PHYSICS

For Joint Entrance Examination (JEE)

Mechanics II

B.M. Sharma

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Physics for JEE:

Mechanics II

B.M. Sharma

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Appendix A.1

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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. A combined national-level engineering entrance examination has finally been proposed by the Ministry of Human Resource Development, Government of India. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroot level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-facetted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.
Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of physics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

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CHAPTER

1

Centre of Mass, Conservation of Linear Momentum and Collision

- The Centre of Mass
- Motion of the Centre of Mass
- Conservation of Linear Momentum
- Impulse
- Collision or Impact
- Variable Mass System

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THE CENTRE OF MASS

In this section, we will discuss the overall motion of a system of particles in terms of a very special point called **the centre of mass** of the system. This notion gives us confidence in the particle model because we will see that the centre of mass accelerates as if all the system's mass were concentrated at that point and all external forces act there.

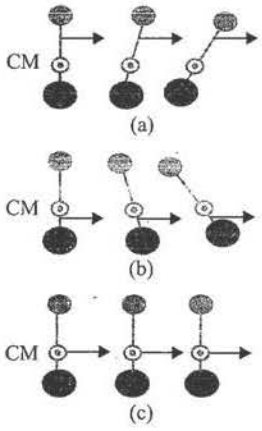


Fig. 1.1

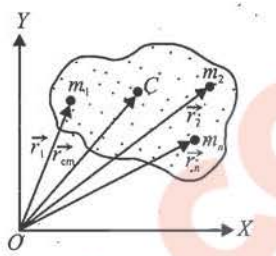
Consider a system consisting of a pair of particles connected by a light rigid rod. The centre of mass as indicated in Fig. 1.1 is located on the rod and is closer to the larger mass in the figure. If a single force is applied at some point on the rod above the centre of mass, the system rotates clockwise as it translates through space [Fig. 1.1(a)]. If the force is applied at a point on the rod below the centre of mass, the system rotates counterclockwise [Fig. 1.1(b)]. If the force is applied exactly at the centre of mass, the system moves in the direction of $\vec{F}$ without rotating [Fig. 1.1(c)] as if the system is behaving as a particle. Therefore, in theory, the centre of mass can be located with this experiment.

If we were to analyse the motion [Fig. 1.1(c)], we would find that the system moves as if all its mass were concentrated at the centre of mass. Furthermore, if the external net force on the system is $\Sigma\vec{F}$ and the total mass of the system is M, the centre of mass moves with an acceleration given by $\vec{a} = \Sigma\vec{F}/M$.

Centre of Mass of a System of 'N' Discrete Particles

Consider a system of N point masses $m_1, m_2, m_3, ..., m_n$ whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$, respectively.

Then, the position vector of the centre of mass C of the system is given by



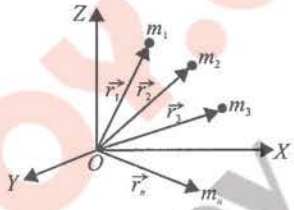


Fig. 1.2

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\Sigma_{i=1}^{n} m_i\vec{r}_i}{\Sigma_{i=1}^{n} m_i}$$

$$= \frac{1}{M}\Sigma_{i=1}^{n} m_i\vec{r}$$

where $M(= \Sigma_{i=1}^{n} m_i)$ is the total mass of the system.

Consider a system of point masses $m_1, m_2, m_3, ...$ located at the coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), ..., respectively. The centre of mass of this system of masses is a point whose coordinates are (x_{cm}, y_{cm}, z_{cm}), which are given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots}, \quad y_{cm} = \frac{m_1y_1 + m_2y_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$z_{cm} = \frac{m_1z_1 + m_2z_2 + \cdots}{m_1 + m_2 + \cdots}$$

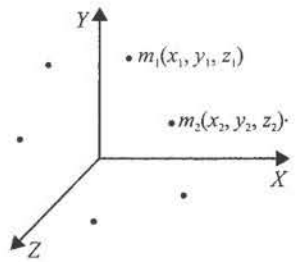


Fig. 1.3

Illustration 1.1 **Four particles of masses 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices *A*, *B*, *C* and *D*, respectively, of a square of side 1 m. Find the position of centre of mass of the particles.**

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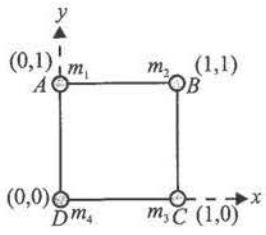

Fig. 1.4

Sol. Assuming D as the origin, DC as x-axis and DA as y-axis, we have

$$m_1 = 1 \text{ kg}, (x_1, y_1) = (0, 1 \text{ m})$$
$$m_2 = 2 \text{ kg}, (x_2, y_2) = (1 \text{ m}, 1 \text{ m})$$
$$m_3 = 3 \text{ kg}, (x_3, y_3) = (1 \text{ m}, 0)$$
$$m_4 = 4 \text{ kg}, (x_4, y_4) = (0, 0)$$

Coordinates of their CM are calculated as follows.

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 \quad m_4}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4} = \frac{5}{10} = \frac{1}{2} \text{ m} = 0.5 \text{ m}$$

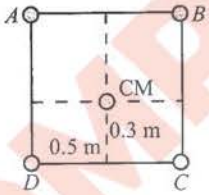

Fig. 1.5

Similarly, $y_{CM} = \dfrac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$

$$= \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1 + 2 + 3 + 4} = \frac{3}{10} \text{ m}$$

$$= 0.3 \text{ m}$$

$$\therefore \quad (x_{CM}, y_{CM}) = (0.5 \text{ m}, 0.3 \text{ m})$$

Illustration 1.2 **Consider a two-particle system with the particles having masses m_1 and m_2. If the first particle is pushed towards the centre of mass through a distance d, by what distance should the second particle be moved so as to keep the centre of mass at the same position?**

Sol. Consider Fig. 1.6. Suppose the distance of m_1 from the centre of mass C is x_1 and that of m_2 from C is x_2. Suppose mass m_2 is moved through a distance d' towards C so as to keep the centre of mass at C.

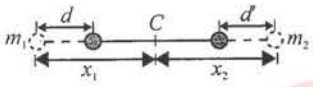

Fig. 1.6

Then, $m_1x_1 = m_2x_2$ (i)

and $m_1(x_1 - d) = m_2(x_2 - d')$ (ii)

Subtracting Eq. (ii) from Eq. (i), $m_1d = m_2d'$ or $d' = (m_1/m_2)\,d$.

Centre of Mass of a Continuous Mass Distribution

For a continuous mass distribution, the centre of mass can be located by replacing summation sign with an integral sign. Proper limits for the integral are chosen according to the situation.

$$x_{CM} = \frac{\int x dm}{\int dm}, y_{CM} = \frac{\int y dm}{\int dm}, z_{CM} = \frac{\int z dm}{\int dm}$$

$\int dm = M$ (mass of the body)

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r} dm$$

Note: ***If an object has symmetric uniform mass distribution about x-axis, then y-coordinate of CM is zero and vice versa.***

Centre of Mass of a Uniform Rod

Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$.

Mass per unit length of the rod $= \dfrac{M}{L}$

Hence, the mass of the element dx situated at $x = x$ is

$$dm = \frac{M}{L} dx$$

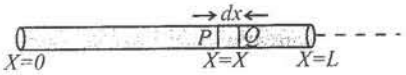

Fig. 1.7

The coordinates of the element PQ are $(x, 0, 0)$. Therefore, x-coordinate of CM of the rod will be

$$x_{CM} = \frac{\int_0^L x dm}{\int dm} = \frac{\int_0^L x \frac{M}{L} dx}{M} = \frac{1}{L}\int_0^L x\, dx = \frac{L}{2}$$

The y-coordinate of CM is $y_{CM} = \dfrac{\int y dm}{\int dm} = 0$

Similarly, $z_{CM} = 0$.

i.e., the coordinates of CM of the rod are $(L/2, 0, 0)$, or it lies at the centre of the rod.

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The following points regarding centre of mass can be noted.

1. Centre of mass of a uniform rectangular, square or circular plate lies at its geometrical centre.

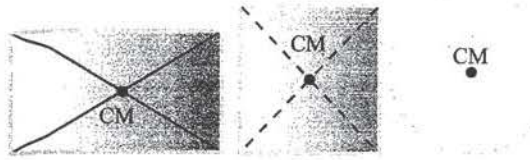

Fig. 1.8

2. For a lamina type (two-dimensional) body with uniform negligible thickness, the formulae for finding the position of centre of mass are as follows:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$= \frac{\rho A_1 t\vec{r}_1 + \rho A_2 t\vec{r}_2 + \cdots}{\rho A_1 t + \rho A_2 t + \cdots}, (m = \rho A t)$$

$$= \frac{A_1\vec{r}_1 + A_2\vec{r}_2 + \cdots}{A_1 + A_2 + \cdots}$$

Here, A stands for area and r for density.

3. If some mass is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formulae:

(i) $\vec{r}_{CM} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$ or $\vec{r}_{CM} = \frac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$

(ii) $x_{CM} = \frac{m_1x_1 - m_2x_2}{m_1 - m_2}$ or $x_{CM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$

(iii) $y_{CM} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2}$ or $y_{CM} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$

(iv) $z_{CM} = \frac{m_1z_1 - m_2z_2}{m_1 - m_2}$ or $z_{CM} = \frac{A_1z_1 - A_2z_2}{A_1 - A_2}$

Here, $m_1, A_1, \vec{r}_1, x_1, y_1$ and z_1 are the values for the whole mass while $m_2, A_2, \vec{r}_2, \vec{x}_2, y_2$ and z_2 are the values for the mass which has been removed.

Note: ***We can imagine a rigid body as a system of masses and hence every rigid body has a centre of mass. In case of a regularly shaped uniform rigid body, centre of mass is simply the geometric centre of the body. The centre of mass of some continuous geometrical figures of uniform mass density is given in Fig. 1.9.***

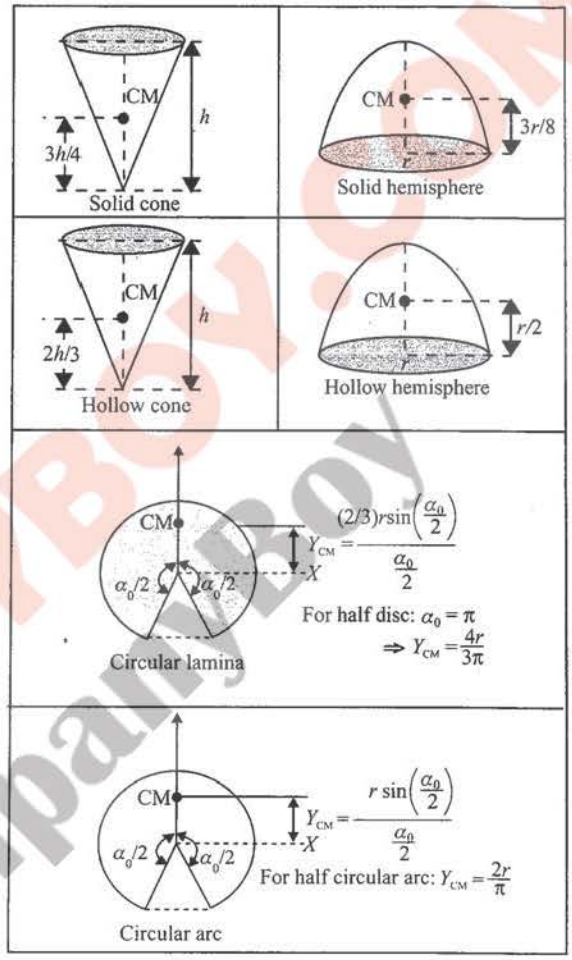

Fig. 1.9

Illustration 1.3 **Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown in Fig. 1.10. The mass of lamina is 3 kg.**

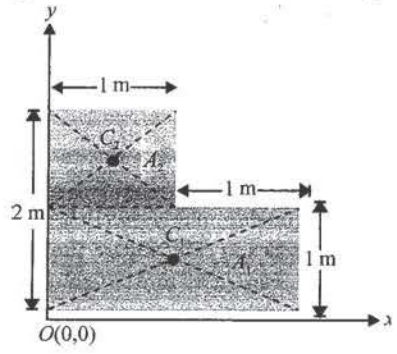

Fig. 1.10

Sol. The plate has uniform density and same thickness everywhere. So its CM will coincide with the centroid.

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Divide the given plate into two parts of area A_1 and A_2 as shown in the figure. We have

$A_1 = 2 \times 1\ m^2$ with its centroid C_1 (1, 1/2) and

$A_2 = 1 \times 1\ m^2$ with its centroid C_2 (1/2, 3/2)

The centroid of the whole plate can be defined as

$$\bar{x} = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} = \frac{2 \times 1 + 1 \times \frac{1}{2}}{2+1} = \frac{5}{6}\ m$$

$$\bar{y} = \frac{A_1y_1 + A_2y_2}{A_1 + A_2} = \frac{2 \times \frac{1}{2} + 1 \times \frac{3}{2}}{2+1} = \frac{5}{6}\ m$$

Illustration 1.4 **Find the position of centre of mass of the uniform lamina shown in Fig. 1.11.**

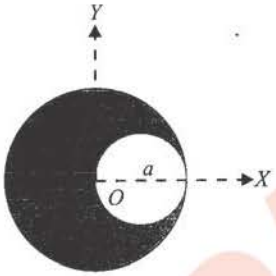

Fig. 1.11

Sol. Here, A_1 = area of complete circle = πa^2

A_2 = area of small circle = $\pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$

(x_1, y_1) = coordinates of centre of mass of the large circle = (0, 0)

(x_2, y_2) = coordinates of centre of mass of the small circle = $\left(\frac{a}{2}, 0\right)$

Using $x_{CM} = (A_1x_1 - A_2x_2)/(A_1 - A_2)$, we get

$$x_{CM} = \frac{\pi a^2 \times 0 - \frac{\pi a^2}{4} \times \frac{a}{2}}{\pi a^2 - \frac{\pi a^2}{4}} = -\frac{a}{6}$$

$y_{CM} = 0$ (as y_1 and y_2 both are zero)

Therefore, coordinates of CM of the lamina shown in Fig. 1.11 are $(-a/6, 0)$.

