MATHEMATICS
for
Joint Entrance Examination
JEE (Advanced)

2nd edition

Vectors and 3D Geometry

G. Tewani

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Mathematics: Vectors and 3D Geometry, 2e

G. Tewani

ISBN-13: 978-81-315-2654-5
ISBN-10: 81-315-2654-2

Cengage Learning India Pvt. Ltd
418, F.I.E., Patparganj
Delhi 110092

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Printed in India

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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

G. Tewani

Chapter 1

Introduction to Vectors

COORDINATE AXES AND COORDINATE PLANES IN THREE-DIMENSIONAL SPACE

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other as shown in the following figure.

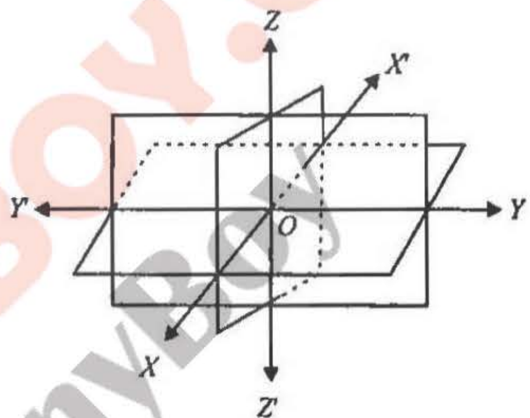
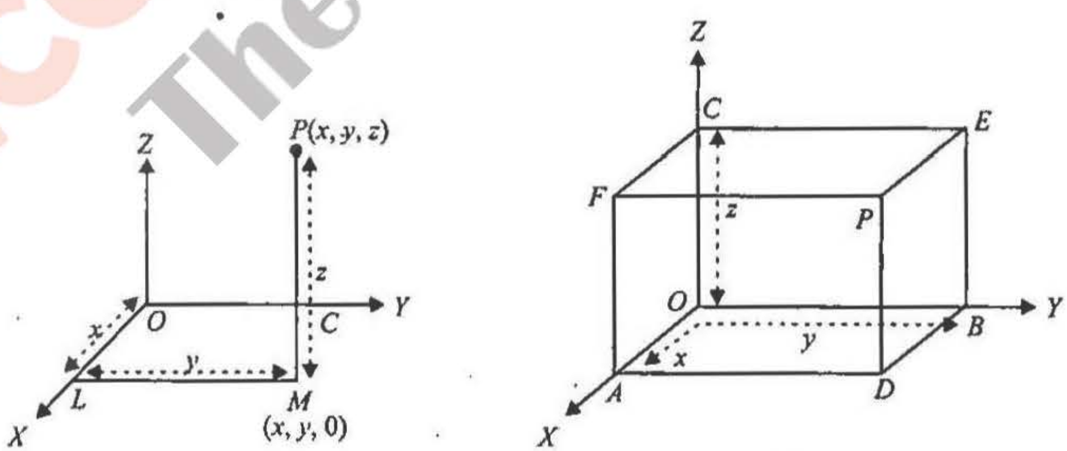


Fig. 1.1

These three planes intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$, called the x-, y- and z-axes, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes XOY, YOZ and ZOX, called respectively, the XY-plane, the YZ-plane and the ZX-plane, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line $Z'OZ$ as perpendicular to the plane XOY. If the plane of the paper is considered to be horizontal, then the line $Z'OZ$ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ are taken as negative. Similarly, the distances measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY' as negative, in front of the YZ-plane along OX as positive and to the back of it along OX' as negative. The point O is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants can be named as $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$ and are denoted by I, II, III, IV, V, VI, VII and VIII, respectively.

Coordinates of a Point in Space



(i) (ii)

Fig. 1.2

Consider a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular. Then, from point M, we draw a perpendicular ML to the x-axis, meeting it at L. Let OL be x, LM be y and MP be z. Then x, y and z are called the x-, y- and z-coordinates, respectively, of point P in the space. In Fig. 1.2, we may note that the point $P(x, y, z)$ lies in the octant $XOYZ$ and so all x, y, z are positive. If P was in any other octant, the signs of x, y and z would change accordingly. Thus, to each point P in the space, there corresponds an ordered triplet (x, y, z) of real numbers.

We observe that if $P(x, y, z)$ is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

Note: The coordinates of the origin O are $(0, 0, 0)$. The coordinates of any point on the x-axis will be $(x, 0, 0)$ and the coordinates of any point in the YZ-plane will be $(0, y, z)$.

The sign of the coordinates of a point determines the octant in which the point lies. The following table shows the signs of the coordinates in the eight octants:

Octant Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelepiped with one diagonal PQ.

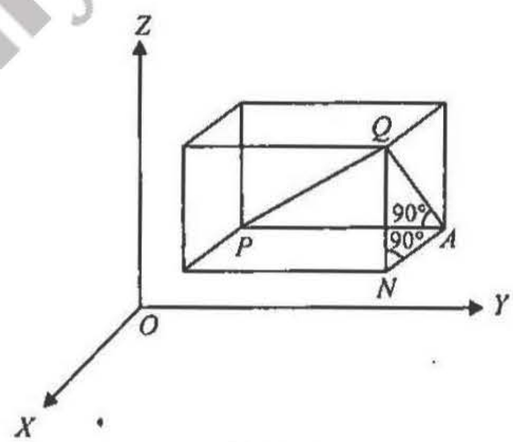


Fig. 1.3

Now, since $\angle PAQ$ is a right angle, it follows that in triangle PAQ,

$$PQ^2 = PA^2 + AQ^2 \quad \text{(i)}$$

Also, triangle ANQ is right-angled with $\angle ANQ$ being the right angle. Therefore,

$$AQ^2 = AN^2 + NQ^2 \quad \text{(ii)}$$

From (i) and (ii), we have

$$PQ^2 = PA^2 + AN^2 + NQ^2$$

Now $\quad PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence, $\quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{x_2^2 + y_2^2 + z_2^2}$, which gives the distance between the origin O and any point $Q(x_2, y_2, z_2)$.

Section Formula

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let point $R(x, y, z)$ divide PQ in the given ratio $m : n$ internally. Draw PL, QM and RN perpendicular to the XY-plane. Obviously $PL \parallel RN \parallel OM$ and the feet of these perpendiculars lie in the XY-plane. Through point R draw a line ST parallel to line LM. Line ST will intersect line LP externally at point S and line MQ at T, as shown in Fig. 1.4.

Also note that quadrilaterals $LNRS$ and $NMTR$ are parallelograms.

The triangles PSR and QTR are similar. Therefore,

$$\frac{m}{n}=\frac{PR}{QR}=\frac{SP}{QT}=\frac{SL-PL}{QM-TM}=\frac{NR-PL}{QM-NR}=\frac{z-z_1}{z_2-z}$$

$$\Rightarrow \qquad z=\frac{mz_2+nz_1}{m+n}$$

Hence, the coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m : n$ are $\frac{mx_2+nx_1}{m+n}$, $\frac{my_2+ny_1}{m+n}$ and $\frac{mz_2+nz_1}{m+n}$.

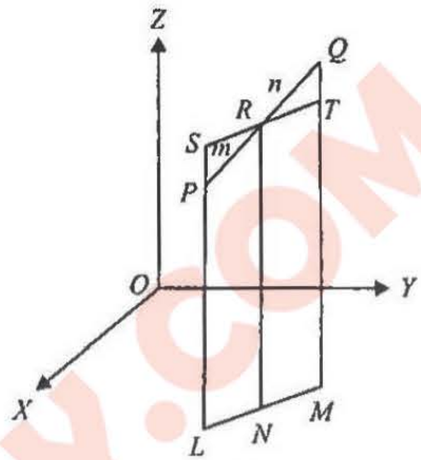


Fig. 1.4

If point R divides PQ externally in the ratio $m : n$, then its coordinates are obtained by replacing n with $-n$ so that the coordinates become $\frac{mx_2-nx_1}{m-n}$, $\frac{my_2-ny_1}{m-n}$ and $\frac{mz_2-nz_1}{m-n}$.

Notes:

1. If R is the midpoint of PQ, then $m : n = 1:1$; so $x=\frac{x_1+x_2}{2}$, $y=\frac{y_1+y_2}{2}$, $z=\frac{z_1+z_2}{2}$.

   These are the coordinates of the midpoint of the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

2. The coordinates of the point R which divides PQ in the ratio $k : 1$ are obtained by taking $k=\frac{m}{n}$, which are given by $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$.

3. If vertices of triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, and $AB = c$, $BC = a$, $AC = b$, then centroid of the triangle is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$ and its incenter is

   $$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right)$$

EVOLUTION OF VECTOR CONCEPT

In our day-to-day life, we come across many queries such as 'What is your height?' and 'How should a football player hit the ball to give a pass to another player of his team?' Observe that a possible answer to the first query may be 1.5 m, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors*. In mathematics, physics and engineering, we frequently come across with both types of quantities, namely scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density and resistance and vector quantities such as displacement, velocity, acceleration, force, momentum and electric field intensity.

Let 'l' be a straight line in a plane or a three-dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a directed line [Fig. 1.5 (i), (ii)].

Now observe that if we restrict the line l to the line segment AB, then a magnitude is prescribed on line (i) with one of the two directions, so that we obtain a directed line segment, Fig. 1.5 (iii). Thus, a directed line segment has magnitude as well as direction.

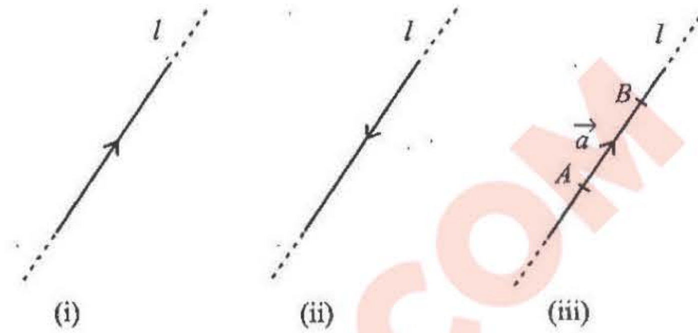


Fig. 1.5

Definition

A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector [Fig. 1.5(iii)], denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

Point A from where vector $\overrightarrow{AB}$ starts is called its initial point, and point B where it ends is called its terminal point. The distance between initial and terminal points of a vector is called the magnitude (or length) of the vector, denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a. The arrow indicates the direction of the vector.

Position Vector

Consider a point P in space having coordinates (x, y, z) with respect to the origin O (0, 0, 0). Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the position vector of the point P with respect to O. Using the distance formula, the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

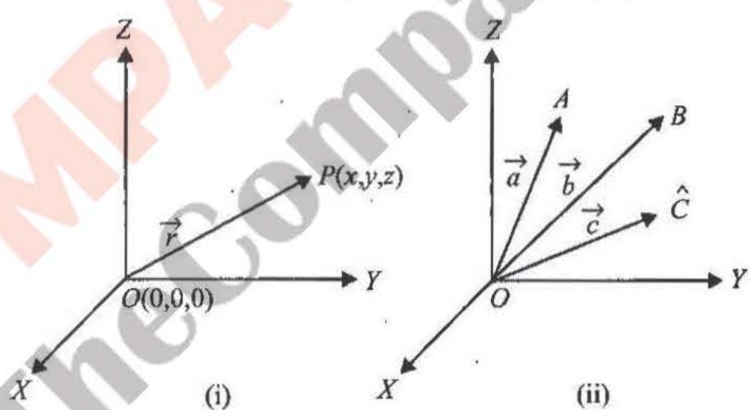


Fig. 1.6

In practice, the position vectors of points A, B, C, etc., with respect to origin O are denoted by $\vec{a}, \vec{b}, \vec{c}$, etc., respectively [Fig. 1.6(ii)].

Direction Cosines

Consider the position vector $\overrightarrow{OP}$ (or $\vec{r}$) of a point P (x, y, z). The angles α, β and γ made by the vector $\vec{r}$ with the positive directions of x-, y- and z-axes, respectively, are called its direction angles. The cosine values of these angles, i.e., $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, are called direction cosines of the vector $\vec{r}$ and are usually denoted by l, m and n, respectively.

From Fig. 1.7, one may note that triangle OAP is right angled, and in it, we have $\cos\alpha = x/r$ (r stands for $|\vec{r}|$). Similarly, from the right-angled triangles OBP and OCP, we may write $\cos\beta = y/r$ and $\cos\gamma = z/r$. Thus, the coordinates of point P may also be expressed as (lr, mr, nr). The numbers lr, mr and nr, proportional to the direction cosines, are called the direction ratios of vector $\vec{r}$ and are denoted by a, b and c, respectively (see this topic in detail in Chapter 3).

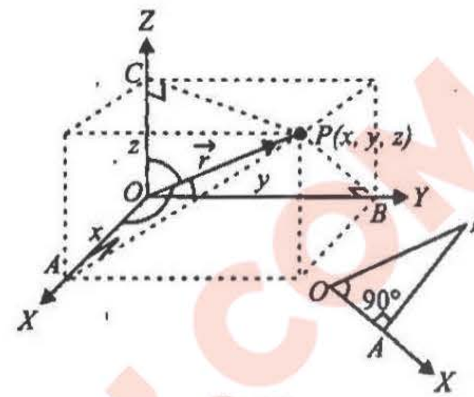


Fig. 1.7

TYPES OF VECTORS

Zero Vector

A vector whose initial and terminal points coincide is called a zero vector (or null vector) and is denoted as $\vec{0}$. A zero vector cannot be assigned a definite direction as it has zero magnitude or, alternatively, it may be regarded as having any direction. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

Unit Vector

A vector of unit magnitude is called a unit vector. Unit vectors are denoted by small letters with a cap on them.

