

MATHEMATICS

Calculus

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Jx - some special segs
R1 - some segs
R2 - Circle
Straight line

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Contents

Chapter 1 Functions

Number System and Inequalities	1.2
Number System	1.2
Intervals	1.2
Some Facts About Inequalities	1.2
Generalized Method of Intervals	1.3
Frequently Used Inequalities	1.3
Sign Scheme of $F(x) = f_1(x)f_2(x)f_3(x)\dots f_n(x)$	1.4
Function	1.4
Definition of Function	1.5
Function as a Set of Ordered Pairs	1.5
Distinction Between a Relation and a Function by Graphs	1.5
Domain, Co-Domain, Range	1.6
Different Types of Functions	1.6
Quadratic Function	1.6
Modulus Function	1.8
Trigonometric Functions	1.10
Inverse Trigonometric Functions	1.13
Exponential and Logarithmic Functions	1.15
Greatest Integer and Fractional Part Function	1.17
Signum Function	1.19
Functions of the Form $f(x) = \max. \{g_1(x), g_2(x), \dots, g_n(x)\}$ or $f(x) = \min. \{g_1(x), g_2(x), \dots, g_n(x)\}$	1.20
Different Types of Mappings (Functions)	1.21
One-One and Many-One Functions	1.21
Onto and Into Functions	1.22
Number of Functions (Mappings)	1.22
Even and Odd Functions	1.24
Even Function	1.24
Odd Function	1.24
Extension of Domain	1.26
Periodic Functions	1.26
Important Facts about Periodic Functions	1.26
Composite Function	1.27
Properties of Composite Functions	1.28
Inverse Functions	1.29
Properties of Inverse Functions	1.29
Identical Function	1.31
Transformation of Graphs	1.32
Miscellaneous Solved Problems	1.35

1.1

Exercises

Subjective Type	1.40
Objective Type	1.40
Multiple Correct Answers Type	1.47
Reasoning Type	1.49
Linked Comprehension Type	1.50
Matrix-Match Type	1.51
Integer Type	1.53
Archives	1.54
Answers and Solutions	1.57
Subjective Type	1.57
Objective Type	1.60
Multiple Correct Answers Type	1.71
Reasoning Type	1.75
Linked Comprehension Type	1.76
Matrix-Match Type	1.79
Integer Type	1.81
Archives	1.84

Chapter 2 Limits

Concept of Limits	2.2
Formal Definition of Limit	2.2
Neighbourhood (NBD) of a Point	2.2
Left- and Right-Hand Limits	2.2
Existence of Limit	2.2
Indeterminate Forms	2.3
Difference between Limit of Function at $x = a$ and $f(a)$	2.3
Algebra of Limits	2.4
Sandwich Theorem for Evaluating Limits	2.5
Use of Expansions in Evaluating Limits	2.6
Some Important Expansions	2.6
Evaluation of Algebraic Limits	2.7
Direct Substitution Method	2.7
Factorization Method	2.8
Rationalization Method	2.8
Evaluation of Algebraic Limit	2.9
Using Some Standard Limits	2.9
Evaluation of Algebraic Limits at Infinity	2.10
Evaluation of Trigonometric Limits	2.11
Evaluation of Exponential and Logarithmic Limits	2.13