Illustration 1.5 **Figure 1.12 shows a uniform disc of radius R, from which a hole of radius $R/2$ has been cut out from left of the centre and is placed on the right of the centre of the disc. Find the CM of the resulting disc.**

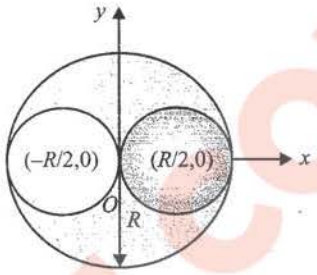

Fig. 1.12

Sol. Mass of the cut-out disc is

$$m = \frac{M}{\pi R^2} \times \pi\left(\frac{R}{2}\right)^2 = \frac{M}{4}$$

Let centre of the disc is at the origin of the coordinates.

Then we can write the CM of the system as

$$x_{CM} = \frac{M\vec{R} - m\vec{r} + m\vec{r}}{M - m + m} = \frac{M \times 0 - \frac{M}{4}\left(\frac{-R}{2}\right) + \frac{M}{4}\left(\frac{R}{2}\right)}{M - \frac{M}{4} + \frac{M}{4}} = \frac{R}{4}$$

$$y_{CM} = 0$$

MOTION OF THE CENTRE OF MASS

Motion of Centre of Mass and Conservation of Momentum: Velocity of Centre of Mass of System

$$\vec{v}_{CM} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \cdots + m_n\frac{d\vec{r}_n}{dt}}{M}$$

$$= \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots}{m_1 + m_2 + \cdots} \quad \Rightarrow \quad \vec{v}_{CM} = \frac{\vec{P}_{total}}{M_{total}}$$

Here numerator of the right-hand-side term is the total momentum of the system, i.e., summation of momentum of the individual components (particles) of the system.

Hence, velocity of centre of mass of the system is the ratio of momentum of the system per unit mass of the system.

Acceleration of Centre of Mass of System

$$\vec{a}_{CM} = \frac{m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \cdots + m_n\frac{d\vec{v}_n}{dt}}{M}$$

$$= \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n}{M}$$

$$= \frac{\vec{F}_{ext}}{M} = \frac{\text{Net external force}}{M}$$

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(Both the action and reaction of an internal force must be within the system. Vector summation will cancel all internal forces and hence net internal force on system is zero.)

$$\therefore \qquad \vec{F}_{ext} = M\vec{a}_{CM}$$

where $\vec{F}_{ext}$ is the sum of the 'external' forces acting on the system. The internal forces which the particles exert on one another play absolutely no role in the motion of the centre of mass.

If no external force is acting on a system of particles, the acceleration of centre of mass of the system will be zero. If $a_{CM} = 0$, it implies that v_{CM} must be a constant and if v_{CM} is a constant, it implies that the total momentum of the system must remain constant. It leads to the principle that the total momentum of the system must remain constant. It leads to the principle of conservation of momentum in the absence of external forces.

If $\vec{F}_{ext} = 0$, then $\vec{P}_{total}$ = constant.

If no external force is acting on the system, net momentum of the system must remain constant.

Centre of Mass at Rest

If $F_{ext} = 0$ and $v_{CM} = 0$, then centre of mass remains at rest. Individual components of a system may move and have non-zero momentum due to mutual forces (internal), but the net momentum of the system remains zero.

Some examples:

- All the particles of the system are at rest.
- Particles are moving such that their net momentum is zero. Example: Net momentum = $2m \times 2 + m(-4) = 0$

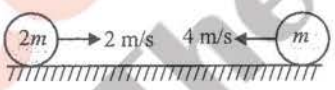
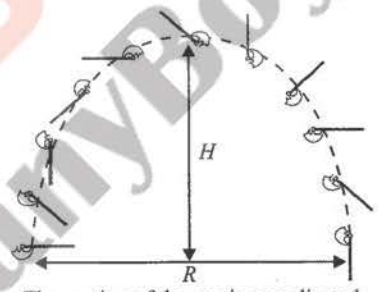


Fig. 1.13

- A bomb at rest suddenly explodes into various smaller fragments, all moving in different directions. Since the explosive forces are internal and there is no external force on the system for explosion, the centre of mass of the bomb will remain at the original position and the fragments fly such that their net momentum remains zero.
- Two men standing on a frictionless platform push each other, then also their net momentum remains zero because the push forces are internal for the two-man system.
- A boat floating in a lake also has the net momentum zero if the people on it change their positions, because the friction force required to move the people is internal of the boat system.
- Objects, initially at rest, if moving under mutual forces (electrostatic or gravitation) also have net momentum zero.
- A light spring of spring constant k is kept compressed between two blocks of masses m_1 and m_2 on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions, such that the net momentum is zero.
- **Centre of mass in motion:** An axe is thrown at some angle with horizontal. Centre of mass of axe will move in such a way that if all the mg force acts on itself. So centre of mass will move in a parabolic path, however the whole motion of the axe will be complicated.



The motion of the axe is complicated but the CM is moving in a parabolic motion.

Fig. 1.14

Illustration 1.6 **A projectile is fired at a speed of 100 m/s at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1:3, the smaller coming to rest. Find the distance from the launching point to the point where the heavier piece lands.**

Sol. Internal forces do not affect the motion of the centre of mass, the centre of mass hits the ground at a position where the original projectile would have landed. The range of the original projectile is

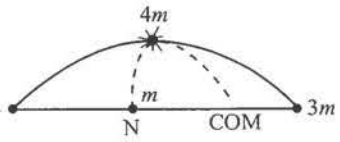


Fig. 1.15

$$x_{CM} = \frac{2u^2 \sin\theta \cos\theta}{g} = \frac{2 \times 10^4 \times \frac{3}{5} \times \frac{4}{5}}{10} \text{ m} = 960 \text{ m}$$

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The centre of mass will hit the ground at this position. As the smaller block comes to rest after breaking, it falls down vertically and hits the ground at half of the range, i.e., at $x = 480$ m. If the heavier block hits the ground at x_2, then

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \Rightarrow 960 = \frac{(m)(480) + (3m)(x_2)}{(m + 3m)}$$

$$\therefore \; x_2 = 1120 \text{ m}$$

Illustration 1.7 **Two balls with masses $m_1 = 3$ kg and $m_2 = 5$ kg have initial velocities $v_1 = v_2 = 5$ m/s in the directions shown in Fig. 1.16. They collide at the origin.**

(a) Find the velocity of the CM 3 s before the collision.

(b) Find the position of the CM 2 s after the collision.

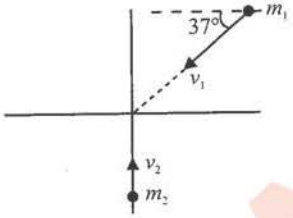

Fig. 1.16

Sol. (a) The given time is of no consequence since v_{CM} is fixed for all times.

$$v_{CM}(x) = \frac{m_1 v_{1x} + m_2 v_{2x}}{m_1 + m_2}$$

$$= \frac{(3)(-5\cos 37°) + (5)(0)}{8 \text{ kg}} = -1.5 \text{ m/s}$$

$$v_{CM}(y) = \frac{m_1 v_{1y} + m_2 v_{2y}}{m_1 + m_2}$$

$$= \frac{(3)(-5\sin 37°) + (5 \times 5)}{8 \text{ kg}} = +2 \text{ m/s}$$

$$\therefore \quad \vec{v}_{CM} = -1.5\hat{i} + 2\hat{j} \text{ m/s}$$

(b) Since the collision occurs at the origin ($\vec{r}_1 = \vec{0}$), the position of the CM 2 s later is $\vec{r}_{CM} = \vec{r}_1 + \vec{v}_{CM} t$

$= \vec{v}_{CM} t = -3\hat{i} + 4\hat{j}$ m

Illustration 1.8 **Two particles of masses 2 kg and 4 kg are approaching towards each other with accelerations 1 m/s^2 and 2 m/s^2, respectively, on a smooth horizontal surface. Then find the acceleration of centre of mass of the system and direction of acceleration of CM.**

Sol: The acceleration of CM of the system is given by

$$a_{CM} = \frac{m_1 a_1 + m_2 a_2}{m_2 + m_2} \Rightarrow a_{CM} = \frac{2\times 1 + 4(-2)}{2+4} = -1 \text{ m/s}^2$$

Negative sign indicates that acceleration of CM will be in the direction of acceleration of 4 kg mass.

Illustration 1.9 **A pulley fixed to the ceiling carried a thread with bodies of masses m_1 and m_2 attached to its ends. The masses of the pulley and the thread are negligible and friction is absent. Find the acceleration of the centre of mass of this system.**

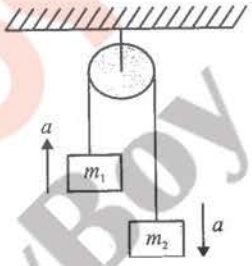

Fig. 1.17

Sol. Let as assume that $m_2 > m_1$. We can see that the masses have equal and opposite acceleration of the same magnitude.

$a = \frac{m_2 - m_1}{m_2 + m_1} g$ and tension in string is $T = \frac{2m_1 m_2 g}{m_2 + m_1}$

Taking downward direction as positive, $a_1 = -a$; $a_2 = +a$

$$a_{CM} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2} = \frac{(m_2 - m_1)\, a}{m_2 + m_1}$$

Substituting for the value of a, we have

$$a_{CM} = \left(\frac{m_2 - m_1}{m_2 + m_1}\right)^2 g$$

Alternative method:

$$a_{CM} = \frac{f_{ext}}{m_1 + m_2} = \frac{(m_1 g + m_2 g) - 2T}{m_1 + m_2} = g - \frac{2T}{m_1 + m_2}$$

$$= g - \frac{4m_1 m_2 g}{(m_2 + m_1)^2} = \left(\frac{m_2 - m_1}{m_2 + m_1}\right)^2 g \text{ (downwards)}$$

Illustration 1.10 **A log of wood of length l and mass M is floating on the surface of a river perpendicular to the banks. One end of the log touches the banks. A man of mass m standing at the other end walks towards the bank. Calculate the displacement of the log when he reaches the nearer end of the log.**

Sol. Let PQ be the log of wood. As there is no external force, the centre of mass of man and the log system remains at rest. Let the bank of the river be the origin A. Initially, the man is at point Q.

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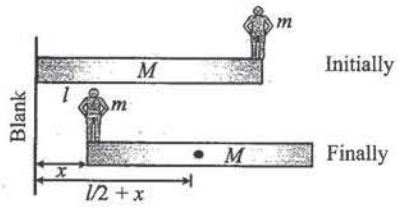

Fig. 1.18

Let m = mass of man,

M = mass of log,

x = displacement of log w.r.t. ground (here water)

$$X_{CM}\text{ (initial)} = \frac{m(l) + M\left(\frac{l}{2}\right)}{m+M}$$

$$X_{CM}\text{ (final)} = \frac{m(x) + M\left(\frac{l}{2}+x\right)}{m+M}$$

Now, $X_{CM}\text{ (initial)} = X_{CM}\text{ (final)}$

$$\Rightarrow \quad ml + \frac{Ml}{2} = mx + \frac{Ml}{2} + Mx$$

$$\Rightarrow \quad x = \frac{ml}{m+M}.$$

Hence, the log moves away from the bank through a distance of $\frac{m\ell}{(m+M)}$.

Alternative method: Displacement of the log $= \Delta x_1 = x$

Displacement of the man $= \Delta x_2 = \ell - x$

Apply $m_1 x_1 = m_2 x_2 \Rightarrow$ (if centre of mas remains at the same place)

$$\Rightarrow Mx = m(l-x) \quad \Rightarrow \quad x = \frac{m\ell}{(m+M)}$$

Illustration 1.11 **A plank of mass M and length L is at rest on a frictionless floor. The top surface of the plank has friction. At one end of it a man of mass m is standing as shown in Fig. 1.19. If the man walks towards the other end, find the distance, which the plank moves (a) till the man reaches the centre of the plank, (b) till the man reaches the other end of the plank.**

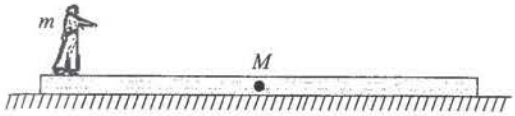

Fig. 1.19

Sol. Method 1: The corresponding situation can be better explained with the help of Fig. 1.19. Consider the man and the plank as a system on which no external force is acting. The centre of mass of the system must remain stationary. The only interaction force between the man and the plank is the friction as shown in Fig. 1.19, due to which the man walks along the plank and the friction on plank would be in opposite direction, due to which plank moves towards left, such that the centre of mass of the plank plus man remains at rest. Just before the motion started, the initial distance of the centre of mass from the centre of the plank is

$$x_c = \frac{m\times\frac{L}{2} + m\times 0}{m+M} = \frac{m\times\frac{L}{2}}{m+M}$$

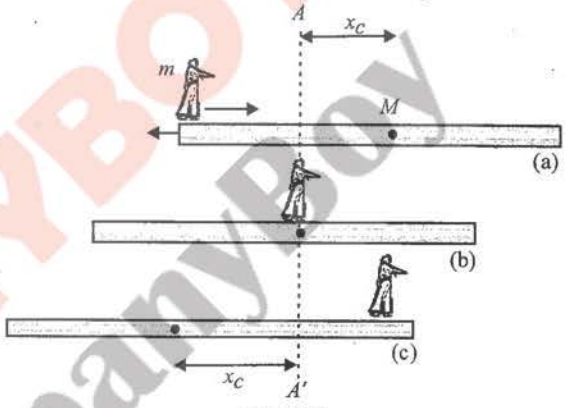

Fig. 1.20

Initially, the centre of mass of the system is on line AA' as shown in Fig. 1.20. During motion of the man, this centre of mass must remain at this line only. As the man moves towards right, the plank will move towards left such that centre of mass remains on AA'. Thus, when the man reaches the centre of the plank, the plank's centre must also reach the same point so that the centre of mass is at the same position. Up to this instant the plank moves by a distance x_c. Similarly, when the man reaches the other end, plank has to move towards left further by x_c to maintain the position of centre of mass.

Method 2: As there is no external force on the system (man + plank) and the system is initially at rest, there should not be movement of centre of mass of the system during the motion of the man.