Thus, $\hat{a}$ is unit vector of $\vec{a}$, where $|\hat{a}| = 1$, i.e., if vector $\vec{a}$ is divided by magnitude $|\vec{a}|$, then we get a unit vector in the direction of $\vec{a}$. Thus, $\frac{\vec{a}}{|\vec{a}|} = \hat{a} \Leftrightarrow \vec{a} = |\vec{a}|\hat{a}$, where $\hat{a}$ is the unit vector in the direction of $\vec{a}$.

Coinitial Vectors

Two or more vectors having the same initial point are called coinitial vectors.

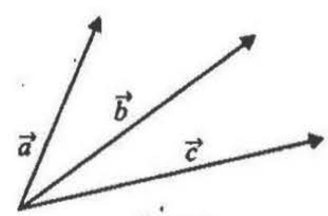


Fig. 1.8

Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction regardless of the positions of their initial points. They are written as $\vec{a} = \vec{b}$.

Negative of a Vector

A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but whose direction is opposite to that of it, is called negative of the given vector. For example, vector $\overrightarrow{BA}$ is negative of vector $\overrightarrow{AB}$ and is written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Free Vectors

Vectors whose initial points are not specified are called free vectors.

Localised Vectors

A vector drawn parallel to a given vector, but through a specified point as the initial point, is called a localised vector.

Parallel Vectors

Two or more vectors are said to be parallel if they have the same support or parallel support.

Parallel vectors may have equal or unequal magnitudes and their directions may be same or opposite as shown in Fig. 1.9.

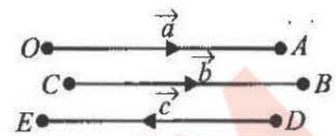
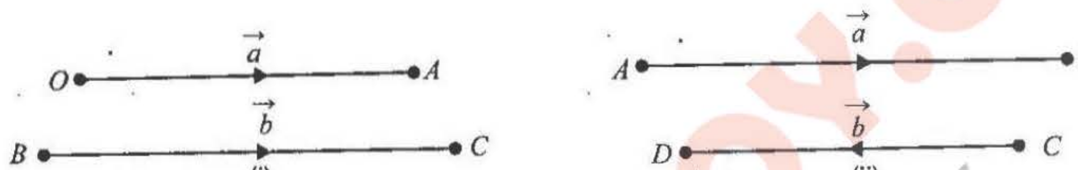


Fig. 1.9

Like and Unlike Vectors

Two parallel vectors having the same direction are called *like vectors* [see Fig. 1.10(i)].

Two parallel vectors having opposite directions are called *unlike vectors* [see Fig. 1.10(ii)].



(i) (ii)

Fig. 1.10

Collinear Vectors

Vectors $\vec{a}$ and $\vec{b}$ are collinear if they have same direction or are parallel or anti-parallel. Since their magnitudes are different, we can find some scalar λ for which $\vec{a} = \lambda\vec{b}$. If $\lambda > 0$, $\vec{a}$ and $\vec{b}$ are in the same direction; if $\lambda < 0$, $\vec{a}$ and $\vec{b}$ are in the opposite directions. Collinear vectors are often called dependent vectors.

Non-collinear Vectors

Two vectors acting in different directions are called non-collinear vectors. Non-collinear vectors are often called independent vectors. Here we cannot write vector $\vec{a}$ in terms of $\vec{b}$, though they have the same magnitude. However, we can find component of one vector in the direction of the other. Two non-collinear vectors describe plane.

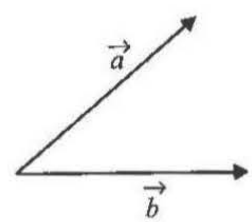


Fig. 1.11

Coplanar Vectors

Two parallel vectors or non-collinear vectors are always coplanar or two vectors $\vec{a}$ and $\vec{b}$ in different directions determine unique plane in space. Now if vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. Generally more than two vectors are coplanar if all are in the same plane.

Three non-coplanar vectors describe space.

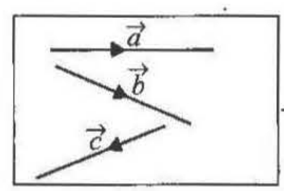


Fig. 1.12

ADDITION OF VECTORS

A vector $\overrightarrow{AB}$ simply means the displacement from point A to point B. Now consider a situation where a boy moves from A to B and then from B to C. The net displacement made by the boy from point A to point C is given by vector $\overrightarrow{AC}$ and is expressed as

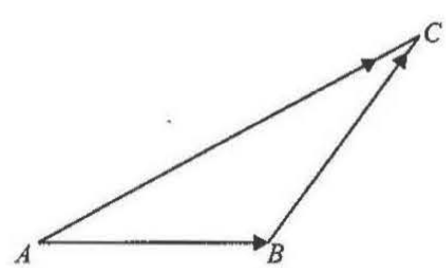


Fig. 1.13

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

This is known as the triangle law of vector addition.

In general, if we have two vectors $\vec{a}$ and $\vec{b}$ [Fig. 1.14(i)], then to add them, they are positioned such that the initial point of one coincides with the terminal point of the other [Fig. 1.14(ii)].

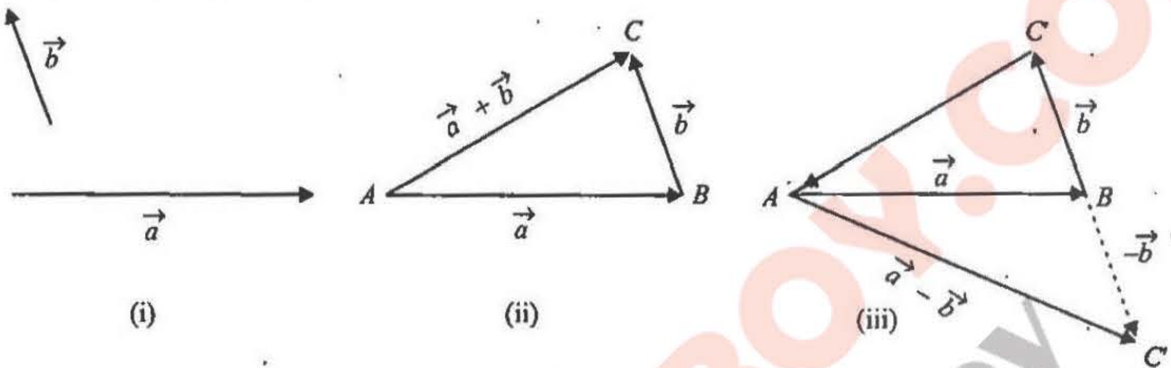


Fig. 1.14

For example, in Fig. 1.14(ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction so that its initial point coincides with the terminal point of $\vec{a}$. Then the vector $\vec{a} + \vec{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$, i.e., in triangle ABC [Fig. 1.14 (ii)], we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Since $\overrightarrow{AC} = -\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided [Fig. 1.14 (iii)].

Now, construct a vector $\overrightarrow{BC}$ so that its magnitude is same as that of vector $\overrightarrow{BC}$, but the direction is opposite to that of $\overrightarrow{BC}$ [Fig. 1.14 (iii)], i.e.,

$$\overrightarrow{BC} = -\overrightarrow{BC'}$$

Then, on applying triangle law from Fig. 1.14(iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

Vector $\overrightarrow{AC'}$ is said to represent the difference of $\vec{a}$ and $\vec{b}$.

Now, consider a boat going from one bank of a river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors—one is the velocity imparted to the boat by its engine and the other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat actually starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

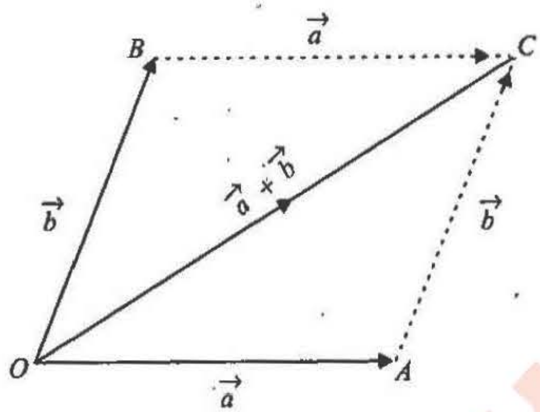


Fig. 1.15

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig. 1.15), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the parallelogram law of vector addition.

Notes:

1. From figure, using the triangle law, one may note that

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

or
$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \qquad (\because \overrightarrow{AC} = \overrightarrow{OB})$$

which is the parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

2. If $\overrightarrow{OA}$ and $\overrightarrow{AC}$ are collinear, their sum is still $\overrightarrow{OC}$. Although in this case we do not have a triangle or a parallelogram in their usual sense.

Fig. 1.16

3. As from the figure:

$\overrightarrow{OA} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n}$ by the polygon law of addition.

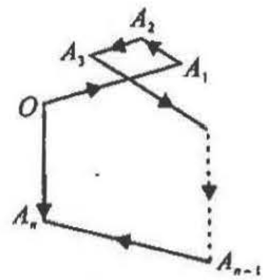


Fig. 1.17

Properties of Vector Addition

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative property)

2. $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associative property)

3. $\vec{a}+\vec{0}=\vec{a}$ (additive identity)

4. $\vec{a}+(-\vec{a})=\vec{0}$ (additive inverse)

5. $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$ and $|\vec{a}-\vec{b}|\ge|\vec{a}|-|\vec{b}|$

Illustration 1.1 If vector $\vec{a}+\vec{b}$ bisects the angle between $\vec{a}$ and $\vec{b}$, then prove that $|\vec{a}|=|\vec{b}|$.

Sol. We know that vector $\vec{a}+\vec{b}$ is along the diagonal of the parallelogram whose adjacent sides are vectors $\vec{a}$ and $\vec{b}$. Now if $\vec{a}+\vec{b}$ bisects the angle between vectors $\vec{a}$ and $\vec{b}$, then the parallelogram must be a rhombus; hence, $|\vec{a}|=|\vec{b}|$.

Illustration 1.2 If $\overline{AO}+\overline{OB}=\overline{BO}+\overline{OC}$, then prove that B is the midpoint of AC.

Sol. $\overline{AO}+\overline{OB}=\overline{BO}+\overline{OC}$

$\Rightarrow$ $\overline{AB}=\overline{BC}$

Thus, vectors $\overline{AB}$ and $\overline{BC}$ are collinear

$\Rightarrow$ Points A, B, C are collinear

Also $|\overline{AB}|=|\overline{BC}|$

$\Rightarrow$ B is the midpoint of AC

Illustration 1.3 $ABCDE$ is a pentagon. Prove that the resultant of forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

Sol. $\vec{R}=\overrightarrow{AB}+\overrightarrow{AE}+\overrightarrow{BC}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}$

$$=(\overrightarrow{AB}+\overrightarrow{BC})+(\overrightarrow{AE}+\overrightarrow{ED}+\overrightarrow{DC})+\overrightarrow{AC}$$

$$=\overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC}=3\overrightarrow{AC}$$

Illustration 1.4 Prove that the resultant of two forces acting at point O and represented by $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is given by $2\overrightarrow{OD}$, where D is the midpoint of BC.

Sol. $\vec{R}=\overrightarrow{OB}+\overrightarrow{OC}$

$$=(\overrightarrow{OD}+\overrightarrow{DB})+(\overrightarrow{OD}+\overrightarrow{DC})$$

$$=2\overrightarrow{OD}+(\overrightarrow{DB}+\overrightarrow{DC})=2\overrightarrow{OD}+\vec{0}=2\overrightarrow{OD}$$

(Since D is the midpoint of BC, we have $\overrightarrow{DB}=-\overrightarrow{DC}$)

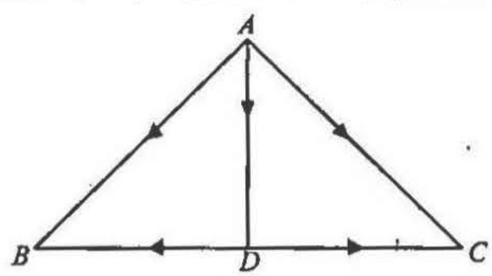


Fig. 1.18

Illustration 1.5 Prove that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

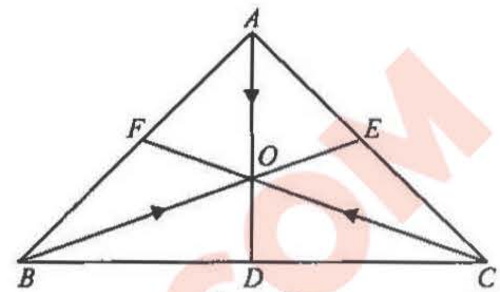


Fig. 1.19

Sol. $\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD}$

$$\overrightarrow{BC} + \overrightarrow{BA} = 2\,\overrightarrow{BE}$$

$$\overrightarrow{CA} + \overrightarrow{CB} = 2\,\overrightarrow{CF}$$

Adding, we get

$$(\overrightarrow{AB} + \overrightarrow{BA}) + (\overrightarrow{AC} + \overrightarrow{CA}) + (\overrightarrow{BC} + \overrightarrow{CB}) = 2\,(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF})$$

or $\quad \vec{0} + \vec{0} + \vec{0} = 2\,(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF})$

or $\quad \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$

Illustration 1.6 ABC is a triangle and P any point on BC. If $\overrightarrow{PQ}$ is the sum of $\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PC}$, show that $ABQC$ is a parallelogram and Q, therefore, is a fixed point.

