrightarrow{AF}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AO}, \quad \overrightarrow{AD} = 2\overrightarrow{AO}, \quad \overrightarrow{AE} = \overrightarrow{AO} + \overrightarrow{AF}$

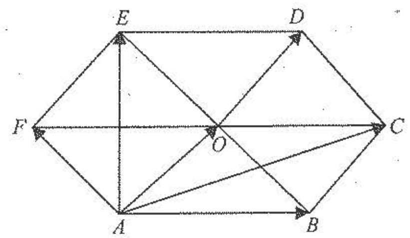


Fig. 2.80

Now, $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$

$= 5\overrightarrow{AO} + \overrightarrow{AB} + \overrightarrow{AF} = 5\overrightarrow{AO} + \overrightarrow{AO} = 6\overrightarrow{AO}$

46. b. Displacement = $\overrightarrow{AB}$, angle between $\vec{r}_1$ and $\vec{r}_2$: $\theta = 75° - 15° = 60°$

From Fig. 2.55, $AB^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$

$= 3^2 + 4^2 - 2 \times 3 \times 4\cos 60° = 13$

$\Rightarrow \quad AB = \sqrt{13}$

47. c. $x + y = 16$. Also, $y^2 = 8^2 + x^2$

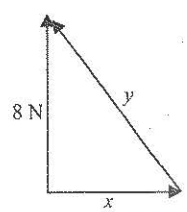


Fig. 2.81

or $y^2 = 64 + (16 - y)^2 \quad [\because \quad x = 16 - y]$

or $y^2 = 64 + 256 + y^2 - 32y$

or $32y = 320$ or $y = 10$ N

$\therefore \quad x + 10 = 16$ or $x = 6$ N

48. c. Graphically:

$\angle ROQ = \theta/2$, $\angle RQO = \theta/2$

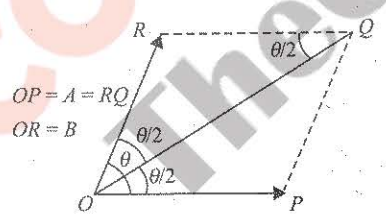


Fig. 2.82

Hence, ΔOQR is isosceles

$\Rightarrow OR = RQ \Rightarrow \quad B = A$

Analytically: $\tan(\theta/2) = \dfrac{B\sin\theta}{A + B\cos\theta}$

$$\Rightarrow \quad \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{2B\sin(\theta/2)\cos(\theta/2)}{A + B(2\cos^2(\theta/2) - 1)}$$

$\Rightarrow A + 2B\cos^2(\theta/2) - B = 2B\cos^2(\theta/2) \Rightarrow A = B.$

49. a. $R_x = 1 + 2\cos 120° + 3\cos 240°$

$R_y = 2\sin 120° + 3\sin 240°$

$$= 2 \times \frac{\sqrt{3}}{2} + 3 \times \left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{3}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \quad \Rightarrow \theta = 30°$$

50. a. $u_x = 8\ \text{ms}^{-1}$, $a_x = 2\ \text{ms}^{-2}$

$$\vec{v}_x = \vec{u}_x + \vec{a}_x t = 8\hat{i} + 2t\hat{i}$$

$$\vec{v}_y = -15\hat{j},\ \vec{V} = \vec{v}_x + \vec{v}_y,\ \vec{V} = [(8 + 2t)\hat{i} - 15\hat{j}]\ \text{ms}^{-1}$$

51. c. You have to try all the options. Let us discuss the correct option only.

$$\hat{i} \times \hat{n} = \mu(\hat{r} \times \hat{n})$$

$(1)(1)\sin(180° - i) = \mu(1)(1)(180° - r)$

$\sin i = \mu \sin r$

52. c. $P + Q = 16$ (i)

$P^2 + Q^2 + 2PQ\cos\theta = 64$ (ii)

$\tan 90° = \dfrac{Q\sin\theta}{P + Q\cos\theta}$ or $\infty = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

$P + Q\cos\theta = 0$ or $Q\cos\theta = -P$

From equation (ii), $P^2 + Q^2 + 2P(-P) = 64$

or $Q^2 - P^2 = 64$ or $Q - P = \dfrac{64}{16} = 4$ (iii)

Adding equations (i) and (iii), we get

$2Q = 20$ or $Q = 10$ units

From equation (i), $P + 10 = 16$ or $P = 6$ units

53. d. The length of the vector is not changed by the rotation of the coordinate axes.