Limits of the Form $\lim_{x \rightarrow a} (f(x))^{g(x)}$	2.15	Objective Type	3.19
Form: $0^0, \infty^0$	2.15	Multiple Correct Answers Type	3.24
Form: 1^∞	2.15	Reasoning Type	3.26
L' Hopital's Rule for Evaluating Limits	2.16	Linked Comprehension Type	3.27
Finding Unknowns When Limit is Given	2.18	Matrix-Match Type	3.28
Miscellaneous Solved Problems	2.18	Integer Type	3.29
<i>Exercises</i>	2.22	Archives	3.29
Subjective Type	2.22	<i>Answers and Solutions</i>	3.33
Objective Type	2.22	Subjective Type	3.33
Multiple Correct Answers Type	2.27	Objective Type	3.36
Reasoning Type	2.28	Multiple Correct Answers Type	3.46
Linked Comprehension Type	2.29	Reasoning Type	3.49
Matrix-Match Type	2.29	Linked Comprehension Type	3.51
Integer Type	2.30	Matrix-Match Type	3.53
Archives	2.31	Integer Type	3.56
<i>Answers and Solutions</i>	2.32	Archives	3.58
Subjective Type	2.32		
Objective Type	2.36		
Multiple Correct Answers Type	2.45		
Reasoning Type	2.47		
Linked Comprehension Type	2.48		
Matrix-Match Type	2.49		
Integer Type	2.51		
Archives	2.54		
Chapter 3 Continuity and Differentiability	3.1	Chapter 4 Methods of Differentiation	4.1
Continuity	3.2	Geometrical Meaning of a Derivative	4.2
Definition of Continuity of a Function	3.2	Standard Derivatives	4.3
Directional Continuity	3.2	Some Standard Substitutions	4.3
Continuity in Interval	3.2	Differentiation of Inverse Trigonometric Functions	4.4
Types of Discontinuity	3.3	Theorems on Derivatives	4.5
Removable Discontinuity	3.3	Differentiation of Composite Functions (Chain Rule)	4.6
Non-removable Discontinuity	3.3	Differentiation of Implicit Functions	4.8
Continuity of Special Types of Functions	3.8	Differentiation of Functions in Parametric Form	4.9
Continuity of Functions in Which Greatest Integer Function is Involved	3.8	Differentiation Using Logarithm	4.10
Continuity of Functions in Which Signum Function is Involved	3.9	Differentiation of One Function w.r.t. Other Function	4.11
Continuity of Functions Involving Limit $\lim_{n \rightarrow \infty} a^n$	3.9	Differentiation of Determinants	4.12
Continuity of Functions in which $f(x)$ is Defined Differently for Rational and Irrational Values of x	3.10	Higher Order Derivatives	4.13
Continuity of Composite Functions	3.10	Problems Based on First Definition of Derivative	4.14
Intermediate Value Theorem	3.12	Miscellaneous Solved Problems	4.15
Differentiability	3.12	<i>Exercises</i>	4.18
Existence of Derivative	3.12	Subjective Type	4.18
Differentiability and Continuity	3.12	Objective Type	4.19
How Can a Function Fail to be Differentiable	3.13	Multiple Correct Answers Type	4.24
Differentiability Using First Definition of Derivatives	3.14	Reasoning Type	4.24
Differentiability Using Theorems on Differentiability	3.14	Linked Comprehension Type	4.25
Differentiability Using Graphs	3.14	Matrix-Match Type	4.26
Differentiability by Differentiation	3.17	Integer Type	4.27
<i>Exercises</i>	3.18	Archives	4.27
Subjective Type	3.18	<i>Answers and Solutions</i>	4.29
		Subjective Type	4.29
		Objective Type	4.33
		Multiple Correct Answers Type	4.42
		Reasoning Type	4.44
		Linked Comprehension Type	4.44
		Matrix-Match Type	4.46
		Integer Type	3.46
		Archives	4.50

CHAPTER 5 Application of Derivatives: Tangents and Normals, Rate Measure

5.1

Equation of Tangents and Normals	5.2
Tangent From an External Point	5.3
Condition for Which Given Line is Tangent or Normal to the Given Curve	5.4
Length of Tangent, Normal, Sub-Tangent and Sub-Normal	5.5
Angle between the Curves	5.6
Miscellaneous Applications	5.7
Shortest Distance between Two Curves	5.8
Interpretation of dy/dx as a Rate Measurer	5.9
Approximations	5.11
Mean Value Theorems	5.12
Rolle's Theorem	5.12
Lagrange's Mean Value Theorem	5.14
Cauchy's Mean Value Theorem	5.15
Exercises	5.16
Subjective	5.16
Objectives	5.17
Multiple Correct Answers Type	5.20
Reasoning Type	5.21
Link Comprehension Type	5.21
Matrix-Match Type	5.22
Integer Type	5.23
Archives	5.23
Answers and Solutions	5.24
Subjective	5.24
Objective	5.28
Multiple Correct Answers Type	5.35
Reasoning Type	5.37
Link Comprehension Type	5.38
Matrix-Match Type	5.39
Integer Type	5.40
Archives	5.42

Chapter 6 Monotonicity and Maxima-Minima of Functions

6.1

Monotonicity: Introduction	6.2
Monotonicity of a Function at a Point	6.2
Monotonicity in an Interval	6.2
Separating the Intervals of Monotonicity	6.5
Point of Inflection	6.6
Inequalities Using Monotonicity	6.7
Extremum	6.8
Introduction	6.8
Basic Theorem of Maxima and Minima	6.8
Tests for Local Maximum/Minimum	6.9
When $f(x)$ is Differentiable at $x = a$	6.9
When $F(x)$ is not Differentiable at $x = a$	6.12
Concept of Global Maximum/Minimum	6.14
Global Maximum/Minimum in $[a, b]$	6.14