Sol. Here $\quad \overrightarrow{PQ} = \overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PC}$

$$\overrightarrow{PQ} - \overrightarrow{PC} = A\,\overrightarrow{AP} + \overrightarrow{PB}$$

$$\overrightarrow{PQ} + \overrightarrow{CP} = A\,\overrightarrow{AP} + \overrightarrow{PB}$$

$$\overrightarrow{CQ} = \overrightarrow{AB} \Rightarrow CQ = AB \text{ and } CQ \parallel AB$$

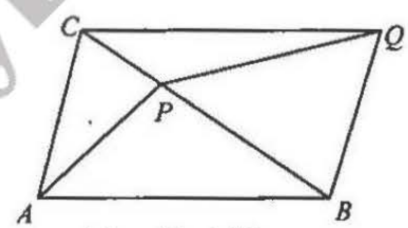


Fig. 1.20

Therefore, $ABQC$ is a parallelogram.
But A, B and C are given to be fixed points and $ABQC$ is a parallelogram. Therefore, Q is a fixed point.

Illustration 1.7 Two forces $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are acting at the vertex A of a quadrilateral $ABCD$ and two forces $\overrightarrow{CB}$ and $\overrightarrow{CD}$ at C. Prove that their resultant is given by $4\,\overrightarrow{EF}$, where E and F are the midpoints of AC and BD, respectively.

Sol. $\overrightarrow{AB} + \overrightarrow{AD} = 2\,\overrightarrow{AF}$, where F is the midpoint of BD.

$$\overrightarrow{CB} + \overrightarrow{CD} = 2\,\overrightarrow{CF}$$

$\therefore \quad \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 2(\overrightarrow{AF} + \overrightarrow{CF})$

$$= -\,2(\overrightarrow{FA} + \overrightarrow{FC})$$

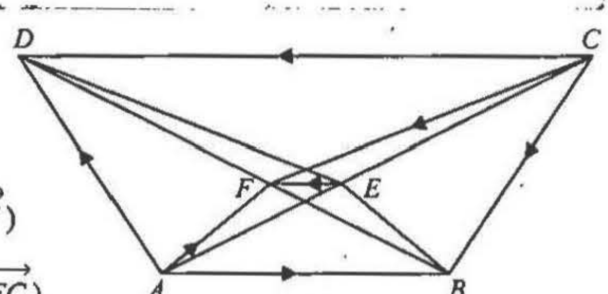


Fig. 1.21

$= -\,2[2\,\overrightarrow{FE}]$, where E is the midpoint of AC

$$= 4\,\overrightarrow{EF}$$

Illustration 1.8 If $O(\vec{0})$) is the circumcentre and O' the orthocentre of a triangle ABC, then prove that

i. $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OO'}$

ii. $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{O'O}$

iii. $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\,\overrightarrow{AO} = \overrightarrow{AP}$

where AP is the diameter through A of the circumcircle.

Sol. O is the circumcentre, which is the intersection of the right bisectors of the sides of the triangle, and O' is the orthocenter, which is the point of intersection of altitudes drawn from the vertices. Also, from geometry, we know that

$$2OD = AO'.$$

$$\therefore \quad 2\,\overrightarrow{OD} = \overrightarrow{AO'} \qquad \text{(i)}$$

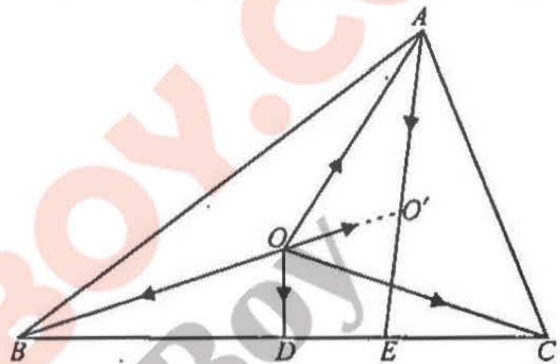


Fig. 1.22

i. To prove: $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OO'}$

Now $\overrightarrow{OB} + \overrightarrow{OC} = 2\,\overrightarrow{OD} = \overrightarrow{AO'}$

$\Rightarrow \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AO'} = \overrightarrow{OO'}$ [by (i)]

ii. To prove: $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{OO'}$

L.H.S. $= 2\,\overrightarrow{DO} + 2\,\overrightarrow{O'D}$ [by (i)]

$= 2(\overrightarrow{O'D} + \overrightarrow{DO}) = 2\,\overrightarrow{O'O}$

iii. To prove: $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\,\overrightarrow{AO} = \overrightarrow{AP}$

L.H.S. $= 2\,\overrightarrow{AO'} - \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C}$

$= 2\,\overrightarrow{AO'} + (\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C}$

$= 2\,\overrightarrow{AO'} + 2\overrightarrow{O'O} = 2\,\overrightarrow{AO}$

$= \overrightarrow{AP}$ (where AP is the diameter through A of the circumcircle).

COMPONENTS OF A VECTOR

Let us take the points $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$ on the x-axis, y-axis and z-axis, respectively.

Then, clearly $|\overrightarrow{OA}| = 1, |\overrightarrow{OB}| = 1$ and $|\overrightarrow{OB}| = 1$

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called unit vectors along the axes OX, OY and OZ, respectively, and are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.

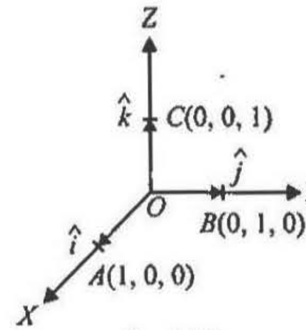


Fig. 1.23

Now, consider the position vector $\overrightarrow{OP}$ of a point $P(x, y, z)$ as shown in Fig. 1.24. Let P_1 be the foot of the perpendicular from P on the plane XOY.

We, thus, see that P_1P is parallel to the z-axis. As $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the x-, y- and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{OP_1} = x\hat{i} + y\hat{j}$ and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

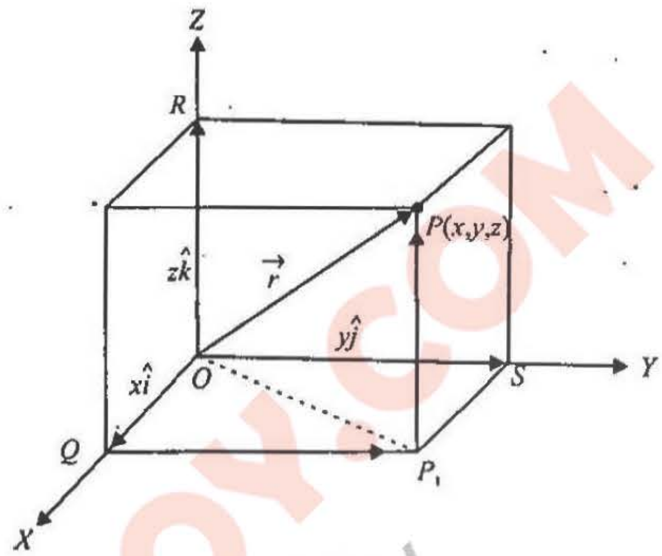


Fig. 1.24

Hence, the position vector of P with reference to O is given by

$$|\overrightarrow{OP}| (\text{or } \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its component form. Here, x, y and z are called the scalar components of $\vec{r}$, and $x\hat{i}, y\hat{j}$ and $z\hat{k}$ are called the vector components of $\vec{r}$ along the respective axes. Sometimes x, y and z are also called rectangular components.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is readily determined by applying the Pythagoras theorem twice. We note that in the right-angled triangle OQP_1,

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2}$$

And in the right-angled triangle OP_1P, we have

$$|\overrightarrow{OP}| = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Notes:

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then

i. The sum (or resultant) of vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

ii. The difference between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_2)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

iii. Vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

$$\vec{b} = \lambda \vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ and $\vec{b}$ be any two vectors, and k and m be any scalars. Then

i. $k\vec{a} + m\vec{a} = (k+m)\vec{a}$

ii. $k(m\vec{a}) = (km)\vec{a}$

iii. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

Remarks

i. One may observe that whatever be the value of λ, vector $\lambda\vec{a}$ is always collinear to vector $\vec{a}$. In fact, two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a non-zero scalar λ such that $\vec{b} = \lambda\vec{a}$. If the vectors $\vec{a}$ and $\vec{b}$ are given in the component form, i.e., $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\Leftrightarrow \quad b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Leftrightarrow \quad b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ b_3 = \lambda a_3$$

$$\Leftrightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

ii. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2, a_3 are also called direction ratios of $\vec{a}$.

iii. In case it is given that l, m, n are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector where α, β and γ are the angles which the vector makes with the x-, y- and z-axes, respectively.

Illustration 1.9 A unit vector of modulus 2 is equally inclined to x- and y-axes at an angle $\pi/3$. Find the length of projection of the vector on the z-axis.

Sol. Given that the vector is inclined at an angle $\pi/3$ with both x- and y-axes. Then

$$\cos\alpha = \cos\beta = \frac{1}{2}$$

Also we know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Therefore,

$$\cos^2\gamma = \frac{1}{2}$$

or $\cos\gamma = \pm\frac{1}{\sqrt{2}}$

Thus, the given vector is

$$2(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}) = 2\left(\frac{\hat{i}}{2} + \frac{\hat{j}}{2} \pm \frac{\hat{k}}{\sqrt{2}}\right) = \hat{i} + \hat{j} \pm \sqrt{2}\hat{k}$$

Hence, the length of projection of vector on the z-axis is $\sqrt{2}$ units.

Illustration 1.10 If the projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively, find the angle at which vector $\vec{a}$ is inclined to the z-axis.

Sol. Since projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively, we have

$$\text{Vector } \vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

Then $\cos\gamma = \frac{2}{3}$ (where γ is the angle of vector $\vec{a}$ with the z-axis), i.e.,

$$\gamma = \cos^{-1}\frac{2}{3}$$

MULTIPLICATION OF A VECTOR BY A SCALAR

Let $\vec{a}$ be a vector and λ a scalar. Then the product of vector $\vec{a}$ by scalar λ, denoted by $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that $\lambda\vec{a}$ is also a vector, collinear to vector $\vec{a}$. Vector $\lambda\vec{a}$ has the direction same (or opposite) as that of vector $\vec{a}$ if the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualization of multiplication of a vector by a scalar is given in the following figure.

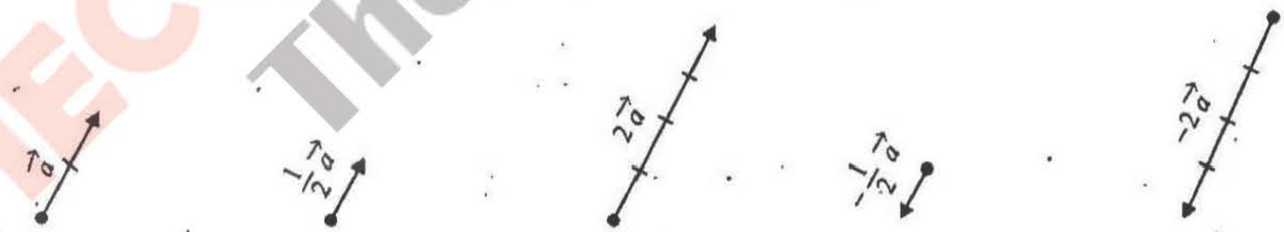


Fig. 1.25

When $\lambda = -1$, $\lambda\vec{a} = -\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$.

Vector $-\vec{a}$ is called the negative (or additive inverse) of vector $\vec{a}$ and we always have $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

Also, if $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$, i.e., $\vec{a}$ is not a null vector, then

$$|\lambda \vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

Illustration 1.11 Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units. **(NCERT)**

Sol. Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$

$\therefore$ $|\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25+1+4} = \sqrt{30}$

$\therefore$ $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}$

Hence, the vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units is given by

$$8\hat{a} = 8\left(\frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$

VECTOR JOINING TWO POINTS

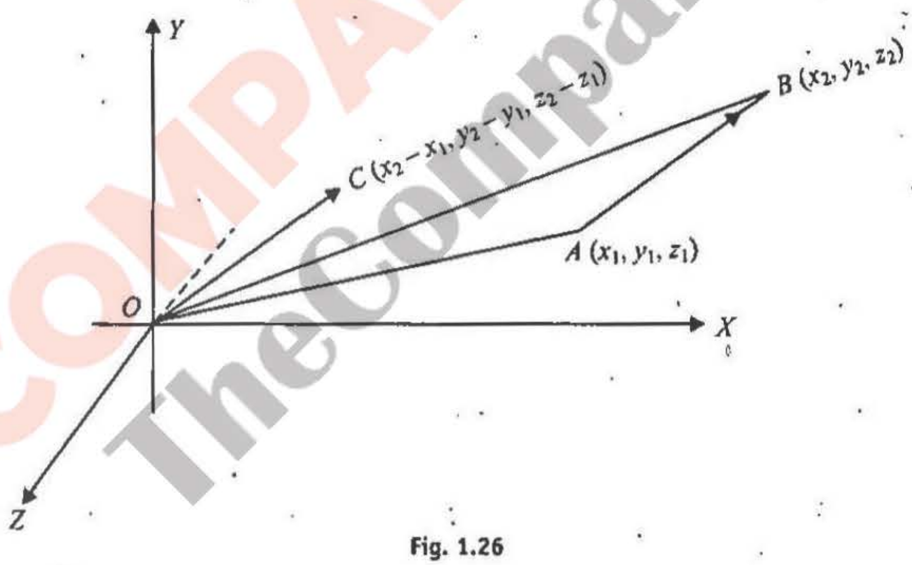


Fig. 1.26

In Fig. 1.26, vector $\vec{AB}$ is shifted without rotation and placed at origin.