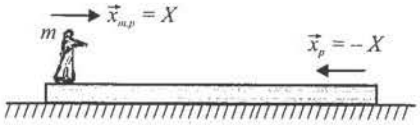

Fig. 1.21

The displacement of centre of mass is Δx_{CM} and is given by

$$\Delta x_{CM} = \frac{m\Delta\vec{x}_m + M\Delta\vec{x}_p}{m+M} \qquad \text{(i)}$$

$\Rightarrow \Delta\vec{x}_m =$ displacement of the man with respect to ground

$\Delta\vec{x}_m = \Delta\vec{x}_{m,p} + \Delta\vec{x}_p = (x - X)$ where $\Delta\vec{x}_p = -X$

(towards left)

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As $\Delta\vec{x}_{CM} = 0$, from Eq. (i),

$$0 = \frac{m(x-X) - MX}{m+M} \Rightarrow X = \frac{mx}{m+M}$$

(a) When $x = \frac{L}{2}$, displacement of block $X = \frac{m\frac{L}{2}}{m+M}$

(b) When $x = L$, displacement of block $X = \frac{mL}{m+M}$

Illustration 1.12 **An explosion blows a rock into three parts. Two pieces go off at right angles to each other; 1.0 kg piece with a velocity of 12 m/s and other 2.0 kg piece with a velocity of 8 m/s. If the third piece flies off with a velocity of 40 m/s, compute the mass of the third piece.**

Sol. Let m_1, m_2 and m_3 be the masses of the three pieces.

$$m_1 = 1.0 \text{ kg}, \quad m_2 = 2.0 \text{ kg}.$$

Let v_1 = 12 m/s, v_2 = 8 m/s, v_3 = 40 m/s. Let v_1 and v_2 be directed along x-and y-axes, respectively, and v_3 be directed as shown.

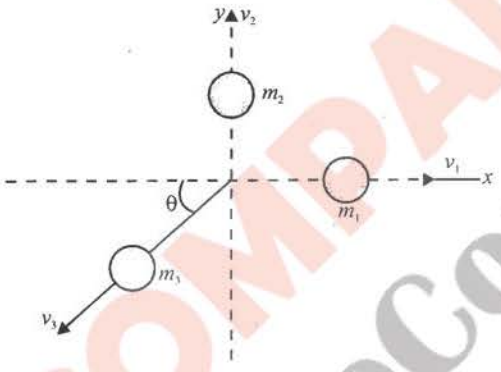

Fig. 1.22

By the principle of conservation of momentum, initial momentum is zero. Hence,

along x-axis: $0 = m_1v_1 - m_3v_3\cos\theta$

along y-axis: $0 = m_2v_2 - m_3v_3\sin\theta$

$\Rightarrow$ $m_1v_1 = m_3v_3\cos\theta$ and $m_2v_2 = m_3v_3\sin\theta$

By squaring and adding, we get $m_1^2v_1^2 + m_2^2v_2^2 = m_3^2v_3^2$

$$\Rightarrow \quad m_3^2 = \frac{1^2(12)^2 + (2)^2(8)^2}{(40)^2} \Rightarrow m_3 = 0.5 \text{ kg}$$

Concept Application Exercise 1.1

1. Two children A and B of same mass (including their caps) M are sitting on a sea-saw as shown in Fig. 1.23. Initially, the beam is horizontal. At once, child B throws away his cap (mass $M/25$) which falls at point Q, midpoint of the left half of the beam, due to this the balance of beam is disturbed. To balance it again what is the mass m required to be put at point P on the right half of the beam?

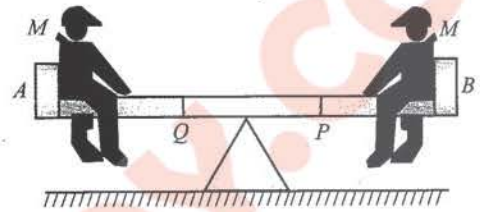

Fig. 1.23

2. Figure 1.24 shows a fixed wedge on which two blocks of masses 2 kg and 3 kg are placed on its smooth inclined surfaces. When the two blocks are released from rest, find the acceleration of centre of mass of the two blocks.

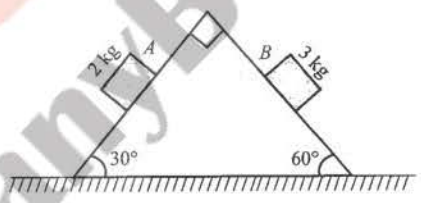

Fig. 1.24

3. Consider a rectangular plate of dimensions $a \times b$. If the plate is considered to be made up of four rectangles of dimensions $\frac{a}{2} \times \frac{b}{2}$ and we now remove one (the lower right) out of the four rectangles, find the position where the centre of mass of the remaining system will be.

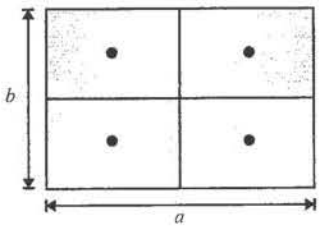

Fig. 1.25

4. There are two masses m_1 and m_2 placed at a distance l apart, let the centre of mass of this system is at a point named C. If m_1 is displaced by l_1 towards C and m_2 is displaced by l_2 away from C, find the distance from C where the new centre of mass will be located.
5. Let there are three equal masses situated at the vertices of an equilateral triangle, as shown in Fig. 1.26. Now particle A starts with a velocity v_1 towards line AB, particle B starts with the velocity v_2 towards line BC and particle C starts with velocity v_3 towards line CA. Find the displacement of the centre of mass of the three particles A, B and C after time t. What would it be if $v_1 = v_2 = v_3$?

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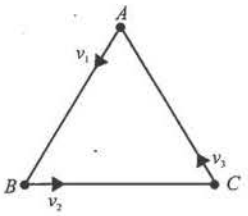

Fig. 1.26

6. Figure 1.27 shows a flat car of mass M on a frictionless road. A small massless wedge is fitted on it as shown. A small ball of mass m is released from the top of the wedge, it slides over it and falls in the hole at distance l from the initial position of the ball. Find the distance the flat car moves till the ball gets into the hole.

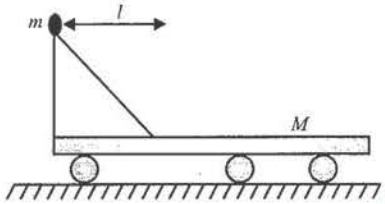

Fig. 1.27

7. Figure 1.28 shows two blocks of masses 5 kg and 2 kg placed on a frictionless surface and connected with a spring. An external kick gives a velocity of 14 m/s to the heavier block in the direction of lighter one. Deduce (a) velocity gained by the centre of mass and (b) the separate velocities of the two blocks in the centre of mass coordinates just after the kick.

Fig. 1.28

8. Two blocks of masses m_1 and m_2 connected by a weightless spring of stiffness k rest on a smooth horizontal plane as shown in Fig. 1.29. Block 2 is shifted a small distance x to the left and then released. Find the velocity of centre of mass of the system after block 1 breaks off the wall.

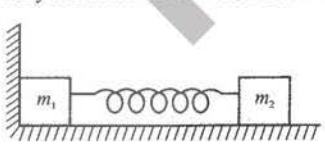

Fig. 1.29

9. Mr. Verma (50 kg) and Mr. Mathur (60 kg) are sitting at the two extremes of a 4 m long boat (40 kg) standing still in water. To discuss a mechanics problem, they come to the middle of the boat. Neglecting friction with water, how far does the boat move in the water during the process?

10. A cart of mass M is at rest on a frictionless horizontal surface and a pendulum bob of mass m hangs from the roof of the cart. The string breaks, the bob falls on the floor, makes several collisions on the floor and finally lands up in a small slot made on the floor. The horizontal distance between the string and the slot is L. Find the displacement of the cart during this process.

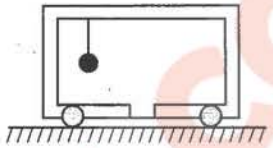

Fig. 1.30

11. Find the displacement of the wedge when 'm' comes out of the wedge. There is no friction anywhere.

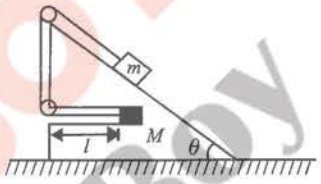

Fig. 1.31

12. A block of mass m is initially lying on a wedge of mass M with an angle of inclination θ, as shown in Fig. 1.32. Calculate the displacement of the wedge when the block is released and reaches to the bottom of the wedge.

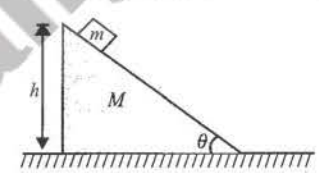

Fig. 1.32

13. Calculate the displacement of the wedge when the ball reaches at the bottom of the groove.

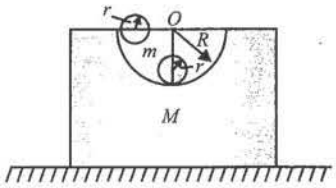

Fig. 1.33

14. A block is released on the convex surface of a hemispherical wedge as shown in Fig. 1.34. Determine the displacement of the wedge when the block reaches the angular position θ.

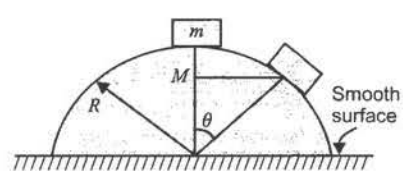

Fig. 1.34

15. Two masses m_1 and m_2 are moving with velocities v_1 and v_2. Find their total kinetic energy in the reference frame of centre of mass.

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16. Figure 1.35 the system is at rest initially with $x = 0$. A man and a woman both are initially at the extreme carrier of the platform. The man and the woman start to move towards each other. Obtain an expression for the displacement s of the platform when the two meet in terms of the displacement x_1 of the man relative to the platform.

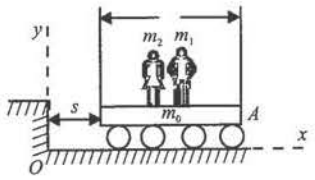

Fig. 1.35

17. A 30 kg projectile moving horizontally with a velocity $\vec{v}_0 = (120 \text{ m/s})\,\hat{i}$ explodes into two fragments A and B of masses 12 kg and 18 kg, respectively. Taking point of explosion as origin and knowing that 3 s later the position of fragment A is (300 m, 24 m, –48 m), determine the position of fragment B at the same instant.

18. Two 20 kg cannon balls are chained together and fired horizontally with a velocity of 200 m/s from the top of a 30 m wall. The chain breaks during the flight of the cannon balls and one of them strikes the ground at $t = 2$ s, at a distance of 250 m from the foot of the wall, and 5 m to the right of the line of fire. Determine the position of the other cannon ball at that instant. Neglect the resistance of air.

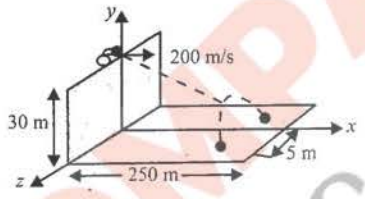

Fig. 1.36

19. A juggler juggles three balls in a continuous cycle. Any one ball is in contact with his hand for one-fifth of the time. Describe the motion of the centre of mass of the three balls. What average force does the juggler exert on one ball while he is touching it?

20. A cannon and a supply of cannon balls are inside a sealed rail road car. The cannon fires to the right, the car recoils to the left. The canon balls remain in the car after hitting the far wall. Show that no matter how the cannon balls are fired, the rail road car cannot travel more than L, assuming it starts from rest.

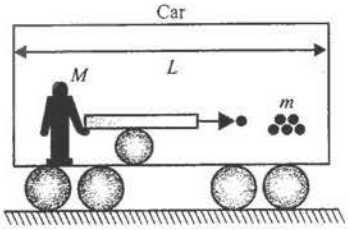

Fig. 1.37

CONSERVATION OF LINEAR MOMENTUM

We were able to solve problems involving these situations by applying a conservation principle, conservation of energy. Consider another situation. A 60 kg archer stands on frictionless ice and fires a 0.50 kg arrow horizontally at 50 m/s. From Newton's third law, we know that the force that the bow exerts on the arrow will be matched by a force in the opposite direction on the bow (and the archer). This force will cause the archer to begin to slide backward on the ice. But with what speed? We cannot answer this question using either Newton's second law or an energy approach because there is not enough information.

Let us conceptualize an isolated system of two particles (Fig. 1.38) with masses m_1 and m_2 and moving with velocities $\vec{v}_1$ and $\vec{v}_2$ at an instant of time. Because the system is isolated, the only force on one particle is that from the other particle, and we can categorize this situation as one in which Newton's laws can be applied. If a force from particle 1 (e.g., a gravitational force) acts on particle 2, there must be a second force, equal in magnitude but opposite in direction, that particle 2 exerts on particle 1. That is, the forces form a Newton's third law action–reaction pair so that $\vec{F}_{12} = -\vec{F}_{21}$. We can express this condition as a statement about the system of two particles as follows:

$$\vec{F}_{12} + \vec{F}_{21} = 0$$

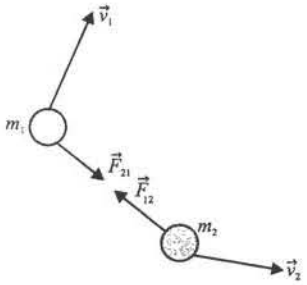

Fig. 1.38

Let us further analyze this situation by incorporating Newton's second law. Over some time interval, the interacting particles in the system will accelerate. Therefore, replacing each force with $m\vec{a}$ gives

$$m_1\vec{a}_1 + m_2\vec{a}_2 = 0 \Rightarrow m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = 0$$

If the masses m_1 and m_2 are constant, we can bring them into the derivatives, which gives

$$\frac{d(m_1\vec{v}_1)}{dt} + \frac{d(m_2\vec{v}_2)}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) = 0 \qquad \text{(i)}$$

Using the definition of momentum, Eq. (i) can be written as

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$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

Because the time derivative of the total system momentum $\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2$ is zero, we conclude that the total momentum $\vec{p}_{tot}$ must remain constant:

$$\vec{p}_{tot} = \text{constant} \quad \text{(ii)}$$

or equivalently,

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \quad \text{(iii)}$$

where $\vec{p}_{1i}$ and $\vec{p}_{2i}$ are initial values and $\vec{p}_{1f}$ and $\vec{p}_{2f}$ are final values of the momentum during a period over which the particles interact. Equation (iii) in component form states that the momentum components of the isolated system in the x, y and z directions are all independently constant; that is,

$$\sum_{\text{system}} p_{ix} = \sum_{\text{system}} p_{fx}, \quad \sum_{\text{system}} p_{iy} = \sum_{\text{system}} p_{fy}, \quad \sum_{\text{system}} p_{iz} = \sum_{\text{system}} p_{fz} \quad \text{(iv)}$$

This result, known as the law of conservation of linear momentum, is the mathematical representation of momentum version of the isolated system model. It is considered one of the most important laws of mechanics. We have generated this law for a system of two interacting particles, but it can be shown to be true for a system of any number of particles. We can state it as follows:

The total momentum of an isolated system remains constant.