Global Maximum/Minimum in (a, b)	6.14
Nature of Roots of Cubic Polynomial	6.15
Application of Extremum	6.16
Optimization	6.18
Plane Geometry	6.18
Coordinate Geometry	6.19
Solid Geometry	6.20
Exercises	6.22
Subjective	6.22
Objective	6.22
Multiple Correct Answers Type	6.28
Reasoning Type	6.30
Link Comprehension Type	6.30
Matrix-Match Type	6.33
Integer Type	6.34
Archives	6.35
Answers and Solutions	6.38
Subjective	6.38
Objective	6.42
Multiple Correct Answers Type	6.54
Reasoning Type	6.59
Link Comprehension Type	6.60
Matrix-Match Type	6.64
Integer Type	6.66
Archives	6.68

Chapter 7 Indefinite Integration

7.1

Integration as Reverse Process of Differentiation	7.2
Elementary Integration	7.2
Fundamental Integration Formulae	7.2
Properties of Indefinite Integration	7.3
Some More Standard Formulae	7.7
Integration by Substitutions	7.8
Bi-quadratic Form	7.13
Some Standard Trigonometric Substitutions	7.14
Integration by Parts	7.16
Theorem	7.16
Integration by Cancellation	7.18
Integration by Partial Fractions	7.20
Some Definitions	7.20
Integrations of Irrational Functions	7.23
Standard Formulae	7.21
Miscellaneous Solved Problems	7.25
Exercise	7.26
Answers and Solutions	7.35
Subjective	7.26
Objective	7.27
Multiple Correct Answers Type	7.33
Reasoning Type	7.34
Link Comprehension Type	7.34
Matrix-Match Type	7.35
Integer Type	7.36
Archives	7.37

Answers and Solutions

Subjective	7.38
Objective	7.38
Multiple Correct Answers Type	7.41
Reasoning Type	7.50
Link Comprehension Type	7.52
Matrix-Match Type	7.52
Integer Type	7.54
Archives	7.55

Chapter 8 Definite Integration / *Level 1 typed* **8.1**

Definite Integration	8.2
Definite Integral as the Limit of a Sum (Integration by First Principle Rule)	8.2
Second Fundamental Theorem of Integral Calculus	8.2
Geometrical Interpretation of the Definite Integral	8.3
Properties of Definite Integrals	8.7
Property I	8.7
Property II	8.7
Property III	8.7
Property IV	8.9
Property V	8.11
Important Result	8.12
Property VI	8.12
Property VII	8.12
Definite Integration of Odd and Even Functions	8.13
Property I	8.13
Property II	8.14
Definite Integration of Periodic Functions	8.15
Property I	8.15
Property II	8.15
Property III	8.15
Leibnitz's Rule	8.17
Inequalities	8.19
Property I	8.19
Property II	8.19
Property III	8.20
Property IV	8.20
Solved Problems	8.21
Exercises	8.25
Subjective Type	8.25
Objective Type	8.26
Multiple Correct Answers Type	8.33
Reasoning Type	8.34
Linked Comprehension Type	8.35
Matrix-Match Type	8.37
Integer Type	8.38
Archives	8.39
Answers and Solutions	8.44
Subjective Type	8.44
Objective Type	8.48
Multiple Correct Answers Type	8.62
Reasoning Type	8.66
Linked Comprehension Type	8.67
Matrix-Match Type	8.70

Integer Type	8.72
Archives	8.75

Chapter 9 Area / *typed* **9.1**

Different Cases of Bounded Area	9.2
Curve Tracing	9.3
Some Standard Area	9.5
Area Bounded by Curves While Integrating Along y-axis	9.6
Exercises	9.7
Subjective Type	9.7
Objective Type	9.7
Multiple Correct Answers Type	9.10
Reasoning Type	9.10
Linked Comprehension Type	9.11
Matrix-Match Type	9.12
Integer Type	9.13
Archives	9.13
Subjective	9.13
Answers and Solutions	9.15
Subjective Type	9.15
Objective Type	9.17
Multiple Correct Answers Type	9.28
Reasoning Type	9.29
Linked Comprehension Type	9.30
Matrix-Match Type	9.34
Integer Type	9.36
Archives	9.40

Chapter 10 Differential Equations / *Level 1 typed* **10.1**

Differential Equations of First Order and First Degree	10.2
Order and Degree of Differential Equation	10.2
Formation of Differential Equations	10.2
Solution of a Differential Equation	10.4
Method of Variable Separation	10.4
Differential Equations Reducible to the Separable Variable Type	10.5
Homogeneous Equations	10.6
Equations Reducible to the Homogenous Form	10.6
Linear Differential Equations	10.7
Bernoulli's Equation	10.8
Differential Equation Reducible to the Linear Form	10.9
General Form of Variable Separation	10.10
Geometrical Applications of Differential Equation	10.10
Finding Orthogonal Trajectories	10.12
Statistical Applications of Differential Equation	10.13
Physical Applications of Differential Equation	10.13
Exercises	10.15
Subjective Type	10.15
Objective Type	10.15
Multiple Correct Answers Type	10.21
Reasoning Type	10.22
Linked Comprehension Type	10.23
Matrix-Match Type	10.24
Integer Type	10.25
Archives	10.25
Subjective	10.25