Now vector $\vec{AB} = \vec{OC}$.

Since $|\vec{AB}| = |\vec{OC}|$, coordinates of point C are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Hence, Vector $\vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Thus,
$$\vec{AB} = \vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$
$$= \vec{OB} - \vec{OA}$$
$$= \text{Position vector of } B - \text{Position vector of } A$$

Also from above, we have $\vec{OB} = \vec{OA} + \vec{AB}$ which describes triangle rule of vector addition.

Further $\vec{OB} = \vec{OA} + \vec{AB} = \vec{OA} + \vec{OC}$ ($\because \vec{OC} = \vec{AB}$), which describes parallelogram rule of vector addition.

Illustration 1.12 Find the unit vector in the direction of vector $\vec{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively. **(NCERT)**

Sol. The given points are P (1, 2, 3) and Q (4, 5, 6). Therefore,

$$\vec{PQ} = (4-1)\,\hat{i} + (5-2)\,\hat{j} + (6-3)\,\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$
$$|\vec{PQ}| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$

Hence, the unit vector in the direction of $\vec{PQ}$ is

$$\frac{\vec{PQ}}{|\vec{PQ}|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Illustration 1.13 For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$. **(NCERT)**

Sol. The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$. Therefore,

$$\vec{a} + \vec{b} = (2-1)\,\hat{i} + (-1+1)\,\hat{j} + (2-1)\,\hat{k} = \hat{i} + \hat{k}$$
$$\therefore \quad |\vec{a} + \vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence, the unit vector in the direction of $(\vec{a} + \vec{b})$ is

$$\frac{(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

Illustration 1.14 Show that the points A, B and C with position vectors $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right-angled triangle. **(NCERT)**

Sol. Position vectors of points A, B and C are, respectively given as

$$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$$

$$\therefore \quad \vec{AB} = \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\vec{BC} = \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\vec{CA} = \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\therefore \quad |\vec{AB}|^2 = (-1)^2 + 3^2 + 5^2 = 35$$

$$|\vec{BC}|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 41$$

$$|\vec{CA}|^2 = 2^2 + (-1)^2 + 1^2 = 6$$

$$\therefore \quad |\vec{AB}|^2 + |\vec{CA}|^2 = 36 + 6 = 41 = |\vec{BC}|^2$$

Hence, ABC is a right-angled triangle.

Illustration 1.15 If $2\overrightarrow{AC} = 3\overrightarrow{CB}$, then prove that $2\overrightarrow{OA} + 3\overrightarrow{OB} = 5\overrightarrow{OC}$ where O is the origin.

Sol. $2\overline{AC} = 3\overline{CB}$ or $2(\overline{OC} - \overline{OA}) = 3(\overline{OB} - \overline{OC})$

or $\quad 2\overline{OA} + 3\overline{OB} = 5\overline{OC}$

Illustration 1.16 Prove that points $\hat{i} + 2\hat{j} - 3\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$ and $2\hat{i} + 5\hat{j} - \hat{k}$ form a triangle in space.

Sol. Given points are $A(\hat{i} + 2\hat{j} - 3\hat{k}), B(2\hat{i} - \hat{j} + \hat{k}), C(2\hat{i} + 5\hat{j} - \hat{k})$

Vectors $\vec{AB} = \hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{AC} = \hat{i} + 3\hat{j} + 2\hat{k}\, k$

Clearly vectors $\vec{AB}$ and $\vec{AC}$ are non-collinear as there does not exist any real λ for which $\vec{AB} = \lambda\vec{AC}$.

Hence, vectors $\vec{AB}$ and $\vec{AC}$ or the given three points form a triangle.

SECTION FORMULA

Internal Division

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively and C be a point dividing AB internally in the ratio $m : n$. Then the position vector of C is given by $\overrightarrow{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.

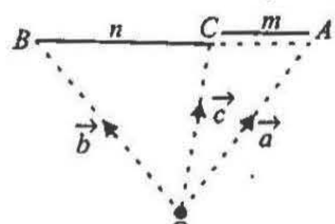


Fig. 1.27

Proof:

Let O be the origin. Then $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of C which divides AB internally in the ratio $m : n$. Then

$$\frac{AC}{CB} = \frac{m}{n}$$

or $\quad n\,\overrightarrow{AC} = m\,\overrightarrow{CB}$

or $\quad n(\text{P.V. of } C - \text{P.V. of } A) = m(\text{P.V. of } B - \text{P.V. of } C)$

or $\quad n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$

or $\quad n\vec{c} - n\vec{a} = m\vec{b} - m\vec{c}$

or $\quad \vec{c}(n+m) = m\vec{b} + n\vec{a}$

or $\quad \vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$ or $\overrightarrow{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

External Division

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and C be a point dividing $\overrightarrow{AB}$ externally in the ratio $m : n$. Then the position vector of $\vec{C}$ is given by $\overrightarrow{OC} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$.

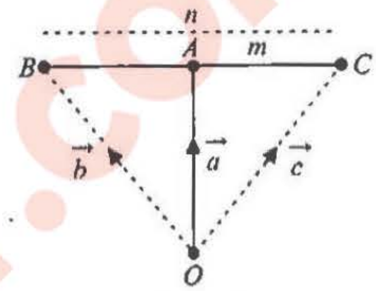


Fig. 1.28

Proof:

Let O be the origin. Then $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of point C dividing AB externally in the ratio $m : n$. Then

Then, $\quad \dfrac{AC}{BC} = \dfrac{m}{n}$

or $\quad nAC = mBC$

or $\quad n\overrightarrow{AC} = m\overrightarrow{BC}$

or $\quad n$(P.V. of C – P.V. of A) = m(P.V. of C– P.V. of B)

or $\quad n(\vec{c} - \vec{a}) = m(\vec{c} - \vec{b})$

or $\quad n\vec{c} - n\vec{a} = m\vec{c} - m\vec{b}$

or $\quad \vec{c}(m-n) = m\vec{b} - n\vec{a}$

or $\quad \vec{c} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$ or $\overrightarrow{OC} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

Notes:

1. If C is the midpoint of AB, then it divides AB in the ratio 1 : 1.

   Therefore, the P.V. of C is $\dfrac{1\cdot\vec{a} + 1\cdot\vec{b}}{1+1} = \dfrac{\vec{a} + \vec{b}}{2}$. Thus, the position vector of the midpoint of AB is $\dfrac{1}{2}(\vec{a} + \vec{b})$.

2. We have $\vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n} = \dfrac{m}{m+n}\vec{b} + \dfrac{n}{m+n}\vec{a}$. Therefore,

   $$\vec{c} = \lambda\vec{a} + \mu\vec{b}, \text{ where } \lambda = \frac{n}{m+n} \text{ and } \mu = \frac{m}{m+n}$$

   Thus, position vector of any point C on $\overrightarrow{AB}$ can always be taken as $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, where $\lambda + m = 1$.

3. We have $\vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$. Therefore,

$$(m+n)\,\vec{c} = m\vec{b} + n\vec{a}$$

$n\,\overrightarrow{OA} + m\,\overrightarrow{OB} = (m+n)\,\overrightarrow{OC}$, where $\vec{C}$ is a point on $\overrightarrow{AB}$ dividing it in the ratio $m : n$.

In ΔABC, having vertices $A(\vec{a}), B(\vec{b})$ and $C(\vec{a})$,

Centroid is $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

Incentre is $\dfrac{BC\,\vec{a} + AC\,\vec{b} + AB\,\vec{c}}{AB + AC + AB}$

Orthocentre is $\dfrac{\tan A\,\vec{a} + \tan B\,\vec{b} + \tan C\,\vec{c}}{\tan A + \tan B + \tan C}$

Circumcentre is $\dfrac{\sin 2A\,\vec{a} + \sin 2B\,\vec{b} + \sin 2C\,\vec{c}}{\sin 2A + \sin 2B + \sin 2C}$

Illustration 1.17 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$, respectively, in the ratio 2 : 1.

i. Internally **ii.** Externally **(NCERT)**

Sol. The position vector of point R dividing the line segment joining two points P and Q in the ratio $m : n$ is given by

i. Internally: $\dfrac{m\vec{b} + n\vec{a}}{m+n}$ **ii.** Externally: $\dfrac{m\vec{b} - n\vec{a}}{m-n}$

Position vectors of P and Q are given as

$$\overrightarrow{OP} = \hat{i} + 2\hat{j} - \hat{k} \text{ and } \overrightarrow{OQ} = -\hat{i} + \hat{j} + \hat{k}$$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2 : 1 is given by

$$\overrightarrow{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1} = \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})}{3}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2 : 1 is given by

$$\overrightarrow{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2-1} = (-2\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})$$

$$= -3\hat{i} + 3\hat{k}$$

Illustration 1.18 If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the position vectors of points A, B, C and D, respectively referred to the same origin O such that no three of these points are collinear and $\vec{a} + \vec{c} = \vec{b} + \vec{d}$, then prove that quadrilateral $ABCD$ is a parallelogram.

Sol. Since $\vec{a} + \vec{c} = \vec{b} + \vec{d}$, we have

$$\frac{\vec{a} + \vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2}$$

i.e., Midpoint of AC and BD coincide.

Hence, quadrilateral $ABCD$ is a parallelogram.

Illustration 1.19 Find the point of intersection of AB and CD, where $A(6, -7, 0)$, $B(16, -19, -4)$, $C(0, 3, -6)$ and $D(2, -5, 10)$.

Sol. Let AB and CD intersect at P.

Let P divides AB in ratio $\lambda:1$ and CD in ratio $\mu:1$.

Then coordinates of P are $\left(\frac{16\lambda + 6}{\lambda + 1}, \frac{-19\lambda - 7}{\lambda + 1}, \frac{-4\lambda}{\lambda + 1}\right)$ or $\left(\frac{2\mu}{\mu + 1}, \frac{-5\mu + 3}{\mu + 1}, \frac{10\mu - 6}{\mu + 1}\right)$

Comparing we have $\lambda = -\frac{1}{3}$ or $\mu = 1$.

Using these values, we get point of intersection as $(1, -1, 2)$.

Here it is also proved that lines AB and CD intersect or points A, B, C and D are coplanar.

Illustration 1.20 Find the angle of vector $\vec{a} = 6\hat{i} + 2\hat{j} - 3\hat{k}$ with x-axis.

Sol. $\vec{a} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

or $|\vec{a}| = \sqrt{(6)^2 + (2)^2 + (-3)^2} = 7$

Hence, the angle of vector with the x-axis is $\cos^{-1}\frac{6}{7}$.

Illustration 1.21

i. Show that the lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.

ii. Show that the joins of the midpoints of the opposite edges of a tetrahedron intersect and bisect each other.

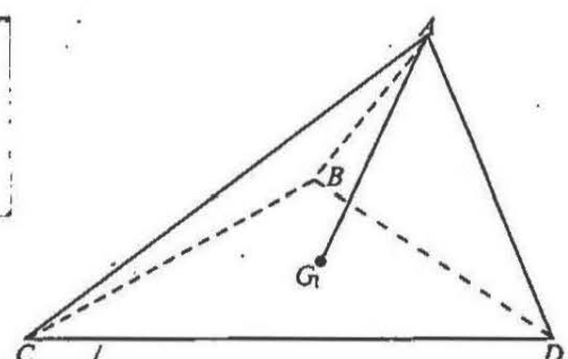


Fig. 1.29

Sol.

i. G_1, the centroid of ΔBCD, is $\frac{\vec{b} + \vec{c} + \vec{d}}{3}$ and A is $\vec{a}$.

The position vector of point G which divides AG_1 in the ratio $3 : 1$ is

$$\frac{3 \cdot \frac{\vec{b} + \vec{c} + \vec{d}}{3} + 1 \cdot \vec{a}}{3 + 1} = \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4}$$

The symmetry of the result shows that this point will also lie on BG_2, CG_3 and DG_4 (where G_2, G_3, G_4 are centroids of faces ACD, ABD and ABC, respectively). Hence, these four lines are concurrent at point $\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$, which is called the centroid of the tetrahedron.

ii. The midpoint of DA is $\dfrac{\vec{a}+\vec{d}}{2}$ and that of BC is $\dfrac{\vec{b}+\vec{c}}{2}$ and the midpoint of these midpoints is $\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$ and symmetry of the result proves the fact.

Illustration 1.22 The midpoints of two opposite sides of a quadrilateral and the midpoints of the diagonals are the vertices of a parallelogram. Prove this using vectors.

Sol. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of vertices A, B, C and D, respectively.