Notice that we have made no statement concerning the nature of the forces acting between members of the system. The only requirement is that the forces must be internal to the system. Therefore, momentum is conserved for an isolated system regardless of the nature of the internal forces, even if the forces are non-conservative.

As linear momentum depends on frame of reference, observers in different frames would find different values of linear momentum of a given system but each would agree that his own value of linear momentum does not change with time provided the system is isolated and closed, i.e., law of conservation of linear momentum is independent of the frame of reference though linear momentum depends on the frame of reference.

Conservation of linear momentum is equivalent to Newton's third law of motion.

This law is universal, i.e., it applies to both macroscopic and microscopic systems. It holds good even in atomic and nuclear physics where classical mechanics fails. Further it is more generally applicable than the law of 'conservation of mechanical energy' because 'internal forces' are often non-conservative and so mechanical energy is not conserved but momentum is (if $F_{ext} = 0$). Principal applications of conservation of linear momentum are in the field of collisions.

Note: ***Remember that the momentum of an isolated system is conserved. The momentum of one particle within an isolated system is not necessarily conserved because other particles in the system may be interacting with it. Always apply conservation of momentum to an isolated system.***

Illustration 1.13 **A man of mass m moves on a plank of mass M with a constant velocity u with respect to the plank, as shown in Fig. 1.39.**

(i) If the plank rests on a smooth horizontal surface, determine the velocity of the plank.

(ii) If the man travels a distance L with respect to the plank, find the distance travelled by the plank with respect to the ground.

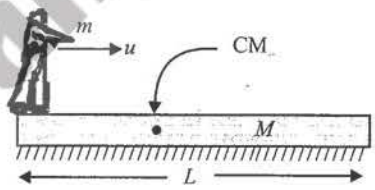

Fig. 1.39

Sol. As no external forces are acting on the system in horizontal direction, so its linear momentum remains constant in that direction. Also there will be no shift in position of centre of mass, i.e., $\Delta x_{CM} = 0$.

(i) Let v_1 and v_2 be the velocities of the man and the plank w.r.t. ground. Then, we have

$$[\vec{v}_{man}]_{plank} = [\vec{v}_{man}]_{ground} - [\vec{v}_{plank}]_{ground}$$

Then, $$[\vec{v}_{man}]_{ground} = [\vec{v}_{man}]_{plank} + [\vec{v}_{plank}]_{ground}$$

or $$v_1 = u - v_2$$

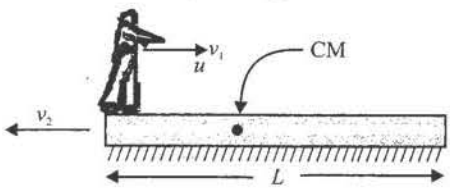

Fig. 1.40

Initially, the system is at rest. Therefore,

$$0 = mv_1 - Mv_2$$

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or $0 = m(u - v_2) - Mv_2$

or $v_2 = \frac{mu}{m+M}$

ii. Let x be the displacement of plank in backward direction, the displacement of mass is $L - x$ in forward dirction.

$$\Delta x_{CM} = \frac{M(-x) + m(L-x)}{M+m}$$

$$\Rightarrow \quad 0 = -Mx + m(L-x) \Rightarrow x = \frac{mL}{M+m}$$

Illustration 1.14 **A shell is fired from a cannon with a speed of 100 m/s at an angle 60° with the horizontal (positive x-direction). At the highest point of its trajectory, the shell explodes into two equal fragments. One of the fragments moves along the negative x-direction with a speed of 50 m/s. What is the speed of the other fragment at the time of explosion.**

Sol. The velocity of the shell at the highest point of trajectory is

$$v_M = u\cos\theta = 100\cos 60° = 50 \text{ m/s}$$

Let v_1 be the speed of the fragment which moves along the negative x-direction and the other fragment has speed v_2, which must be along the positive x-direction. Now from momentum conservation, we have

$$mv = \frac{-m}{2}v_1 + \frac{m}{2}v_2 \text{ or } 2v = v_2 - v_1$$

or $v_2 = 2v + v_1 = (2 \times 50) + 50 = 150$ m/s

Illustration 1.15 **A man of mass m is standing at one end of a boat of mass M and length l. The man walks to the other end. What is the displacement of the centre of mass? What is the displacement of boat?**

Sol. In the process of walking, no external force acts on the system (boat + man) in the horizontal direction. So displacement of centre of mass will be zero. The horizontal momentum of the system is conserved.

(Since initially the momentum of the system is zero as it was at rest)

$$m\vec{v}_m + M\vec{v}_b = 0 \Rightarrow m(\vec{v}_{mb} + \vec{v}_b) + M\vec{v}_b = 0$$

$$v_b = \frac{mv_{mb}}{M+m} \Rightarrow \int_0^t v_b dt = \frac{m}{M+m}\int_0^t V_{mb}dt$$

$$X_b = \frac{m}{m+M}X_{mb}$$

X_{mb} = displacement of man relative to boat = l

$$X_b = \frac{ml}{M+m}$$

Illustration 1.16 **Two blocks of masses m_1 and m_2, interconnected with a spring of stiffness K, are kept on a smooth horizontal surface. Find out the ratio of velocity, displacement and acceleration of block with mass m_1 to block with mass m_2.**

Sol. Here x is the displacement, v is the speed, p is the momentum, a is the acceleration, KE is the kinetic energy, F is the force.

$$(F_{ext})_{system} = 0 \Rightarrow \vec{p}_1 + \vec{p}_2 = 0$$

$$m_1v_1 = m_2v_2 \Rightarrow \frac{v_1}{v_2} = \frac{m_2}{m_1}$$

$$m_1\int_0^t v_1 dt = m_2\int_0^t v_2 dt$$

where t is the time interval of motion of each block.

$$m_1x_1 = m_2x_2 \Rightarrow \frac{x_1}{x_2} = \frac{m_2}{m_1}$$

$$\frac{KE_1}{KE_2} = \frac{P_1^2/2m_1}{P_2^2/2m_2} = \frac{m_2}{m_1} \quad (\because p_1 = p_2)$$

$$\Rightarrow \quad \frac{F_1}{F_2} = \frac{kx}{kx}$$

where x is the deformation of the spring.

$$\therefore \quad \frac{F_1}{F_2} = 1 \Rightarrow \frac{m_1a_1}{m_2a_2} = 1 \Rightarrow \frac{a_1}{a_2} = \frac{m_2}{m_1}$$

Illustration 1.17 **A flat car of mass M is at rest on a frictionless floor with a child of mass m standing at its edge. If the child jumps off from the car towards right with an initial velocity u, with respect to the car, find the velocity of the car after its jump.**

Sol. Let the car attain a velocity v. The net velocity of the child with respect to the earth will be $u - v$, as u is its velocity with respect to the car.

Initially, the system was at rest, thus according to momentum conservation, momentum after jump must be zero. Hence,

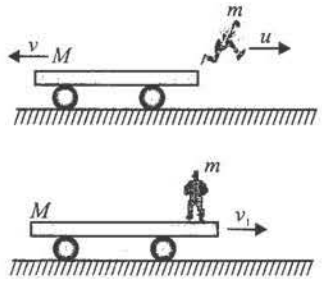

Fig. 1.41

$$m(u - v) = Mv \Rightarrow v = \frac{mu}{m+M}$$

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Illustration 1.18 **A flat car of mass M with a child of mass m is moving with a velocity v_1. The child jumps in the direction of motion of car with a velocity u with respect to the car. Find the final velocities of the child and that of the car after jump.**

Sol. This case is similar to the previous example, except now the car is moving before jump. Here, also no external force is acting on the system in horizontal direction, hence momentum remains conserved in this direction. After the jump, car attains a velocity v_2 in the same direction, which is less than v_1, due to backward push of the child for jumping. After the jump, child attains a velocity $u + v_2$ in the direction of motion of car with respect to ground.

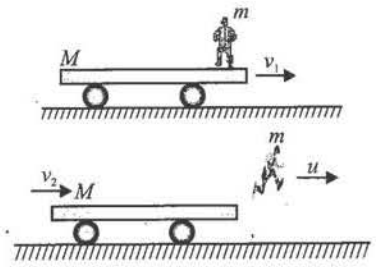

Fig. 1.42

According to momentum conservation,

$$(M+m)v_1 = Mv_2 + m(u+v_2)$$

Velocity of car after jump is $v_2 = \dfrac{(M+m)v_1 - mu}{M+m}$

Velocity of child after jump is $u + v_2 = \dfrac{(M+m)v_1 + Mu}{M+m}$

Illustration 1.19 **Each of the blocks shown in figure has mass 1 kg. The rear block moves with a speed of 2 m/s towards the front block kept at rest. The spring attached to the front block is light and has a spring constant 50 N/m. Find the maximum compression of the spring.**

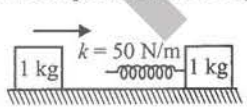

Fig. 1.43

Sol. Maximum compression will take place when the blocks move with equal velocity. As no net external horizontal force acts on the system of the two blocks, the total linear momentum will remain constant. If V is the common speed at maximum compression, we have

$$(1\text{ kg})(2\text{ m/s}) = (1\text{ kg})\,V + (1\text{ kg})\,V \text{ or } V = 1\text{ m/s}.$$

Initial kinetic energy $= \frac{1}{2}(1\text{ kg})(2\text{ m/s})^2 = 2\text{ J}$

Final kinetic energy $= \frac{1}{2}(1\text{ kg})(1\text{ m/s})^2 + \frac{1}{2}(1\text{ kg})(1\text{ m/s})^2 = 1\text{ J}$

The kinetic energy lost is stored as the elastic energy in the spring. Hence,

$$\frac{1}{2}(50\text{ N/m})\,x^2 = 2\text{ J} - 1\text{ J} = 1\text{ J or } x = 0.2\text{ m}$$

Illustration 1.20 **A light spring of spring constant k is kept compressed between two blocks of masses m and M on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions. The spring loses contact with the blocks when it acquires natural length. If the spring was initially compressed through a distance x, find the final speeds of the two blocks.**

Sol. Consider the two blocks plus the spring to be the system. No external force acts on this system in horizontal direction. Hence, the linear momentum will remain constant. Suppose, the block of mass M moves with a speed V and the other block with a speed v after losing contact with the spring. From conservation of linear momentum in horizontal direction, we have

$$MV - mv = 0 \text{ or } V = \frac{mv}{M} \qquad \text{(i)}$$

Initially, the energy of the system $= \frac{1}{2}kx^2$

Finally, the energy of the system $= \frac{1}{2}mv^2 + \frac{1}{2}MV^2$

As there is no friction, mechanical energy will remain conserved. Therefore,

$$\frac{1}{2}mv^2 + \frac{1}{2}MV^2 = \frac{1}{2}kx^2 \qquad \text{(ii)}$$

Solving Eqs. (i) and (ii), we get $v = \left[\dfrac{kM}{m(M+m)}\right]^{\frac{1}{2}} x$

and $V = \left[\dfrac{km}{M(M+m)}\right]^{\frac{1}{2}} x$

Illustration 1.21 **A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane and are at rest. The spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.**

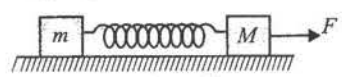

Fig. 1.44

Sol. We solve the situation in the reference frame of centre of mass. As only F is the external force acting on the system, due to this force, the acceleration of the centre of mass is $F/(M+m)$. Thus, with respect to centre of mass, there is a pseudoforce on the two masses in the opposite direction, the free body diagram of m and M with respect to centre of mass (taking centre of mass at rest) is shown in Fig. 1.45.

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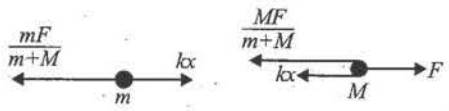

Fig. 1.45

Taking centre of mass at rest, if m moves maximum by a distance x_1 and M moves maximum by a distance x_2, then the work done by external forces (including pseudoforce) will be

$$W = \frac{mF}{m+M}\, x_1 + \left(F - \frac{MF}{m+M}\right) x_2$$

$$= \frac{mF}{m+M}(x_1 + x_2)$$

This work is stored in the form of potential energy of the spring as $U = (1/2)\, k\, (x_1 + x_2)^2$.

Thus, on equating we get the maximum extension in the spring, as after this instant the spring starts contracting.

$$\frac{1}{2} k (x_1 + x_2)^2 = \frac{mF}{m+M} (x_1 + x_2)$$

$$\Rightarrow \qquad x_{max} = x_1 + x_2 = \frac{2mF}{k(m+M)}$$

Concept Application Exercise 1.2

1. A bomb, initially at rest, explodes into several pieces. (a) Is linear momentum of the system conserved? (b) Is kinetic energy of the system conserved? Explain.
2. You are standing perfectly still and then you take a step forward. Before the step your momentum was zero, but afterward you have some momentum. Is the principle of conservation of momentum violated in this case?
3. Does the centre of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.
4. Discuss the possibility of conservation of linear momentum of a block moving on a rough inclined plane if $\mu = \tan\theta$.

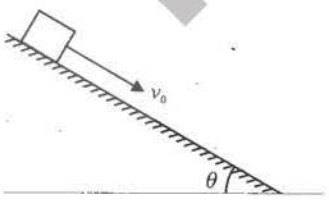

Fig. 1.46

5. A shell is fired from a cannon with a speed of 100 m/s at an angle 30° with the vertical (y-direction). At the highest point of its trajectory, the shell explodes into two fragments of masses in the ratio 1:2. The lighter fragment moves vertically upwards with an initial speed of 200 m/s. What is the speed of the heavier fragment at the time of explosion?
6. Figure 1.47 shows a block A of mass 6 m having a smooth semicircular groove of radius a placed on a smooth horizontal surface. A block B of mass m is released from a position in groove where its radius is horizontal. Find the speed of the bigger block when the smaller block reaches its bottommost position.

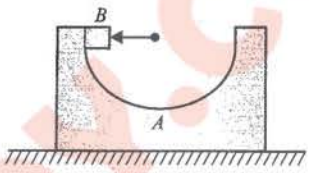

Fig. 1.47

7. Two friends A and B (each weighing 40 kg) are sitting on a frictionless platform some distance d apart. A rolls a ball of mass 4 kg on the platform towards B which B catches. Then B rolls the ball towards A and A catches it. The ball keeps on moving back and forth between A and B. The ball has a fixed speed of 5 m/s on the platform.
   a. Find the speed of A after he rolls the ball for the first time.
   b. Find the speed of A after he catches the ball for the first time.
   c. Find the speed of A and B after the ball has made five round trips and is held by A.
   d. How many times can A roll the ball?
   e. Where is the centre of mass of the system $A + B$ + ball at the end of the nth trip?
8. A smooth wedge of mass M rests on a smooth horizontal surface. A block of mass m is projected from its lowermost point with velocity v_0. What is the maximum height reached by the block?