Objective	10.26	Matrix-Match Type	10.48
<i>Answers and Solutions</i>	10.27	Integer Type	10.49
Multiple Correct Answers Type	10.44	Archives	10.51
Reasoning Type	10.46		
Linked Comprehension Type	10.46		
		Appendix: Solutions to Concept Application Exercises	A.1-A.65

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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

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CHAPTER

1

Functions

- Number System and Inequalities
- Function
- Different Types of Functions
- Different Types of Mappings (Functions)
- Even and Odd Functions
- Periodic Functions
- Composite Function
- Inverse Functions
- Identical Function
- Transformation of Graphs

NUMBER SYSTEM AND INEQUALITIES

Number System

Natural Numbers

The set of numbers $\{1, 2, 3, 4, \dots\}$ is called natural numbers, and is denoted by N , i.e., $N = \{1, 2, 3, \dots\}$.

Integers

The set of numbers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is called integers, and the set is denoted by I or Z .

Here, we represent

- Positive integers $= \{1, 2, 3, 4, \dots\} = \text{Natural numbers}$
- Negative integers $= \{\dots, -4, -3, -2, -1\}$
- Non-negative integers (or N_0) $= \{0, 1, 2, 3, 4, \dots\} = \text{Whole numbers}$
- Non-positive integers $= \{\dots, -3, -2, -1, 0\}$

Rational Numbers

A number which can be written as $\frac{a}{b}$, where a and b are integers, $b \neq 0$ and H.C.F. of a and b is 1, is called a rational number, and a set of rational numbers is denoted by Q .

Note:

- Every integer is a rational number as it could be written as $Q = \frac{a}{b}$ (where $b = 1$).
- All recurring decimals are rational numbers; e.g., $n = 0.3333\dots = 1/3$.
- "Two consecutive rational numbers" is meaningless.
- The set of rational numbers cannot be expressed in roster form.

Irrational Numbers

Those values which could be neither terminated nor expressed as recurring decimals are irrational numbers (i.e., such numbers cannot be expressed in $\frac{a}{b}$ form). Their set is denoted by Q^c (i.e., complement of Q), e.g., $\sqrt{2}, \pi, -\frac{1}{\sqrt{3}}, 2 + \sqrt{2}, \dots$

Note:

- "Two consecutive irrational numbers" is meaningless.
- The set of irrational numbers cannot be expressed in roster form.

Real Numbers

The set of numbers that contains both rational and irrational numbers is called real numbers and is denoted by R . As from the above definitions, it could be shown that real numbers can be expressed on number line with respect to origin as

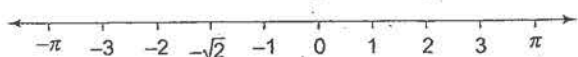


Fig. 1.1

Note:

- The set R represents the set of continuous values (not discrete values).
- Between any two irrational numbers, there exist infinite rational numbers and between two rational numbers there exist infinite irrational numbers.

Intervals

The set of numbers between any two real numbers is called interval. The following are the types of interval.

Closed Interval

$$x \in [a, b] \equiv \{x : a \leq x \leq b\}$$



Fig. 1.2

Open Interval

$$x \in (a, b) \text{ or }]a, b[\equiv \{x : a < x < b\}$$



Fig. 1.3

Semi-Open or Semi-Closed Interval

$$x \in [a, b[\text{ or }]a, b] \equiv \{x : a \leq x < b\}$$



$$x \in]a, b] \text{ or } (a, b] \equiv \{x : a < x \leq b\}$$



Fig. 1.4

Note:

- A set of all real numbers can be expressed as $(-\infty, \infty)$
- $x \in (-\infty, a) \cup (b, \infty) \Rightarrow x \in R - [a, b]$
- $x \in (-\infty, a] \cup [b, \infty) \Rightarrow x \in R - (a, b)$

Some Facts About Inequalities

The following are some very useful points to remember:

- $a \leq b \Rightarrow$ either $a < b$ or $a = b$
- $a < b$ and $b < c \Rightarrow a < c$
- $a < b \Rightarrow -a > -b$, i.e., the inequality sign reverses if both sides are multiplied by a negative number
- $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$
- $a < b \Rightarrow ka < kb$ if $k > 0$ and $ka > kb$ if $k < 0$
- $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$
- $a + \frac{1}{a} \geq 2$ for $a > 0$ and equality holds for $a = 1$
- $a + \frac{1}{a} \leq -2$ for $a < 0$ and equality holds for $a = -1$
- If $x > 2 \Rightarrow 0 < \frac{1}{x} < \frac{1}{2}$
- If $x < -3 \Rightarrow -\frac{1}{3} < \frac{1}{x} < 0$
- If $x < 2$, then we must consider $-\infty < x < 0$ or $0 < x < 2$ (as for $x = 0$, $\frac{1}{x}$ is not defined), then

$$\lim_{x \rightarrow \infty} \frac{1}{x} > \frac{1}{x} > \lim_{x \rightarrow 0^-} \frac{1}{x} \text{ or } \lim_{x \rightarrow 0^+} \frac{1}{x} > \frac{1}{x} > \frac{1}{2}$$

$$\Rightarrow 0 > \frac{1}{x} > -\infty \text{ or } \infty > \frac{1}{x} > \frac{1}{2}$$

$$\Rightarrow \frac{1}{x} \in (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$$

1. Squaring an inequality:

If $a < b$, then $a^2 < b^2$ does not follow always:

Consider the following illustrations:

$$2 < 3 \Rightarrow 4 < 9, \text{ but } -4 < 3 \Rightarrow 16 > 9$$

Also if $x > 2 \Rightarrow x^2 > 4$, but for $x < 2 \Rightarrow x^2 \geq 0$

$$\text{If } 2 < x < 4 \Rightarrow 4 < x^2 < 16$$

$$\text{If } -2 < x < 4 \Rightarrow 0 \leq x^2 < 16$$

$$\text{If } -5 < x < 4 \Rightarrow 0 \leq x^2 < 25$$

Generalized Method of Intervals

$$\text{Let } F(x) = (x - a_1)^{k_1} (x - a_2)^{k_2} \dots (x - a_{n-1})^{k_{n-1}} (x - a_n)^{k_n}$$

Here $k_1, k_2, \dots, k_n \in \mathbb{Z}$ and a_1, a_2, \dots, a_n are fixed real numbers satisfying the condition

$$a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n$$

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm:

- We mark the numbers a_1, a_2, \dots, a_n on the number axis and put plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n .
- Then we put plus sign in the interval on the left of a_n if k_n is an even number and minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule:
 - When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we put plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we put minus sign.

Frequently Used Inequalities

- $(x - a)(x - b) < 0 \Rightarrow x \in (a, b)$, where $a < b$
- $(x - a)(x - b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$
- $x^2 \leq a^2 \Rightarrow x \in [-a, a]$
- $x^2 \geq a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$
- If $ax^2 + bx + c < 0$, ($a > 0$) $\Rightarrow x \in (\alpha, \beta)$, where α, β ($\alpha < \beta$) are the roots of the equation $ax^2 + bx + c = 0$
- If $ax^2 + bx + c > 0$, ($a > 0$) $\Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where α, β ($\alpha < \beta$) are the roots of the equation $ax^2 + bx + c = 0$

Example 1.1 Solve $(2x + 1)(x - 3)(x + 7) < 0$.

$$\text{Sol. } (2x + 1)(x - 3)(x + 7) < 0$$

Sign scheme of $(2x + 1)(x - 3)(x + 7)$ is as follows:

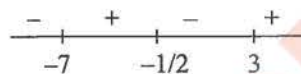


Fig. 1.5

Hence, solution is $(-\infty, -7) \cup (-1/2, 3)$.

Example 1.2 Solve $\frac{2}{x} < 3$.

$$\text{Sol. } \frac{2}{x} < 3$$

$$\Rightarrow \frac{2}{x} - 3 < 0$$

(we cannot cross multiply with x , as x can be negative or positive)

$$\Rightarrow \frac{2 - 3x}{x} < 0$$

$$\Rightarrow \frac{3x - 2}{x} > 0$$

$$\Rightarrow \frac{(x - 2/3)}{x} > 0$$

Sign scheme of $\frac{(x - 2/3)}{x}$ is as follows:

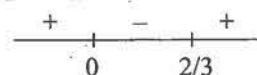


Fig. 1.6

$$\Rightarrow x \in (-\infty, 0) \cup (2/3, \infty)$$

Example 1.3 Solve $\frac{2x - 3}{3x - 5} \geq 3$.

$$\text{Sol. } \frac{2x - 3}{3x - 5} \geq 3$$

$$\Rightarrow \frac{2x - 3}{3x - 5} - 3 \geq 0$$

$$\Rightarrow \frac{2x - 3 - 9x + 15}{3x - 5} \geq 0$$

$$\Rightarrow \frac{-7x + 12}{3x - 5} \geq 0$$

$$\Rightarrow \frac{7x - 12}{3x - 5} \leq 0$$

Sign scheme of $\frac{7x - 12}{3x - 5}$ is as follows:

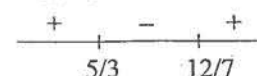


Fig. 1.7

$$\Rightarrow x \in (5/3, 12/7]$$

$x = 5/3$ is not included in the solution as at $x = 5/3$, denominator becomes zero.