Let E, F, G and H be the midpoints of AB, CD, AC and BD, respectively.

$$\text{P.V. of } E = \frac{\vec{a}+\vec{b}}{2}$$

$$\text{P.V. of } F = \frac{\vec{c}+\vec{d}}{2}$$

$$\text{P.V. of } G = \frac{\vec{a}+\vec{c}}{2}$$

$$\text{P.V. of } H = \frac{\vec{b}+\vec{d}}{2}$$

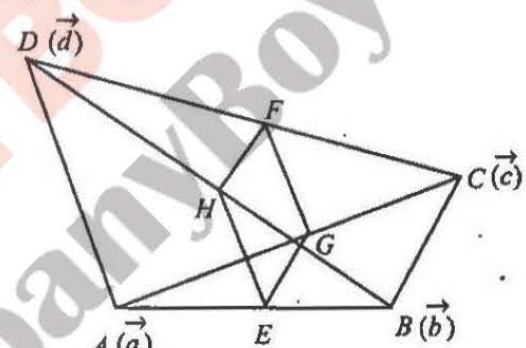


Fig. 1.30

$$\overrightarrow{EG} = \text{P.V. of G} - \text{P.V. of } E = \frac{\vec{a}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}}{2} = \frac{\vec{c}-\vec{b}}{2}.$$

$$\overrightarrow{HF} = \text{P.V. of F} - \text{P.V. of } H = \frac{\vec{c}+\vec{d}}{2} - \frac{\vec{b}+\vec{d}}{2} = \frac{\vec{c}-\vec{b}}{2}$$

$\therefore$ $\overrightarrow{EG} = \overrightarrow{HF} \Rightarrow EG \parallel HF$ and $EG = HF$

Hence, $EGHF$ is a parallelogram.

SOME MORE SOLVED EXAMPLES

Illustration 1.23 Check whether the three vectors $2\hat{i}+2\hat{j}+3\hat{k}, -3\hat{i}+3\hat{j}+2\hat{k}$ and $3\hat{i}+4\hat{k}$ form a triangle or not.

Sol. If vectors $\vec{a} = 2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b} = -3\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{c} = 3\hat{i}+4\hat{k}$ form a triangle, then we must have

$$\vec{a}+\vec{b}+\vec{c} = 0.$$

But for given vectors, $\vec{a}+\vec{b}+\vec{c} \neq 0$. Hence, these vectors do not form a triangle.

Illustration 1.24 Find the resultant of vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 4\hat{k}$. Find the unit vector in the direction of the resultant vector.

Sol. Let the resultant vector of $\vec{a}$ and $\vec{b}$ is $\vec{a} + \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k} = \vec{c}$

Now unit vector in the direction of $\vec{c}$ is $\hat{c} = \dfrac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{(2)^2 + (1)^2 + (-2)^2}} = \dfrac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$

Illustration 1.25 If in parallelogram $ABCD$, diagonal vectors are $\vec{AC} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{BD} = -6\hat{i} + 7\hat{j} - 2\hat{k}$, then find the adjacent side vectors $\vec{AB}$ and $\vec{AD}$.

Sol. Let $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$. Then

$$\vec{AC} = \vec{a} + \vec{b} \text{ and } \vec{BD} = \vec{b} - \vec{a}$$

$$\Rightarrow \quad \vec{b} = \frac{\overline{AC} + \overline{BD}}{2} \text{ and } \vec{a} = \frac{\overline{AC} - \overline{BD}}{2}$$

$$\Rightarrow \quad \vec{AB} = -2\hat{i} + 35 + \hat{k} \text{ and } \vec{AD} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

Illustration 1.26 If two sides of a triangle are $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{k}$, then find the length of the third side.

Sol. Given sides of the triangle are $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = \hat{i} + \hat{k}$.

If vector along the third side is $\vec{c}$, then we must have $\vec{a} + \vec{b} + \vec{c} = 0$. Then

$$\vec{c} = -(\hat{i} + 2\hat{j}) - (\hat{i} + \hat{k}) = -2\hat{i} - 2\hat{j} - \hat{k}$$

Therefore, the length of the third side $|\vec{c}|$ is $\sqrt{(-2)^2 + (-2)^2 + (-1)^2} = 3$.

Illustration 1.27 Three coinitial vectors of magnitudes a, $2a$ and $3a$ meet at a point and their directions are along the diagonals of three adjacent faces of a cube. Determine their resultant R. Also prove that the sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through the same corner, the vectors being directed from the corner, is twice the vector determined by the diagonal of the cube.

Sol. Let the length of an edge of the cube be taken as unity and the vectors represented by OA, OB and OC (let the three coterminous edges of unit be $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively). OR, OS and OT are the three diagonals of the three adjacent faces of the cube along which act the forces of magnitudes a, $2a$ and $3a$, respectively. To find the vectors representing these forces, we will first find unit vectors in these directions and then multiply them by the corresponding given magnitudes of these forces.

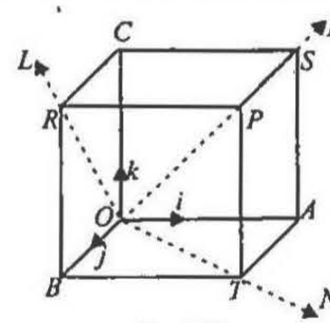


Fig. 1.31

Since $\overrightarrow{OR} = \hat{j} + \hat{k}$, the unit vector along OR is $\dfrac{1}{\sqrt{2}}(\hat{j} + \hat{k})$.

Hence, force $\vec{F_1}$ of magnitude a along OR is given by

$$\vec{F_1} = \frac{a}{\sqrt{2}}(\hat{i} + \hat{k})$$

Similarly, force $\vec{F_1}$ of magnitude $2a$ along OS is $\frac{2a}{\sqrt{2}}(\hat{k} + \hat{i})$ and force $\vec{F_3}$ of magnitude $3a$ along OT is $\frac{3a}{\sqrt{2}}(\hat{i} + \hat{j})$.

If $\vec{R}$ is their resultant, then

$$\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3}$$

$$= \frac{a}{\sqrt{2}}(\hat{j} + \hat{k}) + \frac{2a}{\sqrt{2}}(\hat{k} + \hat{i}) + \frac{3a}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$= \frac{5a}{\sqrt{2}}\hat{i} + \frac{4a}{\sqrt{2}}\hat{j} + \frac{3a}{\sqrt{2}}\hat{k}$$

Again, $\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = \hat{j} + \hat{k} + \hat{i} + \hat{k} + \hat{i} + \hat{j}$

$$= 2(\hat{i} + \hat{j} + \hat{k})$$

Also $\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{TP} = (\hat{i} + \hat{j} + \hat{k})$ $\quad (\because \overrightarrow{OT} = \hat{i} + \hat{j}$ and $\overrightarrow{TP} = \overrightarrow{OC} = \hat{k})$

$$\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = 2\,\overrightarrow{OP}$$

Illustration 1.28 The axes of coordinates are rotated about the z-axis through an angle of $\pi/4$ in the anticlockwise direction and the components of a vector are $2\sqrt{2}$, $3\sqrt{2}$, 4. Prove that the components of the same vector in the original system are -1, 5, 4.

Sol. If $\hat{i}, \hat{j}, \hat{k}$ are the new unit vectors along the coordinate axes, then

$$\vec{a} = 2\sqrt{2}\,\hat{i} + 2\sqrt{2}\,\hat{j} + 4\hat{k} \quad \text{(i)}$$

$\hat{i}, \hat{j}, \hat{k}$ are obtained by rotating by 45° about the z-axis.

Then $\hat{i}$ is replaced by $\hat{i}\cos 45° + \hat{j}\sin 45° = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$

and $\hat{j}$ is replaced by

$$-\hat{i}\cos 45° + \hat{j}\sin 45° = \frac{-\hat{i} + \hat{j}}{\sqrt{2}}$$

$$\hat{k} = \hat{k},$$

$$\vec{a} = 2\sqrt{2}\left[\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right] + 3\sqrt{2}\left[\frac{-\hat{i} + \hat{j}}{\sqrt{2}}\right] + 4\hat{k}$$

$$= (2-3)\,\hat{i} + (2+3)\,\hat{j} + 4\hat{k}$$

$$= -\hat{i} + 5\hat{j} + 4\hat{k}$$

Illustration 1.29 If the resultant of two forces is equal in magnitude to one of the components and perpendicular to it in direction, find the other component using the vector method.

Sol. Let P be horizontal in the direction of unit vector $\hat{i}$. The resultant is also P but perpendicular to it in the direction of unit vector $\hat{j}$. If Q is the other force making an angle θ (obtuse) as the resultant is perpendicular to P, then the two forces are $P\hat{i}$ and $Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j}$. Their resultant is $P\hat{j}$. Therefore,

$$P\hat{j} = P\hat{i} + (Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j})$$

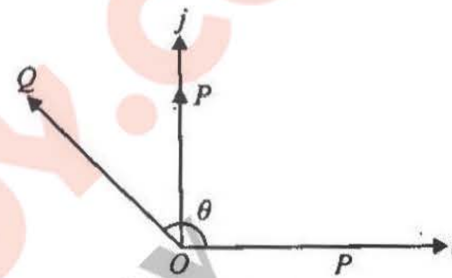


Fig. 1.32

Comparing the coefficients of $\hat{i}$ and $\hat{j}$, we get

$$P + Q\cos\theta = 0 \text{ and } Q\sin\theta = P$$

or $$Q\cos\theta = -P \text{ and } Q\sin\theta = P$$

Squaring and adding $Q = P\sqrt{2}$ and dividing give

$$\tan\theta = -1$$
$$\theta = 135°$$

Illustration 1.30 A man travelling towards east at 8 km/h finds that the wind seems to blow directly from the north. On doubling the speed, he finds that it appears to come from the north-east. Find the velocity of the wind.

Sol. Velocity of wind relative to man
= Actual velocity of wind – Actual velocity of man (i)

Let $\hat{i}$ and $\hat{j}$ represent unit vectors along east and north. Let the actual velocity of wind be given by $x\hat{i} + y\hat{j}$.

In the first case, the man's velocity is $8\hat{i}$ and that of the wind blowing from the north relative to the man is $-p\hat{j}$. Therefore,

$$-p\hat{j} = (x\hat{i} + y\hat{j}) - 8\hat{i} \quad \text{[from Eq. (i)]}$$

Comparing coefficients, $x - 8 = 0, y = -p$ (ii)

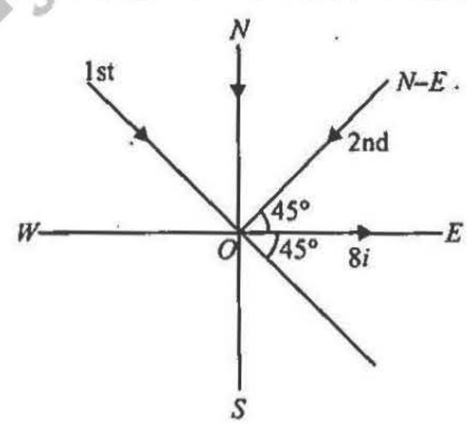


Fig. 1.33

In the second case, when the man doubles his speed, wind seems to come from the north-east direction, i.e.,

$$-q(\hat{i} + \hat{j}) = (x\hat{i} + y\hat{j}) - 16\hat{i}$$

$\therefore$ $$x - 16 = -q, y = -q \quad \text{(iii)}$$

Putting $x = 8$, we get $q = 8$

$$y = -8$$

Hence, the velocity of wind is $x\hat{i} + y\hat{j} = 8(\hat{i} - \hat{j})$

Its magnitude is $\sqrt{(8^2 + 8^2)} = 8\sqrt{2}$ and $\tan\theta = -1$ or $\theta = -45°$

Hence, its direction is from the north to the west.

Illustration 1.31 *OABCDE* is a regular hexagon of side 2 units in the *XY*-plane in the first quadrant. *O* being the origin and *OA* taken along the *x*-axis. A point *P* is taken on a line parallel to the *z*–axis through the centre of the hexagon at a distance of 3 units from *O* in the positive *Z* direction. Then find vector $\overline{AP}$.

Sol. $G \equiv (\hat{i} + \sqrt{3}\,\hat{j})$

Let position vector of *P* be $\bar{p}$

$\because \quad \overline{GP} \parallel \hat{k}$

Then $\quad \bar{p} - (\hat{i} + \sqrt{3}\,\hat{j}) = \lambda\hat{k}$

$\therefore \quad \bar{p} = \hat{i} + \sqrt{3}\,\hat{j} + \lambda\hat{k}$

Also $\quad \left|\overline{OP}\right| = 3$

$\Rightarrow \quad \sqrt{1+3+\lambda^2} = 3$

or $\quad \lambda^2 = 5$

or $\quad \lambda = \pm\sqrt{5}$

$\Rightarrow \quad \bar{p} = \hat{i} + \sqrt{3}\,\hat{j} \pm \sqrt{5}\,\hat{k}$

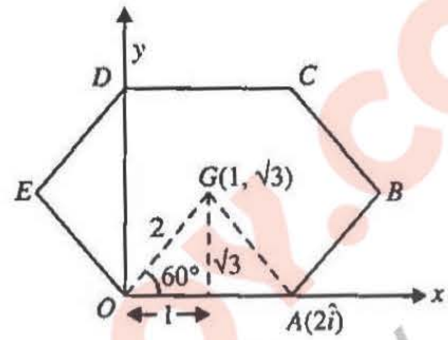


Fig. 1.34

For positive *z*-axis, $\bar{p} = \hat{i} + \sqrt{3}\,\hat{j} + \sqrt{5}\,\hat{k}$

So $\quad \overline{AP} = \bar{p} - 2\hat{i} = -\hat{i} + \sqrt{3}\,\hat{j} + \sqrt{5}\,\hat{k}$

VECTOR ALONG THE BISECTOR OF GIVEN TWO VECTORS

We know that the diagonal in a parallelogram is not necessarily the bisector of the angle formed by two adjacent sides. However, the diagonal in a rhombus bisects the angle between two adjacent sides.