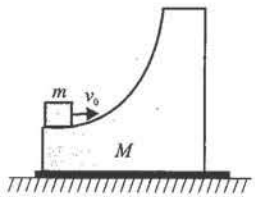

Fig. 1.48

9. Two identical buggies 1 and 2 with one man in each move along parallel rails. When the buggies are opposite to each other, the men jump in a direction perpendicular to the direction of motion of buggies, so as to exchange their places. As a consequence, buggy 1 stops and buggy 2 keeps moving in the same direction with its final velocity v. Find the initial velocities v_1 and v_2 of buggies. Mass of each buggy (without man) equals M, mass of each man is m; ignore frictional effects anywhere and the buggies are constrained to move along the rails only.

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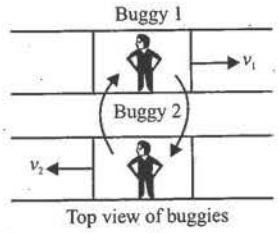

Fig. 1.49

10. In Fig. 1.50 a man stands on a boat floating in still water. The mass of the man and the boat is 60 kg and 120 kg, respectively.

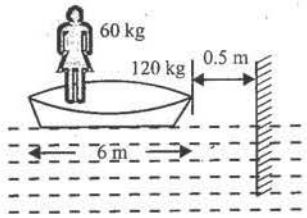

Fig. 1.50

a. If the man walks to the front of the boat and stops, what is the separation between the boat and the pier now?

b. If the man moves at a constant speed of 3 m/s relative to the boat, what is the total kinetic energy of the system (boat + man)? Compare this energy with the kinetic energy of the system if the boat was tied to the pier.

11. Two blocks of masses $m_1 = 2$ kg and $m_2 = 5$ kg are moving in the same direction along a frictionless surface with speeds 10 m/s and 3 m/s, respectively, m_2 being ahead of m_1. An ideal spring with $k = 1120$ N/m is attached to the back side of m_2. Find the maximum compression of the spring when the blocks collide. What are the final velocities of the blocks when they separarate?

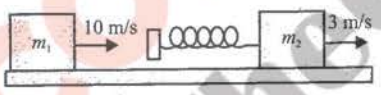

Fig. 1.51

12. An 80 kg boy and his 40 kg sister, both wearing roller blades, face each other at rest. The girl pushes the boy hard, sending him backward with velocity 3.0 m/s towards the west. Ignore friction. (a) Describe the subsequent motion of the girl. (b) How much chemical energy is converted into mechanical energy in the girl's muscles? (c) Is the momentum of the boy–girl system conserved in the pushing apart process? How can it be with no motion beforehand and plenty of motion afterward?

13. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one of them and the blocks are pushed together with the spring between them. A cord initially holding the blocks together is burned; after that, the block of mass $3M$ moves to the right with a speed of 2.00 m/s. (a) What is the velocity of the block of mass M. (b) Find the system's original elastic potential energy, taking $M = 0.350$ kg. (c) Is the original energy in the spring or in the cord? Explain your answer. (d) Is momentum of the system conserved in the bursting apart process? How can it be with large forces acting? How can it be with no motion beforehand and plenty of motion afterward?

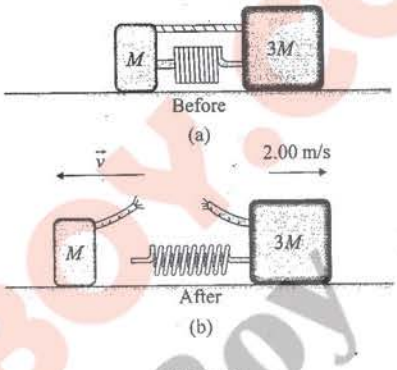

Fig. 1.52

14. A pendulum bob of mass 10^{-2} kg is raised to a height 5×10^{-2} m and then released. At the bottom of its swing, it picks up a mass 10^{-3} kg. To what height will the combined mass rise?

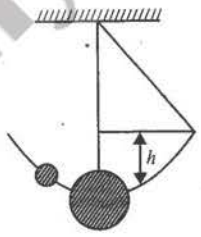

Fig. 1.53

15. A rifle man, who together with his rifle has a mass of 100 kg, stands on a smooth surface and fires 10 shots horizontally. Each bullet has a mass 10 g and a muzzle velocity of 800 m/s.

a. What velocity does the rifle man acquire at the end of 10 shots?

b. If the shots are fired in 10 s, what will be the average force exerted on him?

c. Compare his kinetic energy with that of 10 bullets.

16. A projectile of mass 50 kg is shot vertically upwards with an initial velocity of 100 m/s. After 5 s, it explodes into two fragments, one of which having a mass of 20 kg travels vertically up with a velocity of 150 m/s.

a. What is the velocity of the other fragment at that instant?

b. Calculate the sum of momenta of fragments 3 s after the explosion. What would have been the momentum of the projectile at this instant if there had been no explosion?

17. a. A rail road flat car of mass M can roll without friction along a straight horizontal track (Fig. 1.54). Initially, a man of mass m is standing on the car which is moving to the right with speed v_0. What is the change in velocity of the car if the man runs to the left so that his speed relative to the car is v_{rel} just before he jumps off at the left end?

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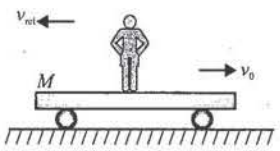

Fig. 1.54

b. If there are n men each of mass m on the car, should they all run and jump off together or should they run and jump one by one in order to give a greater velocity to the car?

18. i. A rail road car of mass M is moving without friction on a straight horizontal track with a velocity u. A man of mass m lands on it normally from a helicopter. What will be the new velocity of the car?

ii. If now the man begins to run on it with speed v_{rel} with respect to car in a direction opposite to motion of the car, what will be the new velocity of the car?

19. A shell of mass 2 kg moving at a rate of 4 m/s suddenly explodes into two equal fragments. The fragments go in directions inclined with the original line of motion with equal velocities. If the explosion imparts 48 J of translational kinetic energy to the fragments, find the velocity and direction of each fragment.

20. A mud ball at rest explodes into three fragments of masses in the ratio 1:2:1. The two equal masses move with velocities $2\hat{i}+5\hat{j}-6\hat{k}$ and $-4\hat{i}+3\hat{j}+2\hat{k}$. Find the velocity of the third mass.

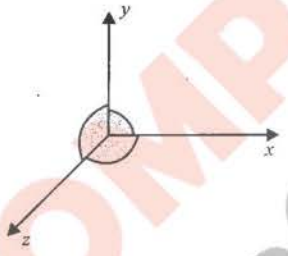

Fig. 1.55

IMPULSE

Impulse of a force F acting on a body is defined as

$$J = \int_{t_i}^{t_f} F\,dt$$

$$= \int F dt = \int m\frac{dv}{dt}\,dt = \int m\,dv$$

It is also defined as change in momentum

$$\vec{J} = \Delta\vec{P} \text{ (impulse–momentum theorem)}$$

Instantaneous Impulse

There are many occasions when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown, the value of their product (i.e., impulse) can be known by measuring the initial and final momenta. Thus, we can write

$$\vec{J} = \int \vec{F} dt = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$$

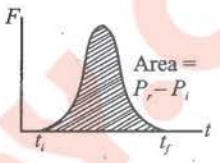

Fig. 1.56

Regarding the impulse, it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under the force–time (F–t) graph in the same time interval.

Points to Remember

1. Impulse is a vector quantity.
2. Dimensions are $[MLT^{-1}]$.
3. SI unit is kg m/s.
4. Direction is along the change in momentum.
5. Magnitude is equal to area under the F–t graph.
6. $J = \int F dt = F_{av}\Delta t$.
7. It is not a property of any particle, but it is a measure of the degree to which an external force changes the momentum of the particle.

Illustration 1.22 **The hero of a stunt film fires 50 g bullets from a machine gun, each at a speed of 1.0 km/s. If he fires 20 bullets in 4 s, what average force does he exert against the machine gun during this period?**

Sol. The momentum of each bullet = (0.050 kg) (1000 m/s) = 50 kg m/s.

The gun is imparted this much of momentum by each bullet fired. Thus, the rate of change of momentum of the gun is (50 kg m/s × 20)/4 s = 250 N

In order to hold the gun, the hero must exert a force of 250 N against the gun.

Impulsive Force

A force of relatively higher magnitude and acting for relatively shorter time is called impulsive force.

An impulsive force can change the momentum of a body in a finite magnitude in a very short time interval.

Impulsive force is a relative term. There is no clear boundary between an impulsive and a non-impulsive force.

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Note: ***Usually colliding forces are impulsive in nature.***

Since the application time is very small, hence, very little motion of the particle takes place.

Points to Remember

1. Gravitational force and spring force are always non-impulsive.
2. Normal, tension and friction are case dependent.
3. An impulsive force can only be balanced by another impulsive force.

1. **Impulsive normal:** In case of collision, normal forces at the surface of collision are always impulsive.

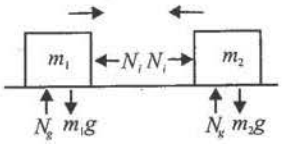

Fig. 1.57

N_i is impulsive; N_g is non-impulsive.

In Fig. 1.58, both normals are impulsive.

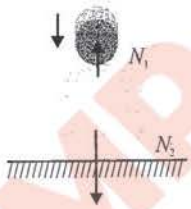

Fig. 1.58

In Fig. 1.59, N_1 and N_3 are impulsive; N_2 is non-impulsive.

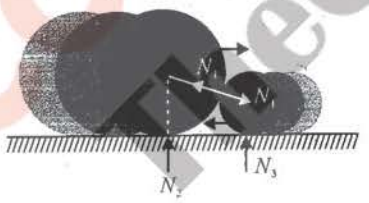

Fig. 1.59

In Fig. 1.60, both normals are impulsive.

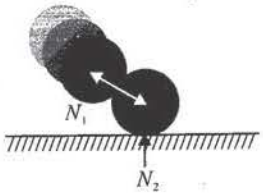

Fig. 1.60

2. **Impulsive friction:** If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.

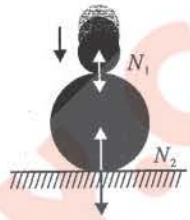

Fig. 1.61

In Fig. 1.61, friction at both surfaces is impulsive.

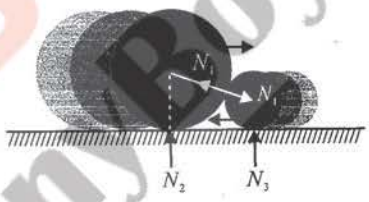

Fig. 1.62

In Fig. 1.62, friction due to N_2 is non-impulsive and that due to N_3 is impulsive.

3. **Impulsive tension:** When a string jerks, equal and opposite tension acts suddenly at each end. Consequently, equal and opposite impulses act on the bodies attached with the string in the direction of the string. There are two cases to be considered.

   (a) One end of the string is fixed: The impulse which acts at the fixed end of the string cannot change the momentum of the fixed object there. The object attached to the free end, however, will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string where no impulsive forces act.

   (b) Both ends of the string are attached to movable objects: In this case, equal and opposite impulses act on the two objects, producing equal and opposite changes in momentum. The total momentum of the system, therefore, remains constant, although the momentum of each individual object is changed in the direction of the string. However, no impulse acts perpendicular to the string and the momentum of each particle in this direction is unchanged.

Note: ***In case of rod, tension is always impulsive. In case of spring, tension is always non-impulsive.***

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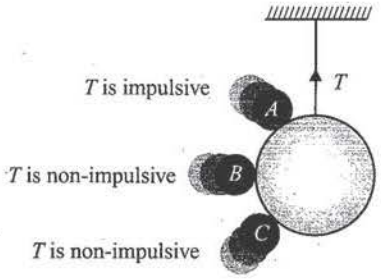

All normal are impulsive but tension T is impulsive only for ball A.

Fig. 1.63

Illustration 1.23 **A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially, the system is at rest; then a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v, find the speed with which the system moves just after the collision.**

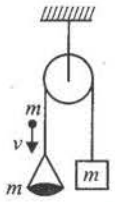

Fig. 1.64

Sol. Let the required speed be V. Further, let J_1 be the impulse between the particle and the pan and J_2 be the impulse imparted to the block and the pan by the string.

Using impulse as the change in momentum, we have the following equations.

For particle,

$$J_1 = mv - mV \quad \text{(i)}$$

For pan,

$$J_1 - J_2 = mV \quad \text{(ii)}$$

For block,

$$J_2 = mV \quad \text{(iii)}$$

Solving Eqs. (i)–(iii), we get $V = v/3$.

Illustration 1.24 **Two identical blocks A and B, connected by a massless string, are placed on a frictionless horizontal plane. A bullet having the same mass, moving with speed u, strikes block B from behind as shown. If the bullet gets embedded into block B, then find**

(a) the velocity of A, B, C after collision;

(b) impulse on A due to tension in the string;

(c) impulse on C due to normal force of collision;

(d) impulse on B due to normal force of collision.

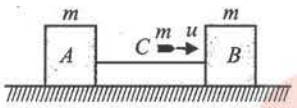

Fig. 1.65

Sol. (a) By conservation of linear momentum, $v = u/3$.

(b) $\int T dt = \frac{mu}{3}$

(c) $\int N dt = m\left(\frac{u}{3} - u\right) = \frac{-2mu}{3}$

(-ve sign indicates left direction)

(d) $\int (N - T)\, dt = \int N dt - \int T dt = \frac{mu}{3} \Rightarrow \int N dt = \frac{2mu}{3}$

Illustration 1.25 **A ball of mass 1 kg is attached to an inextensible string. The ball is released from the position shown in Fig. 1.66. Find the impulse imparted by the string to the ball immediately after the string becomes taut.**

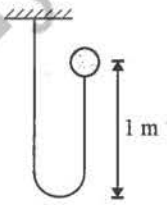

Fig. 1.66

Sol. The string will become taut when the particle will fall through a distance 2 m in downward direction. So the required impulse: $J = mu = 1 \times \sqrt{2 \times 10 \times 2} = \sqrt{40}$ kg m/s.