Example 1.4 Solve $(x - 1)^2(x + 4) < 0$.

$$\text{Sol. } (x - 1)^2(x + 4) < 0$$

Sign scheme of $(x - 1)^2(x + 4)$ is as follows:

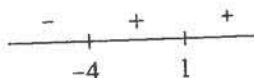


Fig. 1.8

Sign of expression does not change at $x = 1$ as $(x - 1)$ factor has even power.

Hence, solution of (1) is $x \in (-\infty, -4)$.

Example 1.5 Solve $x > \sqrt{1-x}$.

Sol. Given inequality can be solved by squaring both sides. But sometimes squaring gives extraneous solutions that do not satisfy the original inequality. Before squaring, we must restrict x for which terms in the given inequality are well-defined.

$x > \sqrt{1-x}$. Here x must be positive.

Here $\sqrt{1-x}$ is defined only when $1-x \geq 0$ or $x \leq 1$ (1)

Squaring the given inequality we get $x^2 > 1-x$

$$\Rightarrow x^2 + x - 1 > 0 \Rightarrow \left(x - \frac{-1-\sqrt{5}}{2}\right) \left(x - \frac{-1+\sqrt{5}}{2}\right) > 0$$

$$\Rightarrow x < \frac{-1-\sqrt{5}}{2} \text{ or } x > \frac{-1+\sqrt{5}}{2} \quad (2)$$

$$\text{From (1) and (2), } x \in \left(\frac{\sqrt{5}-1}{2}, 1\right] \text{ (as } x \text{ is +ve)}$$

Example 1.6 Find the domain of $f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$.

$$\text{Sol. } f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$$

$$\Rightarrow 1-\sqrt{1-\sqrt{1-x^2}} \geq 0$$

$$\Rightarrow \sqrt{1-\sqrt{1-x^2}} \leq 1$$

$$\Rightarrow 1-\sqrt{1-x^2} \leq 1$$

$$\Rightarrow \sqrt{1-x^2} \geq 0$$

$$\Rightarrow 1-x^2 \geq 0$$

$$\Rightarrow x^2 \leq 1 \Rightarrow x \in [-1, 1]$$

Sign Scheme of

$$F(x) = f_1(x) f_2(x) f_3(x) \dots f_n(x)$$

Put the values of x , which are roots of the equation, $f_1(x) = 0, f_2(x) = 0, \dots, f_n(x) = 0$ on the number line and follow the same procedure explained in the above problems.

Example 1.7 Solve $(x-1)|x+1|\cos x > 0$, for $x \in [-\pi, \pi]$.

Sol. Let $f(x) = (x-1)|x+1|\cos x$

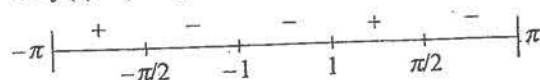


Fig. 1.9

$$\cos x = 0 \Rightarrow x = \pm \pi/2.$$

So, critical points are $-\pi/2, -1, 1, \pi/2$

For $x \in (\pi/2, \pi)$, $\cos x < 0 \Rightarrow f(x) < 0$

At $x = \pi/2$, and $x = 1$, $f(x)$ changes sign as shown in the sign scheme.

At $x = -1$, $f(x)$ does not change sign as $|x+1| > 0$ for all x .

Hence, $f(x) > 0 \Rightarrow x \in (-\pi, -\pi/2) \cup (1, \pi/2)$.

Example 1.8 Find the domain of

$$f(x) = \sqrt{x-4-2\sqrt{(x-5)}} - \sqrt{x-4+2\sqrt{(x-5)}}$$

$$\begin{aligned} \text{Sol. } f(x) &= \sqrt{x-4-2\sqrt{(x-5)}} - \sqrt{x-4+2\sqrt{(x-5)}} \\ &= \sqrt{x-5-2\sqrt{(x-5)}+1} - \sqrt{x-4+2\sqrt{(x-5)}+1} \\ &= \sqrt{(\sqrt{(x-5)}-1)^2} - \sqrt{(\sqrt{(x-5)}+1)^2} \\ &= |\sqrt{(x-5)}-1| - |\sqrt{(x-5)}+1| \end{aligned}$$