Consider vectors $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$ forming a parallelogram *ABCD* as shown in Fig. 1.35.

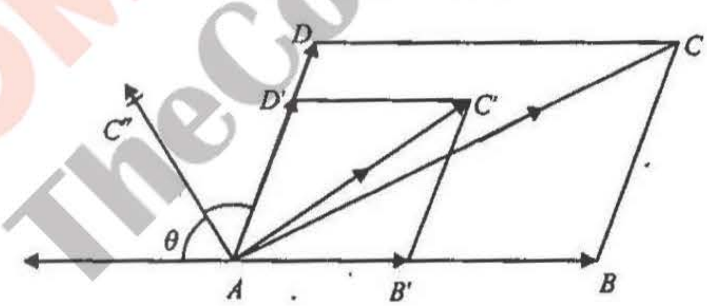


Fig. 1.35

Consider the two unit vectors along the given vectors, which form a rhombus *AB′C′D′*. Now

$$\overrightarrow{AB'} = \frac{\vec{a}}{|\vec{a}|} \text{ and } \overrightarrow{AD'} = \frac{\vec{b}}{|b|}$$

$$\therefore \quad \overrightarrow{AC'} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$$

So any vector along the bisector is $\lambda\left(\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}\right)$.

Similarly, any vector along the external bisector is $\overrightarrow{AC''}=\lambda\left(\frac{\vec{a}}{|\vec{a}|}-\frac{\vec{b}}{|\vec{b}|}\right)$.

Illustration 1.32 If $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$ and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$, determine vector $\vec{c}$ along the internal bisector of the angle between vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{c}| = 5\sqrt{6}$.

Sol. $\hat{a} = \frac{1}{9}(7\hat{i}-4\hat{j}-4\hat{k})$

$$\hat{b} = \frac{1}{3}(-2\hat{i}-\hat{j}+2\hat{k})$$

$$\vec{c} = \lambda[\hat{a}+\hat{b}] = \lambda\frac{1}{9}(\hat{i}-7\hat{j}+2\hat{k}) \quad \text{(i)}$$

$$|\vec{c}| = 5\sqrt{6}$$

$$\Rightarrow \quad \frac{\lambda^2}{81}(1+49+4) = 25\times 6$$

$$\lambda^2 = \frac{25\times 6\times 81}{54} = 225$$

$$\lambda = \pm 15$$

Putting the value of λ in (i), we get

$$\vec{c} = \pm\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$$

Illustration 1.33 Find a unit vector $\vec{c}$ if $-\hat{i}+\hat{j}-\hat{k}$ bisects the angle between vectors $\vec{c}$ and $3\hat{i}+4\hat{j}$.

Sol. Let $\vec{c} = x\hat{i}+y\hat{j}+z\hat{k}$, where $x^2+y^2+z^2=1$. (i)

Unit vector along $3\hat{i}+4\hat{j}$ is $\frac{3\hat{i}+4\hat{j}}{5}$.

The bisector of these two is $-\hat{i}+\hat{j}-\hat{k}$ (given). Therefore,

$$-\hat{i}+\hat{j}-\hat{k} = \lambda\left(x\hat{i}+y\hat{j}+z\hat{k}+\frac{3\hat{i}+4\hat{j}}{5}\right)$$

$$-\hat{i}+\hat{j}-\hat{k} = \frac{1}{5}\lambda[(5x+3)\hat{i}+(5y+4)\hat{j}+5z\hat{k}] \quad \text{(ii)}$$

$$\frac{\lambda}{5}(5x+3) = -1,\ \frac{\lambda}{5}(5y+4) = 1,\ \frac{\lambda}{5}5z = -1$$

$$x=-\frac{5+3\lambda}{5\lambda}, y=\frac{5-4\lambda}{5\lambda}, z=-\frac{1}{\lambda}$$

Putting these values in (i), i.e., $x^2+y^2+z^2=1$, we get

$$(5+3\lambda)^2+(5-4\lambda)^2+25=25\lambda^2$$

$$25\lambda^2-10\lambda+75=25\lambda^2$$

$$\lambda=\frac{15}{2}$$

$$\therefore \quad \vec{c}=\frac{1}{15}(-11\hat{i}+10\hat{j}-2\hat{k})$$

Concept Application Exercise 1.1

1. Find the unit vector in the direction of the vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$. **(NCERT)**
2. Find the direction cosines of the vector $\hat{i}+2\hat{j}+3\hat{k}$. **(NCERT)**
3. Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$ directed from A to B. **(NCERT)**
4. The position vectors of P and Q are $5\hat{i}+4\hat{j}+a\hat{k}$ and $-\hat{i}+2\hat{j}-2\hat{k}$, respectively. If the distance between them is 7, then find the value of a. **(NCERT)**
5. Given three points are $A(-3, -2, 0)$, $B(3, -3, 1)$ and $C(5, 0, 2)$. Then find a vector having the same direction as that of $\overline{AB}$ and magnitude equal to $|\overline{AC}|$
6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors

   $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$. **(NCERT)**
7. Show that the points $A(1, -2, -8)$, $B(5, 0, -2)$ and $C(11, 3, 7)$ are collinear, and find the ratio in which B divides AC. **(NCERT)**
8. If $ABCD$ is a rhombus whose diagonals cut at the origin O, then prove that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=\vec{O}$.
9. Let D, E and F be the middle points of the sides BC, CA and AB, respectively of a triangle ABC. Then prove that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$.
10. Let $ABCD$ be a parallelogram whose diagonals intersect at P and let O be the origin. Then prove that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$.
11. If A, B, C and D are any four points and E and F are the middle points of AC and BD, respectively then prove that $\overrightarrow{CB}+\overrightarrow{CD}+\overrightarrow{AD}+\overrightarrow{AB}=4\overrightarrow{EF}$.
12. If $\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$, then A, B and C are (where O is the origin)

    a. coplanar **b.** collinear **c.** non-collinear **d.** none of these
13. If the sides of an angle are given by vectors $\vec{a}=\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$, then find the internal bisector of the angle.

14. $ABCD$ is a parallelogram. If L and M be the middle points of BC and CD, respectively express $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Also show that $\overrightarrow{AL} + \overrightarrow{AM} = (3/2)\overrightarrow{AC}$.
15. $ABCD$ is a quadrilateral and E the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ is equal to $4\overrightarrow{OE}$, where O is any point.
16. What is the unit vector parallel to $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$? What vector should be added to $\vec{a}$ so that the resultant is the unit vector $\hat{i}$?
17. The position vectors of points A and B w.r.t. the origin are $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 2\hat{k}$, respectively. Determine vector $\overrightarrow{OP}$ which bisects angle AOB, where P is a point on AB.
18. If $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are the position vectors of three collinear points and scalars p and q exist such that $\vec{r}_3 = p\vec{r}_1 + q\vec{r}_2$, then show that $p + q = 1$.
19. If $\bar{a}$ and $\bar{b}$ are two vectors of magnitude 1 inclined at 120°, then find the angle between $\bar{b}$ and $\bar{b} - \bar{a}$.
20. Find the vector of magnitude 3, bisecting the angle between the vectors $\bar{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$.

LINEAR COMBINATION, LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

Linear Combination

A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$, if there exist scalars $m_1, m_2, \cdots, m_n$ such that $\vec{r} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n$.

Linearly Independent

A system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly independent if

$$m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n = \vec{0} \Rightarrow m_1 = m_2 = \cdots = m_n = 0$$

It can be easily verified that

i. A pair of non-collinear vectors is linearly independent.

Proof:

Let $\overline{a_1}$ and $\overline{a_2}$ are non-collinear vectors such that $m_1\overline{a_1} + m_2\overline{a_2} = \overline{O}$

Let $m_1, m_2 \neq 0$

$$\Rightarrow \quad \overline{a_1} = -\frac{m_2}{m_1}\overline{a_2}$$

This means $\overline{a_1}$ and $\overline{a_2}$ are collinear, which contradicts the given fact.

Hence, $m_1, m_2 = 0$

ii. A triad of non-coplanar vector is linearly independent.

Linearly Dependent

A set of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly dependent if there exist scalars $m_1, m_2, \cdots, m_n$, not all zero, such that $m_1 \vec{a}_1 + m_2 \vec{a}_2 + \cdots + m_n \vec{a}_n = \vec{0}$.
It can be easily verified that

i. A pair of collinear vectors is linearly dependent.
ii. A triad of coplanar vectors is linearly dependent.

Theorem 1.1

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then every vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be expressed in one and only one way as a linear combination $x\vec{a} + y\vec{b}$; x and y being scalars.

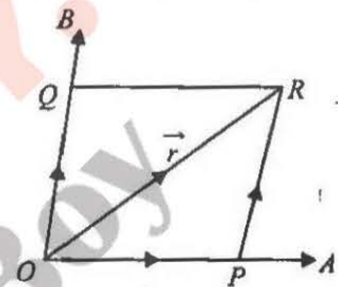


Fig. 1.36

Proof:

i. Let O be any point such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.
As $\vec{r}$ is coplanar with $\vec{a}$ and $\vec{b}$, the lines OA, OB and OR are coplanar. Through R, draw lines parallel to OA and OB, meeting them at P and Q, respectively. Clearly,

$$\overrightarrow{OP} = x\,\overrightarrow{OA} = x\vec{a} \quad (\because \overrightarrow{OP} \text{ and } \overrightarrow{OA} \text{ are collinear vectors})$$

Also $$\overrightarrow{OQ} = y\,\overrightarrow{OB} = y\vec{b} \quad (\because \overrightarrow{OQ} \text{ and } \overrightarrow{OB} \text{ are collinear vectors})$$

$$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \overrightarrow{OQ} \quad (\because \overrightarrow{OQ} \text{ and } \overrightarrow{PR} \text{ are equal})$$

$$= x\vec{a} + y\vec{b} \quad \text{(i)}$$

Thus, $\vec{r}$ can be expressed in one way as a linear combination $x\vec{a} + y\vec{b}$.

ii. To prove that this resolution is unique, let $\vec{r} = x'\vec{a} + y'\vec{b}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}$ and $\vec{b}$. Then,

$$x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b}$$

or $$(x - x')\vec{a} + (y - y')\vec{b} = \vec{0}$$

Since $\vec{a}$ and $\vec{b}$ are non-collinear vectors, we must have

$$x - x' = 0,\ y - y' = 0$$

i.e., $$x = x',\ y = y'$$

Thus, the representation is unique.

Note:

If OA and OB are perpendicular, then these two lines can be taken as the x- and the y-axes, respectively. Let $\hat{i}$ be the unit vector along the x-axis and $\hat{j}$ be the unit vector along the y-axis. Therefore, we have

$$\vec{r} = x\vec{i} + y\vec{j}$$

Also $$r = \sqrt{x^2 + y^2}$$

Theorem 1.2

If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, then any vector $\vec{r}$ can be uniquely expressed as a linear combination $x\vec{a} + y\vec{b} + z\vec{c}$; x, y and z being scalars.

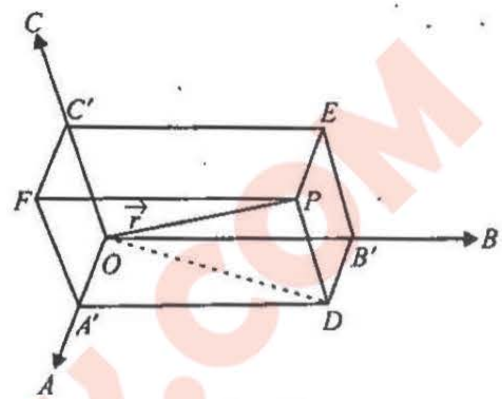


Fig. 1.37

Proof:

i. Take any point O so that

$$\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c} \text{ and } \overline{OP} = \bar{r}.$$

On OP as diagonal, construct a parallelpiped having edges OA', OB' and OC' along OA, OB and OC, respectively. Then there exist three scalars x, y and z such that

$$\overrightarrow{OA'} = x\,\overrightarrow{OA} = x\vec{a}, \overrightarrow{OB'} = y\,\overrightarrow{OB} = y\vec{b}, \overrightarrow{OC'} = z\,\overrightarrow{OC} = z\vec{c}$$

$$\therefore \quad \vec{r} = \overrightarrow{OP}$$

$$= \overrightarrow{OA'} + \overrightarrow{A'P}$$

$$= \overrightarrow{OA'} + \overrightarrow{A'D} + \overrightarrow{DP} \quad \text{(by definition of addition of vectors)}$$

$$= \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}$$

$$x\vec{a} + y\vec{b} + z\vec{c} \quad \text{(i)}$$

Thus $\vec{r}$ can be represented as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

ii. To prove that this representation is unique, let, if possible, $\vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$. (ii)

Then from (i) and (ii), we have

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$$

or $\quad (x - x')\vec{a} + (y - y')\vec{b} + (z - z')\vec{c} = \vec{0}$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are independent, $x - x' = 0$, $y - y' = 0$ and $z - z' = 0$, or $x = x'$, $y = y'$ and $z = z'$. Hence proved.