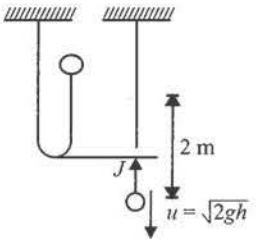

Fig. 1.67

Illustration 1.26 **Two particles A and B of equal mass m are attached by a string of length $2l$ and initially placed over a smooth horizontal table in the position shown in Fig. 1.68. Particle B is projected across the table with speed u perpendicular to AB as shown in the figure. Find the velocities of each particle after the string becomes taut and the magnitude of the impulse tension.**

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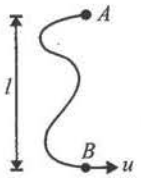

Fig. 1.68

Sol. When the string becomes taut, it can be depicted as in Fig. 1.69. $\theta = 30°$

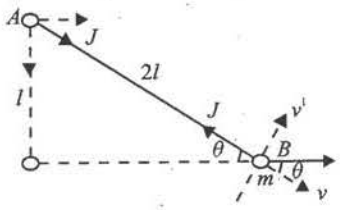

Fig. 1.69

Linear momentum will be conserved in any direction. Let us conserve the linear momentum in the direction of thread. Then,

$$mu\cos\theta = 2\,mv$$

$$\Rightarrow \quad v = \frac{1}{2}u\cos\theta = \frac{1}{2}u\frac{\sqrt{3}}{2} = \left(\frac{\sqrt{3}}{4}\right)u \qquad \text{(i)}$$

Fig. 1.70

From the impulse diagram of A (Fig. 1.70),

$$J = mv \Rightarrow J = \frac{1}{2}mu\cos\theta$$

$$= \frac{1}{2}mu\frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,mu}{4}$$

Velocity of ball B in the direction perpendicular to the string will remain constant. Hence,

$$v' = u\sin\theta = \frac{u}{2}$$

Hence, net velocity of B is

$$v_B = \sqrt{\left(\frac{u}{2}\right)^2 + (v)^2} = \frac{\sqrt{7}\,u}{4} \quad \text{and} \quad v_A = \frac{\sqrt{3}\,u}{4}$$

Illustration 1.27 **A sphere of mass m slides with velocity v on a frictionless surface towards a smooth inclined wall as shown in Fig. 1.71. If the collision with the wall is perfectly elastic, find (a) the impulse imparted by the wall on the sphere, (b) the impulse imparted by the floor on the sphere.**

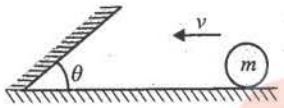

Fig. 1.71

Sol. As the collision is elastic, so the sphere will rebound with the same speed after collision.

$$J_1\cos\theta = J_2 \qquad \text{(i)}$$

$$-mv + J_1\sin\theta = mv \qquad \text{(ii)}$$

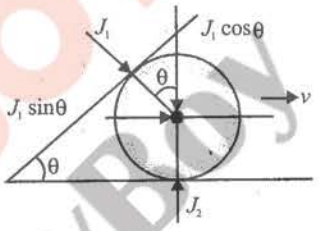

Fig. 1.72

Solving Eqs. (i) and (ii), we get

$$J_2 = 2\,mv\cot\theta$$

$$J_1 = 2\,mv\,\text{cosec}\,\theta$$

COLLISION OR IMPACT

Collision is an isolated event in which a strong force acts between two or more bodies for a short time, which results in change in their velocities. In a collision, a relatively large force acts on each colliding particle for a relatively short time. The basic idea of a 'collision' is that the motion of the colliding particles (or of at least one of them) changes rather abruptly and that we can make a relatively clean separation of times that are 'before the collision' and those that are 'after the collision'.

- *It is not necessary that a physical contact takes place in a collision, e.g., when an alpha particle (He^4) 'collides' with the nucleus of gold (Au^{197}), the force acting between them being repulsive—the particles may not touch, even then it may be called a 'collision'.*
- *When a space probe approaches a large planet, swings around it, and then continues its course with increased speed (a slingshot encounter), that too is a collision. The probe and planet do not actually "touch", but a collision does not require contact, and a collision force does not have to be a force of contact; it can just as easily be a gravitational force, as in this case.*

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Note:

- ***In a collision, particles may or may not come in physical contact.***
- ***The duration of collision Δt is negligible as compared to the usual time intervals of observation of motion.***
- ***In a collision, the effects of external non-impulsive forces such as gravity are not taken into account as due to small duration of collision (Δt). Average impulsive force responsible for collision is much larger than external forces acting on the system.***

The collision is, in fact, a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles.

Line of Impact

The line passing through the common normal to the surfaces in contact during the impact is called line of impact. The force during collision acts along this line on both the bodies.

Direction of line of impact can be determined by

(a) geometry of colliding objects such as spheres, discs, wedge, etc;

(b) direction of change of momentum.

If one particle is stationary before collision, then the line of impact will be along its motion after collision.

Classification of Collisions

(a) On the Basis of Line of Impact

(i) Head-on collision: The velocities of the particles are along the same line before and after the collision.

(ii) Oblique collision: The velocities of the particles are along different lines before and after the collision.

(b) On the Basis of Energy

(i) Elastic collision: In an elastic collision, the particles regain their shape and size completely after collision. That is, no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus, kinetic energy of a system after collision is equal to kinetic energy of a system before collision. Thus, in addition to the linear momentum, kinetic energy also remains conserved before and after collision.

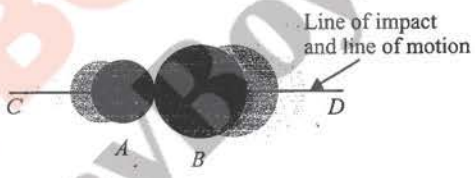
(ii) Inelastic collision: In an inelastic collision, the particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external forces, law of conservation of linear momentum still holds good.

(iii) Perfectly inelastic collision: If velocity of separation just after collision becomes zero, then the collision is perfectly inelastic. Collision is said to be perfectly inelastic if both the particles stick together after collision and move with the same velocity.

Note: ***Actually collision between all real objects is neither perfectly elastic nor perfectly inelastic, its inelastic in nature.***

Illustrations of Line of Impact and Collisions Based on Line of Impact

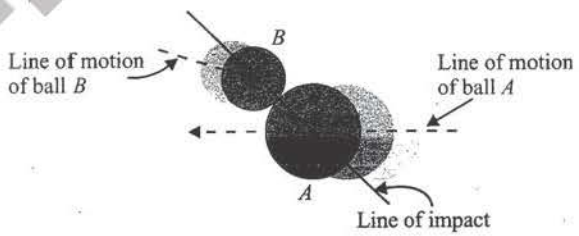
(i) Two balls A and B are approaching each other such that their centres are moving along line CD.



Head-on collision

Fig. 1.73

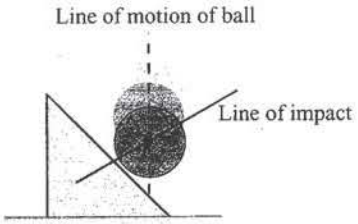
(ii) Two balls A and B are approaching each other such that their centres are moving along dotted lines as shown in Fig. 1.74.



Oblique collision

Fig. 1.74

(iii) Ball is falling on a stationary wedge.



Oblique collision

Fig. 1.75

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Coefficient of Restitution (e)

The coefficient of restitution is defined as the ratio of the impulses of recovery and deformation of either body.

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r\,dt}{\int F_d\,dt}$$

$$= \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

The most general expression for coefficient of restitution is

$$e = \frac{\text{Velocity of separation of points of contact along line of impact}}{\text{Velocity of approach of points of contact along line of impact}}$$

Illustration for Calculation of e

Two smooth balls A and B approach each other such that their centres are moving along the line CD in the absence of external impulsive force. Let the velocities of A and B just before collision be u_1 and u_2, respectively, and the velocities of A and B just after collision be v_1 and v_2, respectively.

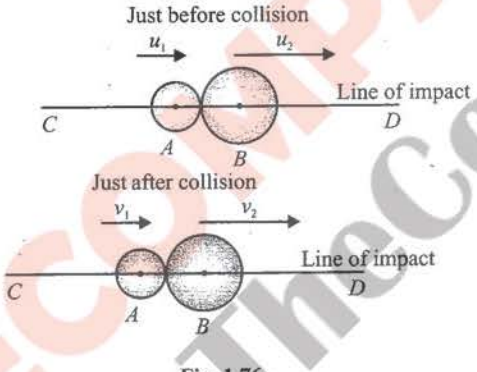

Fig. 1.76

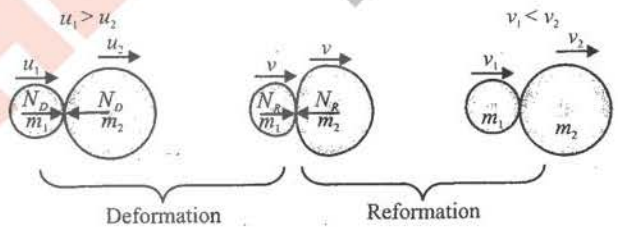

Fig. 1.77

Since, momentum is conserved for the system.

$\therefore \quad F_{ext} = 0$

$$\Rightarrow \quad m_1u_1 + m_2u_2 = (m_1 + m_2)v = m_1v_1 + m_2v_2$$

$$\Rightarrow \quad v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \quad \text{(i)}$$

Impulse of Deformation

J_D = change in momentum of any one body during deformation

$= m_2(v - u_2)$ for m_2

$= m_1(-v + u_1)$ for m_1

Impulse of Reformation

J_R = change in momentum of any one body during reformation

$= m_2(v_2 - v)$ for m_2

$= m_1(v - v_1)$ for m_1

$$e = \frac{\text{Impulse of reformation } (J_R)}{\text{Impulse of deformation } (J_D)} = \frac{v_2 - v}{v - u_2}$$

$$= \frac{v_2 - v_1}{u_1 - u_2}$$

[substituting v from Eq. (i)]

$$= \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

This is also known as Newton's experimental law.

Note: ***e is independent of shape and mass of the object but depends on the material.***

The coefficient of restitution is constant for two particular objects.

(a) For $e = 1$

$\Rightarrow$ Impulse of reformation = Impulse of deformation

$\Rightarrow$ Velocity of separation = Velocity of approach

$\Rightarrow$ Kinetic energy is conserved

$\Rightarrow$ Elastic collision

(b) For $e = 0$

$\Rightarrow$ Impulse of reformation = 0

$\Rightarrow$ Velocity of separation = 0

$\Rightarrow$ Kinetic energy is not conserved

$\Rightarrow$ Perfectly inelastic collision

(c) For $0 < e < 1$

$\Rightarrow$ Impulse of reformation < Impulse of deformation

$\Rightarrow$ Velocity of separation < Velocity of approach

$\Rightarrow$ Kinetic energy is not conserved

$\Rightarrow$ Inelastic collision

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Note:

- ***In case of contact collisions, e is always less than unity.***
- ***u_1, u_2, v_1 and v_2 can be positive, negative or zero.***

Illustration 1.28 **A ball is projected with a velocity u at an elevation from a point distance d from a smooth vertical wall in a plane perpendicular to it. After rebounding from the wall, it returns to the point of projection, prove that $u^2 \sin 2\alpha = gd(1 + 1/e)$. Hence, find the maximum distance d for which the ball can return to the point of projection.**

Sol. The only vertical force on the ball is mg throughout its motion because during impact it experiences a horizontal force from the wall. We can use

$$u_y t - \frac{1}{2} g t^2 = s_y$$

Let t be the total time of flight.

$$\therefore \quad 0 = u \sin \alpha t - \frac{1}{2} g t^2$$

$$\Rightarrow \quad t = \frac{2u \sin \alpha}{g}$$

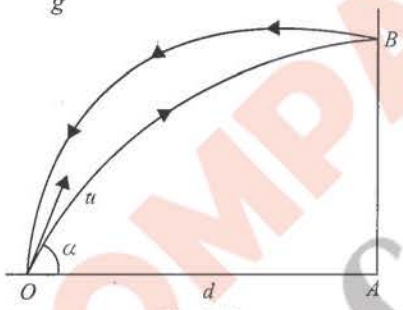

Fig. 1.78

Due to impact with the wall at B, the normal component (i.e., horizontal component) of velocity is reversed and becomes e times.

Horizontal velocity before impact = $u \cos \alpha$

and horizontal velocity after impact = $eu \cos \alpha$

Time taken to reach the wall, $t_1 = d/(u \cos \alpha)$

and time taken to come back to O from B,

$t_2 = d/(eu \cos \alpha)$ we have $t_1 + t_2 = t$

$$\Rightarrow \frac{d}{u \cos \alpha} + \frac{d}{eu \cos \alpha} = \frac{2u \sin \alpha}{g} \Rightarrow u^2 \sin 2\alpha = gd\left[1 + \frac{1}{e}\right]$$

As $\sin 2\alpha \le 1$,

$$\frac{gd}{u^2}\left[1 + \frac{1}{e}\right] \le 1 \Rightarrow d \le \frac{eu^2}{g(1+e)}$$

General Equation for Direct Impact

If u_1, u_2 are the velocities before the impact of the masses m_1, m_2 and v_1, v_2 are the velocities after the impact, then applying conservation of momentum, we get

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \text{ and } v_1 - v_2 = -e(u_1 - u_2)$$

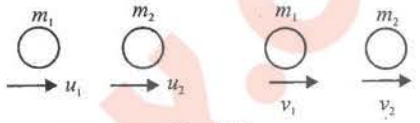

Fig. 1.79

Combining these equations, we get

$$v_1 = \frac{m_1 - em_2}{m_1 + m_2} u_1 + \frac{(1+e)m_2}{m_1 + m_2} u_2; \quad v_2 = \frac{(1+e)m_1}{m_2 + m_1} u_1 + \frac{m_2 - em_1}{m_2 + m_1} u_2$$

For a perfectly elastic collision, we can substitute $e = 1$.

Special case: For $e = 1$ and $m_1 = m_2 = m$, we get

$$v_1 = u_2 \text{ and } v_2 = u_1$$

i.e., when two particles of equal mass collide elastically and the collision is head-on, they exchange their velocities, e.g.,

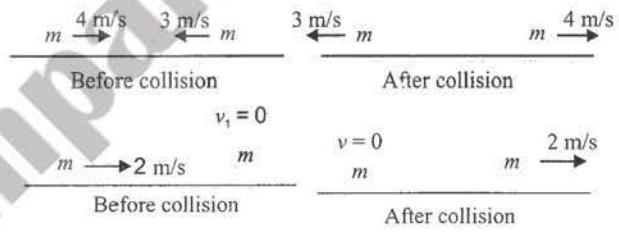

Fig. 1.80

If a body of mass m with initial velocity u strikes on another identical body but at rest and e is the coefficient of restitution, then their velocities v_1 and v_2 after collision will be

$$v_1 = \frac{1-e}{2} u; \; v_2 = \frac{1+e}{2} u$$

If a ball of mass m falls on ground from a vertical height h and rebounds with e as coefficient of restitution between them (Fig. 1.81), then the upward velocity of the ball after nth collision will be $(e^n u)$ and the maximum height attained by the ball after nth collision will be $(e^{2n} h)$.