$\sqrt{(n+1)^2 + 1^2} = \sqrt{n^2 + 3^2}$ or $n^2 + 2n + 2 = n^2 + 9$

or $2n = 7$ or $n = 3.5$

54. b. For collision,

$\vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$ or $\vec{r}_1 - \vec{r}_2 = (\vec{v}_2 - \vec{v}_1)t$

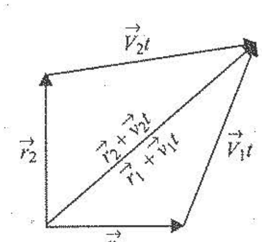


Fig. 2.83

Equating unit vectors, we get

$$\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} = \frac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|}$$

55. a. Net force along x-axis (Fig. 2.84)

$F_x = F_1 + F_2\cos 30° - F_3\cos 30°$

$F_x = 300$ N

$F_y = F_2\sin 30° + F_3\sin 30°$

$= 400 \times \frac{1}{2} + 400 \times \frac{1}{2} = 400$ N

Net force: $F = \sqrt{300^2 + 400^2} = 500$ N

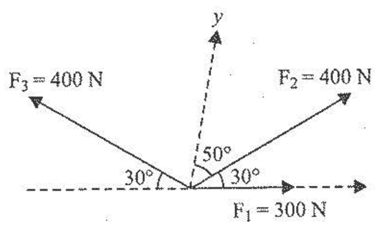


Fig. 2.84

Multiple Correct Answers Type

1. **a., c.** Both x and y components of $\vec{d}_1$ are positive.
   x component of $\vec{d}_2$ is negative and y component is positive.
   Both x and y components of $\vec{d}_1 + \vec{d}_2$ are positive.
2. **a., b.** Component of $\vec{A}$ along $\vec{B}$ is $\left|\vec{A}\right| \cos\theta \hat{B}$ for θ being the angle between the vectors.
   Also $\hat{B} = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$. So, choice (a) is correct.
   The vector $(\hat{i} - \hat{j})$ is perpendicular to the vector $(\hat{i} + \hat{j})$.
   So, the other resolved component is $\left|\vec{A}\right| \sin\theta \left(\frac{\hat{i} - \hat{j}}{\sqrt{2}}\right)$.
3. **a., b., c.** $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$
   $= \hat{i}(1 - 1) - \hat{j}(2 - 1) + \hat{k}(2 - 1) = -\hat{j} + \hat{k}$
   Unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is $\left(\frac{-\hat{j} + \hat{k}}{\sqrt{2}}\right)$. So, choices (a) and (c) are correct.
   Any vector whose magnitude is K (constant) times $(2\hat{i} + \hat{j} + \hat{k})$ is parallel to $\vec{A}$.
   So, unit vector $\frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$ is parallel to $\vec{A}$.
   So, choice (b) is correct.
4. **a., d.** If two vectors are normal to each other, then their dot product is zero.
   $(\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2) = 0 \quad \Rightarrow v_1^2 - v_2^2 = 0$
   $\Rightarrow \quad v_1^2 = v_2^2 \quad \Rightarrow \quad v_1 = v_2$ or $|\vec{v}_1| = |\vec{v}_2|$
5. **a., d.** The resultant of three vectors is zero only if they can form a triangle. But three vectors lying in different planes cannot form a triangle.

CHAPTER

3

Units and Dimensions

- Systems of Units
- Dimensions of a Physical Quantity
- Dimensional Formulae
- Uses of Dimensional Analysis
- Significant Figures
- Errors in Measurements
- Absolute Errors
- Propagation of Combined Errors

To measure a physical quantity, we need a standard known as unit. For example, if length of some metal rod is measured to be 15 cm, then cm is the unit of length. 15 is the numerical part. So

$$\textit{Physical Quantity} = \textit{Numerical Part} \times \textit{Unit}$$

We have three types of units: Fundamental units, Supplementary units and Derived units as illustrated in Fig. 3.1 below.

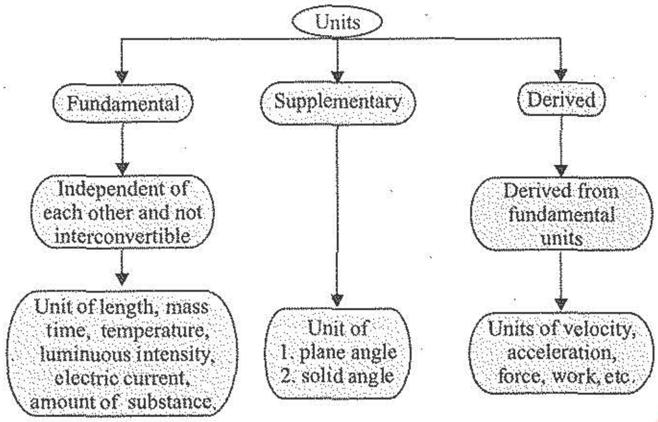


Fig. 3.1

SYSTEMS OF UNITS

It is a complete set of fundamental and derived units. We have four types of systems of units. Generally, a system is named in terms of fundamental units on which it is based.