Theorem 1.3

If vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are coplanar, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Proof:

If vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then there exist scalars λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$. Hence,

$$c_1\hat{i} + c_2\hat{j} + c_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \mu(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

Now $\hat{i}, \hat{j}$ and $\hat{k}$ are non-coplanar and hence independent. Then,

$$c_1 = \lambda a_1 + \mu b_1,\ c_2 = \lambda a_2 + \mu b_2 \text{ and } c_3 = \lambda a_3 + \mu b_3$$

The above system of equations in terms of λ and μ is consistent. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Similarly, if vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$ and $x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$ are coplanar (where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar). Then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$ can be proved with the same arguments.

To prove that four points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ and $D(\vec{d})$ are coplanar, it is just sufficient to prove that vectors $\overrightarrow{AB}, \overrightarrow{BD}$ and $\overrightarrow{CD}$ are coplanar.

Notes:

1. Two collinear vectors are always linearly dependent.
2. Two non-collinear non-zero vectors are always linearly independent.
3. Three coplanar vectors are always linearly dependent.
4. Three non-coplanar non-zero vectors are always linearly independent.
5. More than three vectors are always linearly dependent.
6. Three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear if and only if there exist scalars x, y and z not all zero such that **(i)** $x\vec{a} + y\vec{b} + z\vec{c} = 0$ and **(ii)** $x + y + z = 0$.

Proof:

Let us suppose that points A, B and C are collinear and their position vectors are $\vec{a}, \vec{b}$ and $\vec{c}$, respectively. Let C divide the join of $\vec{a}$ and $\vec{b}$ in the ratio $y : x$. Then,

$$\vec{c} = \frac{x\vec{a} + y\vec{b}}{x + y}$$

or $\quad x\vec{a} + y\vec{b} - (x + y)\vec{c} = \vec{0}$

or $\quad x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $z = -(x + y)$

Also, $\quad x + y + z = x + y - (x + y) = 0.$

Conversely, let $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$. Therefore,

$$x\vec{a} + y\vec{b} = -z\vec{c} = (x + y)\vec{c} \qquad (\because x + y = -z)$$

or $\quad \vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x+y}$

This relation shows that $\vec{c}$ divides the join of $\vec{a}$ and $\vec{b}$ in the ratio $y : x$. Hence, the three points A, B and C are collinear.

7. Four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if there exist scalars x, y, z and w (sum of any two is not zero) such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$ with $x + y + z + w = 0$.

Proof:

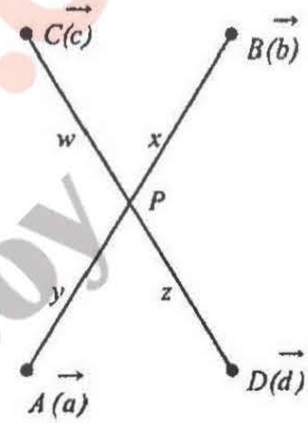


Fig. 1.38

$$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$$

or $\quad x\vec{a} + y\vec{b} = -(z\vec{c} + w\vec{d}) \qquad$ (i)

$$x + y + z + w = 0$$

or $\quad x + y = -(w + z) \qquad$ (ii)

From (i) and (ii), we have $\dfrac{x\vec{a} + y\vec{b}}{x+y} = \dfrac{z\vec{c} + w\vec{d}}{z+w}$

Thus, there is point P

$\Rightarrow \quad \dfrac{x\vec{a} + y\vec{b}}{x+y} = \dfrac{z\vec{c} + w\vec{d}}{z+w} \qquad$ (iii)

$\dfrac{x\vec{a} + y\vec{b}}{x+y}$ is the position vector of a point on AB which divides it in the ratio $y : x$.

$\dfrac{z\vec{c} + w\vec{d}}{z+w}$ is the position vector of a point on CD which divides it in the ratio $w : z$.

From (iii), these points are coincident; hence, the points are coplanar.

Illustration 1.34 The vectors $2\hat{i} + 3\hat{j}, 5\hat{i} + 6\hat{j}$ and $8\hat{i} + \lambda\hat{j}$ have their initial points at (1, 1). Find the value of λ so that the vectors terminate on one straight line.

Sol. Since the vectors $2\hat{i} + 3\hat{j}$ and $5\hat{i} + 6\hat{j}$ have (1, 1) as the initial point, their terminal points are (3, 4) and (6, 7), respectively. The equation of the line joining these two points is $x - y + 1 = 0$. The terminal point of $8\hat{i} + \lambda\hat{j}$ is $(9, \lambda + 1)$. Since the vectors terminate on the same straight line, $(9, \lambda + 1)$ lies on $x - y + 1 = 0$. Therefore,

$$9 - \lambda - 1 + 1 = 0$$

or $\quad \lambda = 9$

Illustration 1.35 If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-zero vectors, no two of which are collinear, $\vec{a} + 2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 3\vec{c}$ is collinear with $\vec{a}$, then find the value of $|\vec{a} + 2\vec{b} + 6\vec{c}|$.

Sol. Given $\vec{a} + 2\vec{b} = \lambda\vec{c}$ (i)

and $\vec{b} + 3\vec{c} = \mu\vec{a}$, (ii)

where no two of $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear vectors.

Eliminating $\vec{b}$ from the above relations, we have

$$\vec{a} - 6\vec{c} = \lambda\vec{c} - 2\mu\vec{a}$$

$$\vec{a}(1+2\mu) = (\lambda+6)\vec{c}$$

$\Rightarrow$ $\mu = -\dfrac{1}{2}$ and $\lambda = -6$ as $\vec{a}$ and $\vec{c}$ are non-collinear.

Putting $\mu = -\dfrac{1}{2}$ in (ii) or $\lambda = -6$ in (i), we get

$$\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$$

or $$\left|\vec{a} + 2\vec{b} + 3\vec{c}\right| = 0$$

Illustration 1.36

i. Prove that the points $\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$ and $-7\vec{b} + 10\vec{c}$ are collinear, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.

ii. Prove that the points $A(1, 2, 3)$, $B(3, 4, 7)$ and $C(-3, -2, -5)$ are collinear. Find the ratio in which point C divides AB.

Sol.

i. Let the given points be A, B and C. Therefore,

$$\overrightarrow{AB} = \text{P.V. of } B - \text{P.V. of } A$$
$$= (2\vec{a} + 3\vec{b} - 4\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$$
$$= \vec{a} + 5\vec{b} - 7\vec{c}$$

$$\overrightarrow{AC} = \text{P.V. of } C - \text{P.V. of } A$$
$$= (-7\vec{b} + 10\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$$
$$= -\vec{a} - 5\vec{b} + 7\vec{c} = -\overrightarrow{AB}$$

Since $\overline{AC} = -\overline{AB}$, it follows that the points A, B and C are collinear.

ii. Let C divide AB in the ratio $k : 1$; then $C(-3, -2, -5) \equiv \left(\dfrac{3k+1}{k+1}, \dfrac{4k+2}{k+1}, \dfrac{7k+3}{k+1}\right)$

$\Rightarrow$ $\dfrac{3k+1}{k+1} = -3, \dfrac{4k+2}{k+1} = -2$ and $\dfrac{7k+3}{k+1} = -5$

$\Rightarrow$ $k = -\dfrac{2}{3}$ from all relations

Hence, C divides AB externally in the ratio 2:3.

Illustration 1.37 Check whether the given three vectors are coplanar or non-coplanar:

$$-2\hat{i}-2\hat{j}+4\hat{k}, -2\hat{i}+4\hat{j}-2\hat{k}, 4\hat{i}-2\hat{j}-2\hat{k}$$

Sol. Given vectors are $-2\hat{i}-2\hat{j}+4\hat{k}, -2\hat{i}+4\hat{j}-2\hat{k}, 4\hat{i}-2\hat{j}-2\hat{k}$

$$\Rightarrow \quad \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = 16+16+16-64+8+8=0$$

Hence, the vectors are coplanar.

Illustration 1.38 Prove that the four points $6\hat{i}-7\hat{j}, 16\hat{i}-19\hat{j}-4\hat{k}, 3\hat{j}-6\hat{k}$ and $2\hat{i}+5\hat{j}+10\hat{k}$ form a tetrahedron in space.

Sol. Given points are $A(6\hat{i}-7\hat{j}), B(16\hat{i}-19\hat{j}-4\hat{k}), C(3\hat{j}-6\hat{k}), D(2\hat{i}+5\hat{j}+10\hat{k})$

Hence, vectors $\vec{AB}=10\hat{i}-12\hat{j}-4\hat{k}$, $\vec{AC}=-6\hat{i}+10\hat{j}-6\hat{k}$ and $\vec{AD}=-4\hat{i}+12\hat{j}+10\hat{k}$

Now determinant of coefficients of $\vec{AB}, \vec{AC}, \vec{AD}$ is

$$\begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix} = 10(100+72)+12(-60-24)-4(-72+40) \neq 0$$

Hence, the given points are non-coplanar and therefore form a tetrahedron in space.

Illustration 1.39 If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, show that points $l_1\vec{a}+m_1\vec{b}, l_2\vec{a}+m_2\vec{b}$ and $l_3\vec{a}+m_3\vec{b}$ are collinear if $\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix}=0$.

Sol. We know that three points having P.V.s $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear if there exists a relation of the form $x\vec{a}+y\vec{b}+z\vec{c}, = \vec{0}$, where $x+y+z=0$.

Now $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$ gives

$$x(l_1\vec{a}+m_1\vec{b})+y(l_2\vec{a}+m_2\vec{b})+z(l_3\vec{a}+m_3\vec{b})=\vec{0}$$

or $$(xl_1+yl_2+zl_3)\vec{a}+(xm_1+ym_2+zm_3)\vec{b}=\vec{0}$$

Since $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, it follows that

$$xl_1+yl_2+zl_3=0 \quad \text{(i)}$$

$$xm_1+ym_2+zm_3=0 \quad \text{(ii)}$$

Because otherwise one is expressible as a scalar multiple of the other which would mean that $\vec{a}$ and $\vec{b}$ are collinear. Also

$$x+y+z=0 \quad \text{(iii)}$$

Eliminating x, y and z from (i), (ii) and (iii), we get

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Alternate method:

$$A(l_1\vec{a} + m_1\vec{b}),\ B(l_2\vec{a} + m_2\vec{b}) \text{ and } C(l_3\vec{a} + m_3\vec{b}) \text{ are collinear.}$$

$\Rightarrow$ Vectors $= (l_2 - l_3)\vec{a} + (m_2 - m_3)\vec{b}$ and $\overrightarrow{AB} = (l_1 - l_2)\vec{a} + (m_1 - m_2)\vec{b}$ are collinear.

$$\Rightarrow \quad \frac{l_1 - l_2}{l_2 - l_3} = \frac{m_1 - m_2}{m_2 - m_3}$$

$$\Rightarrow \quad \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Illustration 1.40 Vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find for what value of x vectors $\vec{c} = (x-2)\vec{a} + \vec{b}$ and $\vec{d} = (2x+1)\vec{a} - \vec{b}$ are collinear?

Sol. Both the vectors $\vec{c}$ and $\vec{d}$ are non-zero as the coefficients of $\vec{b}$ in both are non-zero.

Two vectors $\vec{c}$ and $\vec{d}$ are collinear if one of them is a linear multiple of the other. Therefore,

$$\vec{d} = \lambda\vec{c}$$

or $\quad (2x+1)\vec{a} - \vec{b} = \lambda\{(x-2)\vec{a} + \vec{b}\}$ (i)

or $\quad \{(2x+1) - \lambda(x-2)\}\vec{a} - (1+\lambda)\vec{b} = 0$

The above expression is of the form $p\vec{a} + q\vec{b} = 0$, where $\vec{a}$ and $\vec{b}$ are non-collinear, and hence we have $p = 0$ and $q = 0$. Therefore,

$2x + 1 - \lambda(x-2) = 0$ (ii)

and $\quad 1 + \lambda = 0$ (iii)

From (iii), $\lambda = -1$, and putting this value in (i), we get $x = \frac{1}{3}$

Alternate method:

$$\vec{c} = (x-2)\vec{a} + \vec{b} \text{ and } \vec{d} = (2x+1)\vec{a} - \vec{b} \text{ are collinear.}$$

If $\frac{x-2}{2x+1} = \frac{1}{-1}$, then $x = \frac{1}{3}$

Illustration 1.41 The median AD of the triangle ABC is bisected at E and BE meets AC at F. Find $AF : FC$.

Sol. Taking A at the origin

Let P.V. of B and C be $\vec{b}$ and $\vec{c}$, respectively.

P.V. of D is $\dfrac{\vec{b}+\vec{c}}{2}$ and P.V. of E is $\dfrac{\vec{b}+\vec{c}}{4}$

Let $AF:FC = p:1$.

Then position vector of F is $\dfrac{p\vec{c}}{p+1}$ (i)

Let $BF:EF = q:1$.

The position vector of F is $\dfrac{q\dfrac{(\vec{b}+\vec{c})}{4}-\vec{b}}{q-1}$ (ii)

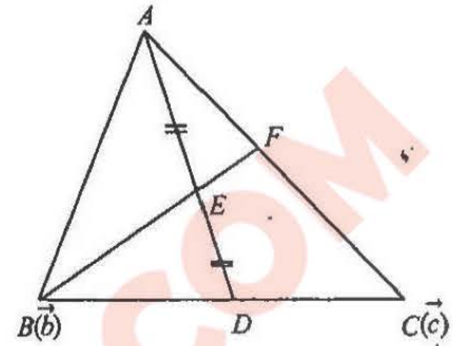


Fig. 1.39

Comparing P.V. of F in (i) and (ii), we have

$$\frac{p\vec{c}}{p+1} = \frac{q\dfrac{(\vec{b}+\vec{c})}{4}-\vec{b}}{q-1}$$

Since vectors $\vec{b}$ and $\vec{c}$ are independent, we have

$$\frac{p}{p+1} = \frac{q}{4(q-1)} \text{ and } \frac{q-4}{4(q-1)} = 0$$

$\Rightarrow$ $p = 1/4$ and $q = 4$

$\Rightarrow$ $AF : FC = 1:2$

Illustration 1.42 Prove that the necessary and sufficient condition for any four points in three-dimensional space to be coplanar is that there exists a linear relation connecting their position vectors such that the algebraic sum of the coefficients (not all zero) in it is zero.