Hence domain is $[5, \infty)$

Concept Application Exercise 1.1

Find the domain of the following functions:

$$1. f(x) = \frac{x-3}{(x+3)\sqrt{x^2-4}}$$

$$2. f(x) = \sqrt{2-x} - \frac{1}{\sqrt{9-x^2}}$$

$$3. f(x) = \sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{1-x}{1+x}}$$

$$4. f(x) = \sqrt{\left(\frac{2}{x^2-x+1} - \frac{1}{x+1} - \frac{2x-1}{x^3+1}\right)}$$

$$5. f(x) = \sqrt{x-\sqrt{1-x^2}}$$

$$6. \text{ Find the range of } f(x) = \frac{x^2+1}{x^2+2}$$

$$7. \text{ Solve } x(e^x-1)(x+2)(x-3)^2 \leq 0.$$

FUNCTION

Roughly speaking, term function is used to define the dependence of one physical quantity on another, e.g., volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. This dependence of V on r would be denoted as $V = f(r)$ and we would simply say that V is a function of r . Here f is purely a symbol (for that matter, any other letter could have been used in place of f), and it is simply used to represent the dependence of one quantity on the other.

Definition of Function

Function can be easily defined with the help of the concept of mapping. Let A and B be any two non-empty sets. "A function from A and B is a rule or correspondence that assigns to each element of set A , one and only one element of set B ". Let the correspondence be f . Then mathematically we write $f: A \rightarrow B$ where $y = f(x)$, $x \in A$ and $y \in B$. We say that y is the image of x under f (or x is the pre-image of y).

- A mapping $f: A \rightarrow B$ is said to be a function if each element in the set A has a image in set B . It is possible that a few elements in the set B are present which are not the images of any element in set A .
- Every element in set A should have one and only one image. That means it is impossible to have more than one image for a specific element in set A . Functions cannot be multi-valued (A mapping that is multi-valued is called a relation from A and B).

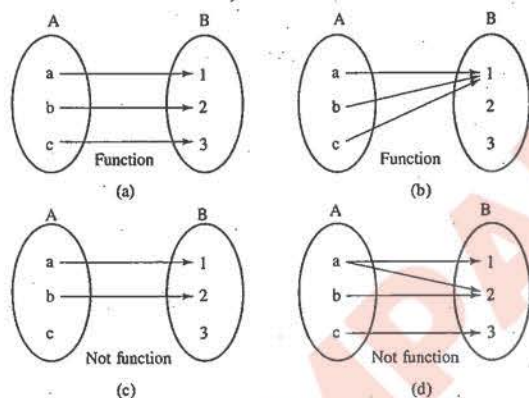


Fig. 1.10

Let us consider some other examples to make the above mentioned concepts clear.

- Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ where $y^2 = x$. This cannot be considered a function as each $x \in \mathbb{R}^+$ would have two images namely $\pm\sqrt{x}$. Hence, it does not represent a function. Thus, it would be a relation.
- Let $f: [-2, 2] \rightarrow \mathbb{R}$, where $x^2 + y^2 = 4$. Here $y = \pm\sqrt{4 - x^2}$, that means for every $x \in [-2, 2]$ we would have two values of y (except when $x = \pm 2$). Hence, it does not represent a function.
- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ where $y = x^3$. Here for each $x \in \mathbb{R}$ we would have a unique value of y in the set \mathbb{R} (as cube of any two distinct real numbers are distinct). Hence, it would represent a function.

Function as a Set of Ordered Pairs

A function $f: A \rightarrow B$ can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and second element is the corresponding element of B .

As such a function $f: A \rightarrow B$ can be considered as a set of ordered pairs $(a, f(a))$ where $a \in A$ and $f(a) \in B$ which is the f image of a . Hence, f is a subset of $A \times B$.

As a particular type of relation, we can define a function as follows:

A relation R from a set A to a set B is called a function if

- each element of A is associated with some element of B
- each element of A has unique image in B

Thus, a function f from a set A to a set B is a subset of $A \times B$ in which each $a \in A$ appears in one and only one ordered pair belonging to f . Hence, a function f is a relation from A to B satisfying the following properties:

- $f \subset A \times B$
- $\forall a \in A \Rightarrow (a, f(a)) \in f$
- $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

Thus, the ordered pairs of f must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first element.

Note:

Every function is a relation but every relation is not necessarily a function.

Distinction between a Relation and a Function by Graphs (Vertical Line Test)

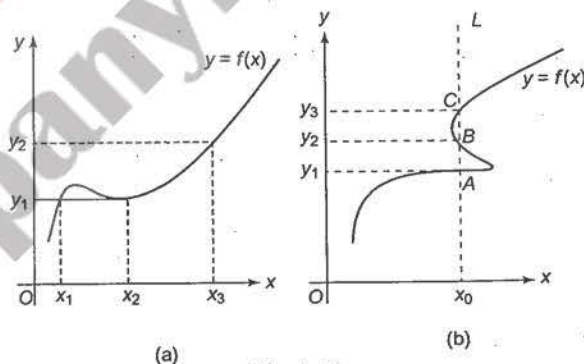


Fig. 1.11

The above figures show the graph of two arbitrary curves. In Fig. 1.11 (a), any line drawn parallel to y -axis would meet the curve at only one point. That means each element of A would have one and only one image. Thus, Fig. 1.11(a) represents the graph of a function.