1. **M.K.S. system:** In this system length, mass and time are taken as fundamental quantities.
2. **C.G.S. system:** It is Gaussian system. In this also length, mass and time are taken as fundamental quantities.
3. **F.P.S. system:** In this also length, mass and time are taken as fundamental quantities. It is British Engineering system.

   M.K.S. and C.G.S. systems are also called metric systems or decimal systems, because multiples and submultiples are related by powers of 10. Example: 1 km = 10^3 m

   F.P.S. system is not used much now a days because of inconvenient multiples and submultiples. The disadvantages of C.G.S. system is that many derived units in this system become unnecessarily small.

   The main drawback of all the above systems is that they are confined to mechanics only. All the physical quantities appearing in physics cannot be described by these systems. So, we need such a system which takes care of all the physical quantities appearing in physics. S.I. system is such a kind of system.
4. **S.I. system:** It was introduced in 1971 by General Conference on Weights and Measures. It is also called as rationalised M.K.S. system because it is made by modifying the M.K.S. system. It is nothing but extended M.K.S. system. It is a comprehensive system (see Table 1).

   This system contains seven fundamental units and two supplementary units as shown in Table 2. It also contains a large number of derived quantities.

Table 1

M.K.S. System	C.G.S. System	F.P.S. System	S.I. Units
(i) Length m (meter)	(i) Length cm (centimeter)	(i) Length ft (foot)	It is an extended form of M.K.S. system. It includes four more fundamental units (in addition to three basic units), which represent fundamental quantities in electricity, magnetism, heat and light.
(ii) Mass kg (kilogram)	(ii) Mass g (gram)	(ii) Mass (pound)	
(iii) Time s (second)	(iii) Time s (second)	(iii) Time s (second)	

Table 2

A. Fundamental Quantities in S.I. System and Their Units

Sr. No.	Physical Quantity	Name of Unit	Symbol of Unit
1.	Mass	kilogram	kg
2.	Length	meter	m
3.	Time	second	s
4.	Temperature	kelvin	K
5.	Luminous intensity	candela	Cd
6.	Electric current	ampere	A
7.	Amount of substance	mole	mol

B. Supplementary Quantities in S.I. System and Their Units

Sr. No.	Physical Quantity	Name of Unit	Symbol of Unit
1.	Plane angle	radian	rad
2.	Solid angle	steradian	sr

Advantage of S.I. system is that it assigns only one unit to various forms of a particular physical quantity. For example, unit of all kinds of energy is J in this system. But in M.K.S. system:
Unit of mechanical energy is joule, that of heat energy is calorie, that of electric energy is Wh (watt hour), etc.

DIMENSIONS OF A PHYSICAL QUANTITY

These are the powers to which the fundamental units of mass, length and time have to be raised to represent a derived unit of the physical quantity under consideration. Dimensions of any derived physical quantity can be represented in the form of fundamental units of mass, length and time. Knowing the units, dimensions can be easily written.

To write the dimensions of a physical quantity, we use following symbols for mass, length and time:
Mass — [M]; Length — [L]; and Time — [T].

DIMENSIONAL FORMULAE

Relations which express physical quantities in terms of appropriate powers of fundamental units are known as dimensional formulae. These formulae tell us about:

1. Fundamental units involved to represent a quantity.
2. The nature of their dependence.

Illustration 3.1 **Obtain the dimensions of acceleration.**

Sol. We know that acceleration:

$$a = \frac{v}{t} = \frac{s/t}{t} = \frac{s}{t^2} \quad (S - \text{distance}, t - \text{time})$$

$$[a] = \frac{[L]}{[T]^2} = [L^1 T^{-2}] = [M^0 L^1 T^{-2}]$$

So, the dimensions of acceleration are 0 in mass, +1 in length and −2 in time.

Some more examples:

1. **Force:** Force = mass × acceleration $= [M] \times [L^1\ T^{-2}]$
$= [M^1 L^1 T^{-2}]$
2. **Momentum:** Momentum = mass × Velocity
$= [M] \times [L^1 T^{-1}] = [M^1 L^1 T^{-1}]$
3. **Work:** Work = force × distance $= [M^1 L^1 T^{-2}] \times [L]$
$= [M^1 L^2 T^{-2}]$

USES OF DIMENSIONAL ANALYSIS

1. Conversion of units of a quantity from one system to another.
2. To check the accuracy of formulae.
3. Derivation of formulae.