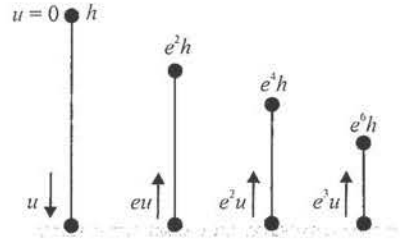

Fig. 1.81

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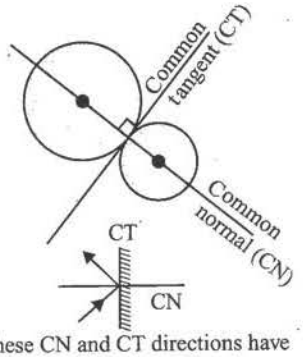
Oblique Collision in Case of Smooth Surfaces

Common Normal (CN)

Force is exerted in CN direction only. Both bodies exert equal and opposite forces (action and reaction) on each other.



These CN and CT directions have nothing to do with the directions of velocity of the two bodies.

Fig. 1.82

Hence, momentum and velocities change accordingly in CN direction.

Apply 'e' (coefficient of restitution) in the CN direction only.

Common Tangent (CT)

$F = 0$ (in case of smooth surfaces)

$F = \mu \times$ (normal reaction) in case of rough surfaces.

Neither momentum nor velocity changes in CT direction.

Illustration 1.29 **What will be the angle of reflection in case of an inelastic collision? and v = ?**

Sol. $v_1 = eu_1$ (CN by definition of e)

$$v_{\parallel} = u_{\parallel} \text{ (CT)}$$

$$\therefore \quad v\cos r = eu\cos i$$

$$v\sin r = u\sin i$$

Squaring and adding, we get

$$v = u\sqrt{\sin^2 i + e^2\cos^2 i}$$

Dividing, we get $\tan r = \dfrac{\tan i}{e}$

$$\Rightarrow \qquad \angle r > \angle i$$

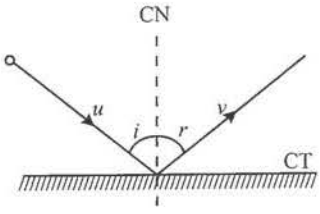


Fig. 1.83

Illustration 1.30 **If a ball strikes with a velocity u_1 at the wall which itself is approaching it with a velocity u_2, then find the velocity of the ball after collision with the wall.**

Sol. As the wall is heavy so after collision it will continue to move with the same velocity u_2. Relative velocity of separation is equal to relative velocity of approach. Hence,

$$v_1 - v_2 = -e(u_1 - u_2)$$

$$\Rightarrow \qquad v_1 - (-u_2) = -e[u_1 - (-u_2)]$$

$$\Rightarrow \qquad v_1 = -e(u_1 + u_2) - u_2$$

In case of perfectly elastic collision, $e = 1$.

$$v_1 = -u_1 - u_2 - u_2 = -(u_1 + 2u_2)$$

–ive sign indicates backward direction.

Illustration 1.31 **A ball drops from a ceiling of a room and after rebounding twice from the floor reaches a height equal to half that of the ceiling. Show that the coefficient of restitution is $\sqrt[4]{1/2}$.**

Sol. Let R = height of ceiling

$\Rightarrow$ Speed before the first impact = $\sqrt{2gh}$

$\Rightarrow$ Speed after the first impact = $e\sqrt{2gh}$

The ball comes back for the second impact.

Before the second impact, speed = $e\sqrt{2gh}$

After the second impact, speed = $e^2\sqrt{2gh}$

Height attained = $\dfrac{u^2}{2g} \Rightarrow \dfrac{h}{2} = \dfrac{[e^2\sqrt{2gh}\,]^2}{2g} \Rightarrow \dfrac{h}{2} = \dfrac{e^4(2gh)}{2g}$

$\Rightarrow e = \dfrac{1}{\sqrt[4]{2}}$

A particle 'B' moving along the dotted line collides with a rod also in state of motion as shown in Fig. 1.84. The particle B comes in contact with point C on the rod.

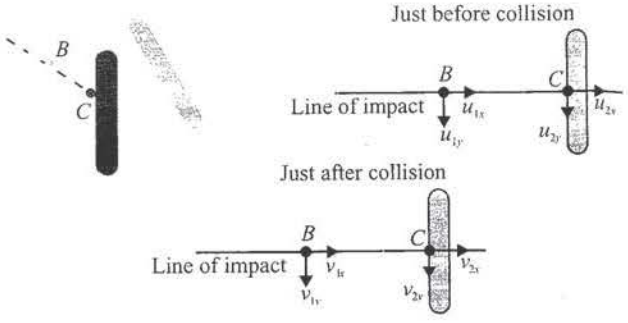


Fig. 1.84

To write down the expression for coefficient of restitution, e, we first draw the line of impact. Then we resolve the components of velocities of points of contact of both the bodies

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along line of impact just before and just after the collision. Then,

$$e = \frac{v_{2x} - v_{1x}}{u_{1x} - u_{2x}}$$

Illustration 1.32 **A ball of mass m hits the floor with a speed v_0 making an angle of incidence α with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.**

Sol. The component of velocity v_0 along common tangent direction $v_0 \sin \alpha$ will remain unchanged. Let v be the component along common normal direction after collision. Applying relative speed of separation = $e \times$ (relative speed of approach) along common normal direction, we get

$$v = ev_0 \cos \alpha$$

Thus, after collision, components of velocity v' are $v_0 \sin \alpha$ and $ev_0 \cos \alpha$.

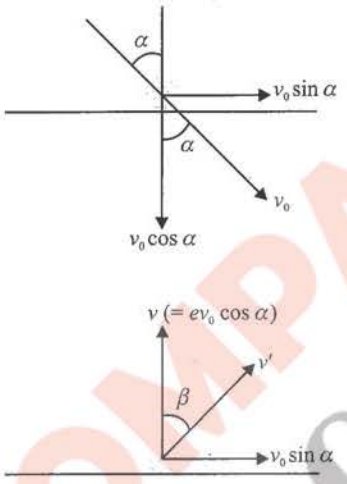


Fig. 1.85

$$\therefore \quad v' = \sqrt{(v_0 \sin \alpha)^2 + (ev_0 \cos \alpha)^2}$$

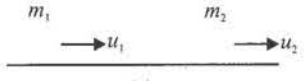
and

$$\tan \beta = \frac{v_0 \sin \alpha}{ev_0 \cos \alpha} = \frac{\tan \alpha}{e}$$

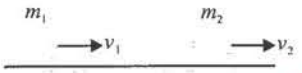
Note: *For elastic collision, $e = 1$.*

$$\therefore \quad v' = v_0 \text{ and } \beta = \alpha$$

Collision in One Dimension (Head-on)



(a)
Before collision



(b)
After collision

Fig. 1.86

$$u_1 > u_2,\ v_2 > v_1$$

$$e = \frac{v_2 - v_1}{u_1 - u_2} \Rightarrow (u_1 - u_2)e = (v_2 - v_1)$$

By momentum conservation, we get

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Now,

$$v_2 = v_1 + e(u_1 - u_2)$$

Hence,

$$v_1 = \frac{m_1u_1 + m_2u_2 - m_2e(u_1 - u_2)}{m_1 + m_2}$$

$$v_2 = \frac{m_1u_1 + m_2u_2 + m_1e(u_1 - u_2)}{m_1 + m_2}$$

Illustration 1.33 **Two identical balls are approaching towards each other on a straight line with velocity 2 m/s and 4 m/s, respectively. Find the final velocities, after elastic collision between them.**

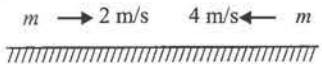


Fig. 1.87

Sol. The two velocities will be exchanged and the final motion is the reverse of the initial motion for both.

Fig. 1.88

Illustration 1.34 **Three balls A, B and C of same mass 'm' are placed on a frictionless horizontal plane in a straight line as shown. Ball A is moved with velocity u towards the middle ball B. If all the collisions are elastic, find the final velocities of all the balls.**

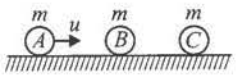


Fig. 1.89

Sol. A collides elastically with B and comes to rest but B starts moving with velocity u.

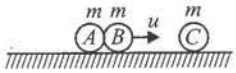


Fig. 1.90

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After a while B collides elastically with C and comes to rest but C starts moving with velocity u.

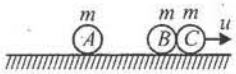

Fig. 1.91

Therefore, final velocities, $V_A = 0$, $V_B = 0$ and $V_C = u$.

Illustration 1.35 **Four identical balls A, B, C and D are placed in a line on a frictionless horizontal surface. A and D are moved with the same speed 'u' towards the middle as shown. Assuming elastic collisions, find the final velocities.**

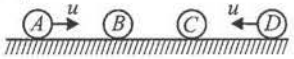

Fig. 1.92

Sol. A and D collide elastically with B and C, respectively, and come to rest but B and C start moving with velocity u towards each other as shown.

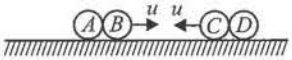

Fig. 1.93

B and C collide elastically and exchange their velocities to move in the opposite directions

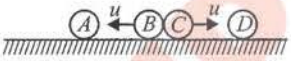

Fig. 1.94

Now, B and C collide elastically with A and D, respectively, and come to rest but A and D start moving with velocity u away from each other as shown.

Fig. 1.95

Therefore, final velocities, $V_A = u\ (\leftarrow)$, $V_B = 0$, $V_C = 0$ and $V_D = u\ (\rightarrow)$.

Illustration 1.36 **Two particles of masses m and $2m$ moving in opposite directions collide elastically with velocity $2v$ and v, respectively. Find their velocities after collision.**

Fig. 1.96

Sol. Let the final velocities of m and $2m$ be v_1 and v_2, respectively, as shown in Fig. 1.97.

Fig. 1.97

By conservation of momentum, we get

$$m(2v) + 2m(-v) = m(v_1) + 2m(v_2)$$

or $$0 = mv_1 + 2mv_2$$

or $$v_1 + 2v_2 = 0 \quad \text{(i)}$$

and since the collision is elastic,

$$v_2 - v_1 = 2v - (-v)$$

or $$v_2 - v_1 = 3v \quad \text{(ii)}$$

Solving the above two equations, we get

$$v_2 = v \text{ and } v_1 = -2v$$

That is, mass $2m$ returns with velocity v while mass m returns with velocity $2v$ in the direction shown in Fig. 1.98.

Fig. 1.98

Illustration 1.37 **A ball of mass m moving at a speed v makes a head-on collision with an identical ball at rest. The kinetic energy of the balls after the collision is 3/4th of the original. Find the coefficient of restitution.**

Sol. As we have seen in the above discussion that under the given conditions:

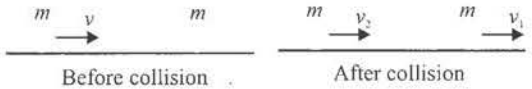

Fig. 1.99

$$v_1 = \left(\frac{1+e}{2}\right)v \text{ and } v_2 = \left(\frac{1-e}{2}\right)v$$

Given that $K_f = \frac{3}{4}K_i$

or $$\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{3}{4}\left(\frac{1}{2}mv^2\right)$$

Substituting the values, we get

$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4}$$

or $(1+e)^2 + (1-e)^2 = 3$ or $2 + 2e^2 = 3$

or $$e^2 = \frac{1}{2} \text{ or } e = \frac{1}{\sqrt{2}}$$

Illustration 1.38 **A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in Fig. 1.100. Assuming collision to be elastic, find the velocity of the ball immediately after the collision.**

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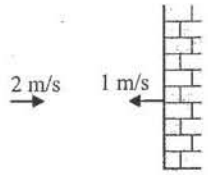

Fig. 1.100

Sol. The speed of the wall will not change after the collision. Let v be the velocity of the ball after collision in the direction shown in Fig. 1.101. Since collision is elastic ($e = 1$),

Speed of separation speed = Speed of approach

or $v - 1 = 2 + 1$ or $v = 4$ m/s

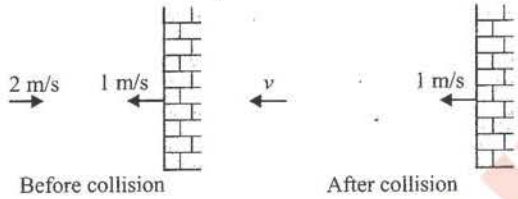

Fig. 1.101

Illustration 1.39 **Two balls of masses 2 kg and 4 kg are moved towards each other with velocities 4 m/s and 2 m/s, respectively, on a frictionless surface. After colliding, the 2 kg ball returns back with velocity 2 m/s. Then find**

(a) velocity of the 4 kg ball after collision

(b) coefficient of restitution e;

(c) impulse of deformation J_D;

(d) maximum potential energy of deformation;

(e) impulse of reformation J_R.

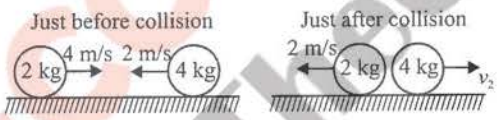

Fig. 1.102

Sol. (a) By momentum conservation,

$$2(4) - 4(2) = 2(-2) + 4(v_2)$$

$$\Rightarrow \quad v_2 = 1 \text{ m/s}$$

(b) $e = \dfrac{\text{Velocity of separation}}{\text{Velocity of approach}} = \dfrac{1 - (-2)}{4 - (-2)} = \dfrac{3}{6} = 0.5$

(c) At the maximum deformed state, by conservation of momentum, common velocity is $v = 0$.

$$J_D = m_2 (v - u_2) = 4[(0 - (-2)] = 8 \text{ N s}$$

(d) Potential energy at the maximum deformed state,

U = loss in kinetic energy during deformation

or $$U = \left(\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2\right) - \frac{1}{2}(m_1 + m_2) v^2$$

$$= \left(\frac{1}{2} 2(4)^2 + \frac{1}{2} 4(2)^2\right) - \frac{1}{2}(2 + 4)(0)^2$$

$$= 24 \text{ J}$$

(e) $J_R = m_2 (v_2 - v) = 4 (1 - 0) = 4$ N s

also $J_R = eJ_D = 0.5 \times 8 = 4$ N s

Illustration 1.40 **Two point particles A and B are placed in line on a frictionless horizontal plane. If particle A (mass 1 kg) is moved with velocity 10 m/s towards stationary particle B (mass 2 kg) and after collision the two move at an angle of 45° with the initial direction of motion, then find**

Fig. 1.103

(a) velocities of A and B just after collision.