Sol. Let us assume that the points A, B, C and D whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, are coplanar. In that case the lines AB and CD will intersect at some point P (it being assumed that AB and CD are not parallel, and if they are, then we will choose any other pair of non-parallel lines formed by the given points). If P divides AB in the ratio $q : p$ and CD in the ratio $n : m$, then the position vector of P written from AB and CD is

$$\frac{p\vec{a}+q\vec{b}}{p+q} = \frac{m\vec{c}+n\vec{d}}{m+n}$$

or $$\frac{p}{p+q}\vec{a} + \frac{q}{p+q}\vec{b} - \frac{m}{m+n}\vec{c} - \frac{n}{m+n}\vec{d} = \vec{0}$$

or $$L\vec{a} + M\vec{b} + N\vec{c} + P\vec{d} = \vec{0}$$

where $$L+M+N+P = \frac{p}{p+q} + \frac{q}{p+q} - \frac{m}{m+n} - \frac{n}{m+n} = 1-1=0$$

Hence, the condition is necessary.

Converse: Let $l\vec{a} + m\vec{b} + n\vec{c} + p\vec{d} = \vec{0}$

where $l + m + n + p = 0$ (i)

We will show that the points A, B, C and D are coplanar.

Now of the three scalars $l + m$, $l + n$ and $l + p$, one at least is not zero, because if all of them are zero, then

$$l + m = 0, l + n = 0, l + p = 0$$

$\therefore$ $m = n = p = -l$

Hence, $l + m + n + p = 0$ or $l - 3l = 0$ or $l = 0$

Hence, $m = n = p = -l = 0$

Thus, $l = 0$, $m = 0$, $n = 0$, $p = 0$, which is against the hypothesis.

Let us suppose that $l + m$ is not zero.

$$l + m = -(n + p) \neq 0, \quad \text{[From (i)]} \quad \text{(ii)}$$

Also from the given relation, we have

$$l\vec{a} + m\vec{b} = -(n\vec{c} + p\vec{d})$$

or

$$\frac{l\vec{a} + m\vec{b}}{l + m} = \frac{n\vec{c} + p\vec{d}}{n + p} \quad \text{[From (ii)]} \quad \text{(iii)}$$

The L.H.S. represents a point which divides AB in the ratio $m : l$ and the R.H.S. represents a point which divides CD in the ratio $p : n$. These points being the same, it follows that a point on AB is the same as a point on CD, showing that the lines AB and CD intersect. Hence, the four points A, B, C and D are coplanar.

Illustration 1.43

i. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that vectors $3\vec{a} - 7\vec{b} - 4\vec{c}$, $3\vec{a} - 2\vec{b} + \vec{c}$ and $\vec{a} + \vec{b} + 2\vec{c}$ are coplanar.

ii. If the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar, then prove that $a = 4$.

Sol.

i. If the given vectors are coplanar, then we should be able to express one of them as a linear combination of the other two.

Let us assume that $3\vec{a} - 7\vec{b} - 4\vec{c} = x(3\vec{a} - 2\vec{b} + \vec{c}) + y(\vec{a} + \vec{b} + 2\vec{c})$,

where x and y are scalars. Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar, equating the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$$3x + y = 3, -2x + y = -7, x + 2y = -4$$

Solving the first two, we find that $x = 2$ and $y = -3$. These values of x and y satisfy the third equation as well.

Hence, the given vectors are coplanar.

ii. Given vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar. Then

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0$$

or $3 - 7a + 25 = 0$

or $a = 4$

Illustration 1.44 If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that the four points $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.

Sol. Let the given points be A, B, C and D. If they are coplanar, then the three coterminous vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ should be coplanar.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{a} - 5\vec{b} + 4\vec{c}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{a} + \vec{b} - \vec{c}$$

and $$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -\vec{a} - 9\vec{b} + 7\vec{c}$$

Since the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar, we must have $\begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ -1 & -9 & 7 \end{vmatrix} = 0$, which is true.

Hence proved.

Illustration 1.45 Let P be an interior point of a triangle ABC and AP, BP, CP meet the sides BC, CA, AB in D, E, F, respectively. Show that $\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$.

Sol. Since A, B, C, P are co-planar, there exists four scalars x, y, z, w not all zero simultaneously such that

$$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{p} = 0$$

where $$x + y + z + w = 0$$

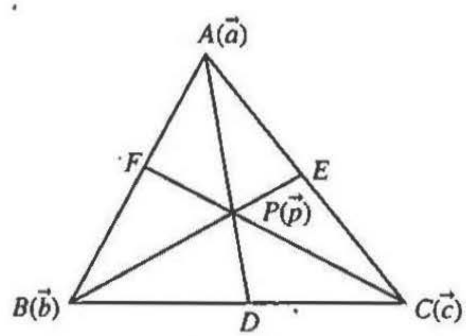


Fig. 1.40

Also, $$\frac{x\vec{a} + w\vec{p}}{x + w} = \frac{y\vec{b} + z\vec{c}}{y + z}$$

Hence, $$\frac{AP}{PD} = -\frac{w}{x} - 1$$

Also $$\frac{x\vec{a} + y\vec{b}}{x + y} = \frac{z\vec{c} + w\vec{p}}{z + w}$$

$\Rightarrow$ $$\frac{AF}{FB} = \frac{y}{x}$$

Similarly, $$\frac{AE}{EC} = \frac{z}{x}$$

Thus, to show that $-\frac{w}{x} - 1 = \frac{y}{x} + \frac{z}{x}$

$\Rightarrow$ $x + y + z + w = 0$ which is true.

Hence proved.

Illustration 1.46 Points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are related as $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ and $x + y + z + w = 0$, where x, y, z and w are scalars (sum of any two of x, y, z and w is not zero). Prove that if A, B, C and D are concyclic, then $|xy|\,|\vec{a} - \vec{b}|^2 = |wz|\,|\vec{c} - \vec{d}|^2$.

Sol. From the given conditions, it is clear that points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are coplanar.

Now, A, B, C and D are concyclic. Therefore,

$$AP \times BP = CP \times DP$$

$$\left|\frac{y}{x+y}\right| |\vec{a}-\vec{b}| \left|\frac{x}{x+y}\right| |\vec{a}-\vec{b}| = \left|\frac{w}{w+z}\right| |\vec{c}-\vec{d}| \left|\frac{z}{w+z}\right| |\vec{c}-\vec{d}|$$

$$|x y| \, |\vec{a}-\vec{b}|^2 = |w z| \, |\vec{c}-\vec{d}|^2$$

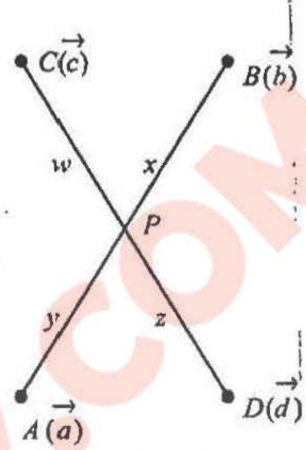


Fig. 1.41

Concept Application Exercise 1.2

1. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors in three-dimensional space with the same initial point and such that $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=\vec{0}$, show that terminals A, B, C and D of these vectors are coplanar. Find the point at which AC and BD meet. Find the ratio in which P divides AC and BD.
2. Show that the vectors $2\vec{a}-\vec{b}+3\vec{c}, \vec{a}+\vec{b}-2\vec{c}$ and $\vec{a}+\vec{b}-3\vec{c}$ are non-coplanar vectors (where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors).
3. Examine the following vectors for linear independence:
   i. $\vec{i}+\vec{j}+\vec{k}, 2\vec{i}+3\vec{j}-\vec{k}, -\vec{i}-2\vec{j}+2\vec{k}$
   ii. $3\vec{i}+\vec{j}-\vec{k}, 2\vec{i}-\vec{j}+7\vec{k}, 7\vec{i}-\vec{j}+13\vec{k}$
4. If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A}=(p+4q)\vec{a}+(2p+q+1)\vec{b}$ and $\vec{B}=(-2p+q+2)\vec{a}+(2p-3q-1)\vec{b}$, and if $3\vec{A}=2\vec{B}$, then determine p and q.
5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three non-coplanar vectors, then prove that points $l_1\vec{a}+m_1\vec{b}+n_1\vec{c}$, $l_2\vec{a}+m_2\vec{b}+n_2\vec{c}, l_3\vec{a}+m_3\vec{b}+n_3\vec{c}, l_4\vec{a}+m_4\vec{b}+n_4\vec{c}$ are coplanar if $\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0$
6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero, non-coplanar vectors, then find the linear relation between the following four vectors: $\vec{a}-2\vec{b}+3\vec{c}, 2\vec{a}-3\vec{b}+4\vec{c}, 3\vec{a}-4\vec{b}+5\vec{c}, 7\vec{a}-11\vec{b}+15\vec{c}$.
7. Let a, b, c be distinct non-negative numbers and the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}, c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then prove that the quadratic equation $ax^2+2cx+b=0$ has equal roots.

Exercises

Subjective Type

1. The position vectors of the vertices A, B and C of a triangle are $\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$ and $\hat{i}+\hat{k}$, respectively. Find a unit vector $\hat{r}$ lying in the plane of ABC and perpendicular to IA, where I is the incentre of the triangle.
2. A ship is sailing towards the north at a speed of 1.25 m/s. The current is taking it towards the east at the rate of 1 m/s and a sailor is climbing a vertical pole on the ship at the rate of 0.5 m/s. Find the velocity of the sailor in space.
3. Given four points P_1, P_2, P_3 and P_4 on the coordinate plane with origin O which satisfy the condition $\overrightarrow{OP}_{n-1}+\overrightarrow{OP}_{n+1}=\frac{3}{2}\overrightarrow{OP}_n$.
   i. If P_1 and P_2 lie on the curve $xy=1$, then prove that P_3 does not lie on the curve.
   ii. If P_1, P_2 and P_3 lie on the circle $x^2+y^2=1$, then prove that P_4 also lies on this circle.
4. $ABCD$ is a tetrahedron and O is any point. If the lines joining O to the vertices meet the opposite faces at P, Q, R and S, prove that $\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}+\frac{OS}{DS}=1$.
5. A pyramid with vertex at point P has a regular hexagonal base $ABCDEF$. Position vectors of points A and B are $\hat{i}$ and $\hat{i}+2\hat{j}$, respectively. The centre of the base has the position vector $\hat{i}+\hat{j}+\sqrt{3}\hat{k}$. Altitude drawn from P on the base meets the diagonal AD at point G. Find all possible position vectors of G. It is given that the volume of the pyramid is $6\sqrt{3}$ cubic units and AP is 5 units.
6. A straight line L cuts the lines AB, AC and AD of a parallelogram $ABCD$ at points B_1, C_1 and D_1, respectively. If $\overrightarrow{AB}_1=\lambda_1\overrightarrow{AB}$, $\overrightarrow{AD}_1=\lambda_2\overrightarrow{AD}$ and $\overrightarrow{AC}_1=\lambda_3\overrightarrow{AC}$, then prove that $\frac{1}{\lambda_3}=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$.
7. The position vectors of the points P and Q are $5\hat{i}+7\hat{j}-2\hat{k}$ and $-3\hat{i}+3\hat{j}+6\hat{k}$, respectively. Vector $\vec{A}=3\hat{i}-\hat{j}+\hat{k}$ passes through point P and vector $\vec{B}=-3\hat{i}+2\hat{j}+4\hat{k}$ passes through point Q. A third vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersects vectors A and B. Find the position vectors of points of intersection.
8. Show that $x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ and $x_3\hat{i}+y_3\hat{j}+z_3\hat{k}$ are non-coplanar if $|x_1|>|y_1|+|z_1|$, $|y_2|>|x_2|+|z_2|$ and $|z_3|>|x_3|+|y_3|$.
9. If $\vec{A}$ and $\vec{B}$ are two vectors and k any scalar quantity greater than zero, then prove that $|\vec{A}+\vec{B}|^2\le(1+k)|\vec{A}|^2+\left(1+\frac{1}{k}\right)|\vec{B}|^2$.
10. Consider the vectors $\hat{i}+\cos(\beta-\alpha)\hat{j}+\cos(\gamma-\alpha)\hat{k}$, $\cos(\alpha-\beta)\hat{i}+\hat{j}+\cos(\gamma-\beta)\hat{k}$ and $\cos(\alpha-\gamma)\hat{i}+\cos(\beta-\gamma)\hat{j}+a\hat{k}$, where α, β and γ are different angles. If these vectors are coplanar, show that a is independent of α, β and γ.
11. In a triangle PQR, S and T are points on QR and PR, respectively, such that $QS=3SR$ and $PT=4TR$. Let M be the point of intersection of PS and QT. Determine the ratio $QM:MT$ using the vector method.