Conversion of Units of a Quantity from One System to Another

Physical quantities can be converted from one system of units to another. Due to this conversion, the numerical part of physical quantity changes but the dimensions and the overall quantity remain the same.

Suppose a physical quantity has the dimensional formula $M^a L^b T^c$.

Let N_1 and N_2 be the numerical values of a quantity in the two systems of units, respectively.

In first system, Physical Quantity $Q = N_1 M_1^a L_1^b T_a^c = N_1 U_1$

In second system, Same Quantity $Q = N_2 M_2^a L_2^b T_2^c = N_2 U_2$

A physical quantity remains the same irrespective of the system of measurement, i.e.,

$$Q = N_1 U_1 = N_2 U_2 \quad \Rightarrow \quad N_1 M_1^a L_1^b T_1^c = N_2 M_2^a L_2^b T_2^c$$

$$\Rightarrow \quad N_2 = N_1 \left[\frac{M_1}{M_2}\right]^a \left[\frac{L_1}{L_2}\right]^b \left[\frac{T_1}{T_2}\right]^c$$

So, knowing the quantities on the right hand side the value of N_2 can be obtained.

Illustration 3.2 **Convert 1 joule into erg.**

Sol. Joule: S.I. system, erg: C.G.S. system

Work = force × distance = mass × acceleration × length

$$= \text{mass} \times \frac{\text{length}}{(\text{time})^2} \times \text{length}$$

Dimensions of work $= [W] = [M^1 L^2 T^{-2}]$

$\therefore \quad a = 1, b = 2, c = -2.$

Now,

S.I. system	$M_1 = 1$ kg	$L_1 = 1$ m	$T_1 = 1$ s	$N_1 = 1$ cm
C.G.S. system	$M_2 = 1$ g	$L_2 = 1$ cm	$T_2 = 1$ s	$N_2 = ?$

$\therefore$ using $\quad N_2 = N_1 \left[\frac{M_1}{M_2}\right]^a \left[\frac{L_1}{L_2}\right]^b \left[\frac{T_1}{T_2}\right]^c$, we have

$$N_2 = 1\left[\frac{1\text{ kg}}{1\text{ g}}\right]^1 \left[\frac{1\text{ m}}{1\text{ cm}}\right]^2 \left[\frac{1\text{ s}}{1\text{ s}}\right]^{-2}$$

$$= 1\left[\frac{1000\text{ g}}{1\text{ g}}\right]^1 \left[\frac{100\text{ cm}}{1\text{ cm}}\right]^2 = 10^7$$

So, 1 joule $= 10^7$ erg.

Illustration 3.3 **Convert 54 kmh^{-1} into ms^{-1}.**

Sol. Let $v = 54$ kmh^{-1} $= n_2$ ms^{-1}

$[v] = LT^{-1}$, $a = 0$, $b = 1$, $c = -1$

$$n_2 = 54\left[\frac{\text{kg}}{\text{g}}\right]^0 \left[\frac{\text{km}}{\text{m}}\right]^1 \left[\frac{\text{h}}{\text{s}}\right]^{-1}$$

$$= 54 \times 1 \times 1000 \times [3600]^{-1} = \frac{54 \times 1000}{3600} = 15$$

Hence, 54 kmh^{-1} = 15 ms^{-1}.

To Check the Accuracy of Formulae

The accuracy of the expression of any physical quantity can be checked by using the *principle of homogeneity*. According to this principle, dimensions of various quantities as a whole on both sides of an expression (related to a physical quantity) are equal.

Illustration 3.4 **Check the accuracy of the relation $v^2 - u^2 = 2as$, where v and u are final and initial velocities, a is acceleration and s is the distance.**

Sol. We have $v^2 - u^2 = 2as$

Checking the dimensions on both sides, we get

L.H.S $= [LT^{-1}]^2 - [LT^{-1}]^2 = [L^2T^{-2}] - [L^2T^{-2}] = [L^2T^{-2}]$

R.H.S $= [L^1T^{-2}][L] = [L^2T^{-2}]$

Comparing L.H.S. and R.H.S., we find L.H.S. = R.H.S. Hence, the formula is dimensionally correct.

Illustration 3.5 **Check whether the relation $S = ut + \frac{1}{2}at^2$ is dimensionally correct or not, where symbols have their usual meaning.**