(b) coefficient of restitution.

Sol. The very first step to solve such problems is to find the line of impact which is along the direction of force applied by A on B, resulting the stationary particle B to move. Thus, by watching the direction of motion of B, line of impact can be determined. In this case, the line of impact is along the direction of motion of B, i.e., 45° with the initial direction of motion of A.

(a) Let us apply the principle of conservation of momentum.

Along x-direction:

$$m_A u_A = m_A v_A \cos 45° + m_B v_B \cos 45°$$

or $$1(10) = 1(v_A \cos 45°) + 2(v_B \cos 45°)$$

or $$v_A + 2v_B = 10\sqrt{2} \qquad \text{(i)}$$

Along y-direction:

$$0 = m_A v_A \sin 45° - m_B v_B \sin 45°$$

or $$0 = 1(v_A \sin 45°) - 2(v_B \sin 45°)$$

or $$v_A = 2v_B \qquad \text{(ii)}$$

Solving the two equations, $v_A = \dfrac{10}{\sqrt{2}}$ m/s and $v_B = \dfrac{5}{\sqrt{2}}$ m/s

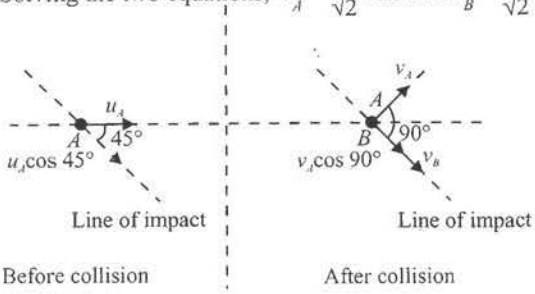

Fig. 1.104

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(b) $e = \dfrac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$

$$= \frac{v_B - v_A \cos 90°}{u_A \cos 45°} = \frac{\frac{5}{\sqrt{2}} - 0}{\frac{10}{\sqrt{2}}} = \frac{1}{2}$$

Hence, P is constant.

Illustration 1.41 **A bullet of mass 50 g is fired from below into the bob of mass 450 g of a long simple pendulum as shown in Fig. 1.105. The bullet remains inside the bob and the bob rises through a height of 1.8 m. Find the speed of the bullet.**

Fig. 1.105

Sol. Let the speed of the bullet be v. Let the common velocity of the bullet and the bob, after the bullet is embedded into the bob, be V. By the principle of conservation of linear momentum,

$$V = \frac{(0.05 \text{ kg})\, v}{0.45 \text{ kg} + 0.05 \text{ kg}} = \frac{v}{10}$$

The string becomes loose and the bob will go up with a deceleration of $g = 10 \text{ m/s}^2$. As it comes to rest at a height of 1.8 m, using the equation $v^2 = u^2 + 2ax$, we get

$$1.8 \text{ m} = \frac{\left(\frac{v}{10}\right)^2}{2 \times 10 \text{ m/s}^2} \quad \text{or} \quad v = 60 \text{ m/s}$$

Illustration 1.42 **A small ball of mass m collides with a rough wall having coefficient of friction μ at an angle θ with the normal to the wall. If after collision the ball moves with angle α with the normal to the wall and the coefficient of restitution is e, then find the reflected velocity v of the ball just after collision.**

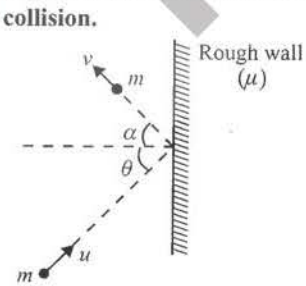

Fig. 1.106

Sol. $mv \cos\alpha - [m(-u\cos\theta)] = \int N dt$

$$mv \sin\alpha - mu \sin\theta = -\mu \int N dt$$

and $e = \dfrac{v\cos\alpha}{u\cos\theta} \Rightarrow v\cos\alpha = eu\cos\theta$ (i)

or $mv\sin\alpha - mu\sin\theta = -\mu(mv\cos\alpha + mu\cos\theta)$ (ii)

From Eqs. (i) and (ii), $v = \dfrac{u}{\sin\alpha}[\sin\theta - \mu\cos\theta\,(e+1)]$

Collision in Two Dimensions (Oblique)

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remains constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.
2. No component of impulse acts along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remains unchanged along this direction.
3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remains conserved before and after collision in any direction.
4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction we can apply.

Relative speed of separation = e × (relative speed of approach)

If the velocities of colliding masses are not linear, then it is known as oblique collision.

From law of conservation of momentum, we have the following.

Along x-axis:

$$m_1u_1\cos\alpha + m_2u_2\cos\beta = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2$$

Along y-axis:

$$m_1u_1\sin\alpha + m_2u_2\sin\beta = m_1v_1\sin\theta_1 + m_2v_2\sin\theta_2$$

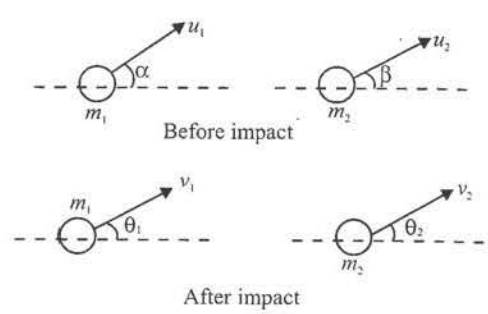

Fig. 1.107

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For Oblique Impact

When two bodies collide obliquely, their relative velocity, resolved along their common normal after the impact, is in a constant ratio to their relative velocity before impact (resolved along common normal) and is in the opposite direction.

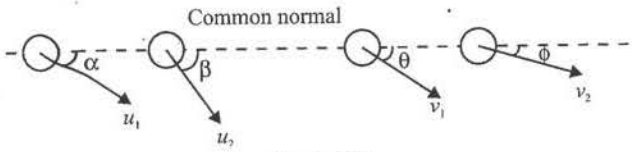

Fig. 1.108

$$\frac{v_1 \cos\theta - v_2 \cos\phi}{u_1 \cos\alpha - u_2 \cos\beta} = -e$$

$$\Rightarrow v_1 \cos\theta - v_2 \cos\phi = -e(u_1 \cos\alpha - u_2 \cos\beta)$$

Illustration 1.43 **A ball of mass *m* makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.**

Sol. Method 1: In head-on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along common normal direction, $v \cos\theta$, becomes zero after collision, while that of ball 2 becomes $v \cos\theta$. While the components along common tangent direction of both the particles remain unchanged. Thus, the components along common tangent and common normal direction of both the balls in tabular form are given in Table 1.1.

Table 1.1

Ball	Component along common tangent direction		Component along common normal direction	
	Before collision	After collision	Before collision	After collision
1	$v \sin\theta$	$v \sin\theta$	$v \cos\theta$	0
2	0	0	0	$v \cos\theta$

From Table 1.1 and Fig. 1.109, we see that both the balls move at right angle after collision with velocities $v \sin\theta$ and $v \cos\theta$.

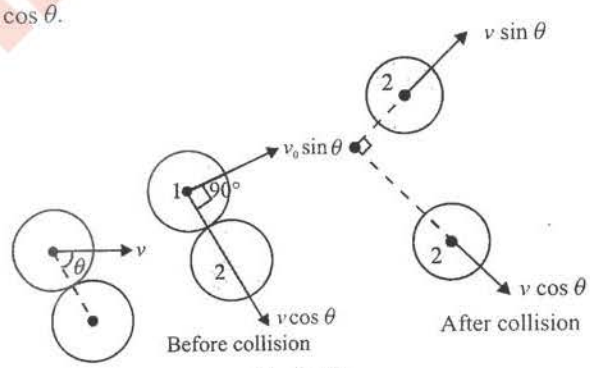

Fig. 1.109

Method 2: $mu = mv_1 \cos\theta_1 + mv_2 \cos\theta_2$ (i)

$0 = mv_1 \sin\theta_1 + mv_2 \sin\theta_2$ (ii)

$\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$ (iii)

Squaring and adding Eqs. (i) and (ii), we get

$$u^2 = v_1^2 + v_2^2 + 2v_1v_2\cos(\theta_1 + \theta_2)$$

Using Eq. (iii), we have $u^2 = v_1^2 + v_2^2$.

Note:

- ***When two identical bodies have an oblique elastic collision, with one particle at rest before collision, then the two particles will go in perpendicular directions.***
- ***The colliding balls of the same mass in an elastic direct impact interchange their velocities.***

Illustration 1.44 **Two spheres are moving towards each other. Both have the same radius but their masses are 2 kg and 4 kg. If the velocities are 4 m/s and 2 m/s, respectively, and coefficient of restitution is $e = 1/3$, find**

(a) the common velocity along the line of impact;

(b) final velocities along line of impact;

(c) impulse of deformation;

(d) impulse of reformation;

(e) maximum potential energy of deformation;

(f) loss in kinetic energy due to collision.

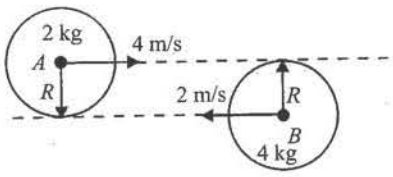

Fig. 1.110

Sol. In ΔABC, $\sin\theta = \frac{BC}{AB} = \frac{R}{2R} = \frac{1}{2} \Rightarrow \theta = 30°$

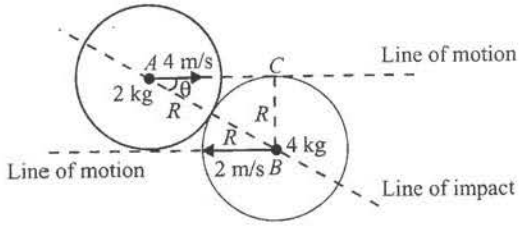

Fig. 1.111

(a) By conservation of momentum along line of impact

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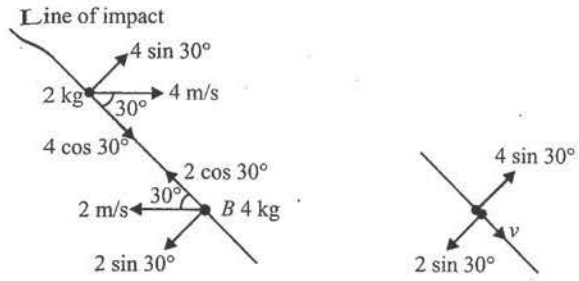

Fig. 1.112

$$2(4\cos 30°) - 4(2\cos 30°) = (2+4)v$$

or $v = 0$ (common velocity along line of impact)

(b)

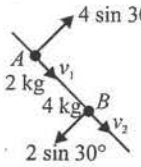

Fig. 1.113

Let v_1 and v_2 be the final velocities of A and B, respectively, then by conservation of momentum along the line of impact,

$$2(4\cos 30°) - 4(2\cos 30°) = 2(v_1) + 4(v_2)$$

or $0 = v_1 + 2v_2$ (i)

Coefficient of restitution,

$$e = \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

or $\frac{1}{3} = \frac{v_2 - v_1}{4\cos 30° + 2\cos 30°}$

or $v_2 - v_1 = \sqrt{3}$ (ii)

From the above two equations,

$$v_1 = \frac{-2}{\sqrt{3}} \text{ m/s and } v_2 = \frac{1}{\sqrt{3}} \text{ m/s}$$

(c) $J_D = m_1(v - u_1) = 2(0 - 4\cos 30°) = -4\sqrt{3}$ N s

(d) $J_R = eJ_D = \frac{1}{3}(-4\sqrt{3}) = -\frac{4}{\sqrt{3}}$ N s

(e) Maximum potential energy of deformation is equal to loss in kinetic energy during deformation up to maximum deformed state. Hence,

$$U = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \frac{1}{2}(m_1+m_2)v^2$$

$$= \frac{1}{2}\times 2(4\cos 30°)^2 + \frac{1}{2}\times 4(-2\cos 30°)^2 - \frac{1}{2}\times(2+4)(0)^2$$

or $U = 18$ J

(f) Loss in kinetic energy,

$$\Delta KE = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$= \frac{1}{2}\times 2(4\cos 30°)^2 + \frac{1}{2}\times 4(-2\cos 30°)^2 - \left[\frac{1}{2}\times 2\left(\frac{2}{\sqrt{3}}\right)^2 + \frac{1}{2}\times 4\left(\frac{1}{\sqrt{3}}\right)^2\right]$$

$\Delta KE = 16$ J

Illustration 1.45 **Two equal spheres of mass m are in contact on a smooth horizontal table. A third identical sphere impinges symmetrically on them and is reduced to rest. Prove that $e = 2/3$ and find the loss in KE.**

Sol. Let u be the velocity of sphere A before impact. As the spheres are identical, the triangle ABC formed by joining their centres is equilateral. The spheres B and C will move in directions AB and AC after impact making an angle of 30° with the original line of motion of sphere A.

Let v be the speed of the other spheres after impact.

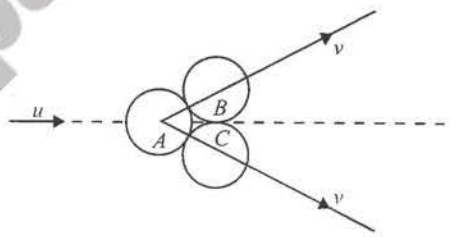

Fig. 1.114

From momentum conservation,

$$mu = mv\cos 30° + mv\cos 30°$$

$$u = v\sqrt{3} \quad \text{(i)}$$

From Newton's experimental law, for an oblique collision, we have to take components along the normal, i.e., along AB for spheres A and B. Hence,

$$v_B - v_A = -e(u_B - u_A)$$

$$\Rightarrow \quad v - 0 = -e(0 - u\cos 30°)$$

$$v = eu\cos 30° \quad \text{(ii)}$$

Combining Eqs. (i) and (ii), we get $e = 2/3$.

$$\text{Loss in KE} = \frac{1}{2}mu^2 - 2\left(\frac{1}{2}mv^2\right)$$

$$= \frac{1}{2}mu^2 - m\left(\frac{u}{\sqrt{3}}\right)^2 = \frac{1}{6}mu^2$$

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