In Fig. 1.11 (b), certain line parallel to y -axis, (e.g., line L) would meet the curve in more than one points (A, B and C). Thus, element x_0 of A would have three distinct images. Thus, this curve does not represent a function.

Hence, if $y = f(x)$ represents a function, lines drawn parallel to y -axis through different points corresponding to points of set X should meet the curve in one and only one point.

Consider the graph of following relations:

Equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a relation, which is a

combination of two functions $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$.

The upper branch represents function $y = b\sqrt{1 - \frac{x^2}{a^2}}$ and the

lower branch represents the function $y = -b\sqrt{1 - \frac{x^2}{a^2}}$.

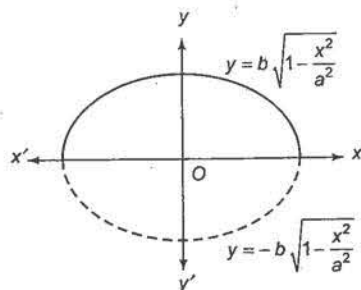


Fig. 1.12

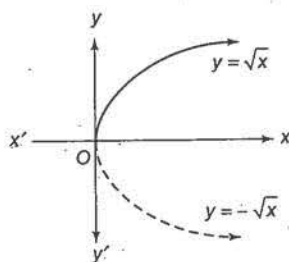
Graph of a parabola $y^2 = x$ 

Fig. 1.13

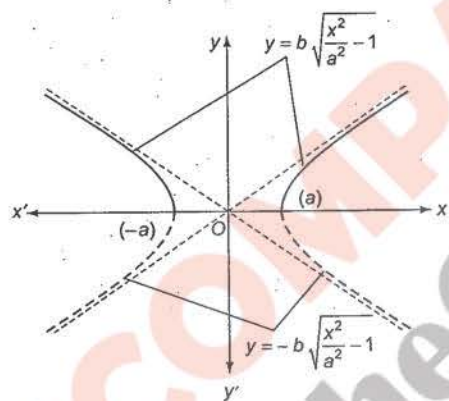
Graph of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 

Fig. 1.14

Domain, Co-Domain, Range

Let $f: A \rightarrow B$ be a function. In general, sets A and B could be any arbitrary non-empty sets. But at this level, we would confine ourselves only to real-valued functions. That means it would be invariably assumed that A and B are the subsets of real numbers.

Set A is called domain of the function f .

Set B is called co-domain of the function f .

The set of images of different elements of set A is called the range of the function f . It is obvious that a range would be a subset of co-domain as we may have few elements in co-domain which are not the images of any element of set A (of course, these elements of co-domain will not be included in the range). Range is also called domain of variation. Domain of function f is normally represented as $\text{Domain}(f)$. Range is represented as $\text{Range}(f)$. Note that sometimes domain of the function is not explicitly defined. In these cases, domain would mean the set of values of x

for which $f(x)$ assumes real values. For example, if $y = f(x)$ then $\text{Domain}(f) = \{x: f(x) \text{ is a real number}\}$.

Rules for the Domain of a Function

- $\text{Domain}(f(x) + g(x)) = \text{Domain } f(x) \cap \text{Domain } g(x)$
- $\text{Domain}(f(x) \times g(x)) = \text{Domain } f(x) \cap \text{Domain } g(x)$
- $\text{Domain}\left(\frac{f(x)}{g(x)}\right) = \text{Domain } f(x) \cap \text{Domain } g(x) \cap \{x: g(x) \neq 0\}$
- $\text{Domain } \sqrt{f(x)} = \text{Domain } f(x) \cap \{x: f(x) \geq 0\}$

Some Important Definitions

- Polynomial function:** If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$ where n is a non-negative integer and $a_0, a_1, a_2, \dots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n .
- Algebraic function:** y is an algebraic function of x , if it is a function that satisfies an algebraic equation of the form $P_0(x)y^n + P_1(x)y^{n-1} + \dots + P_{n-1}(x)y + P_n(x) = 0$ where n is a positive integer and $P_0(x), P_1(x), P_2(x)$ are polynomials in x . For example, $x^3 + y^3 - 3xy = 0$ or $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that all polynomial functions are algebraic but converse is not true.

A function that is not algebraic is called *Transcendental function*.

- Rational function:** A function that can be written as the quotient of two polynomial function is said to be a rational function.

If $P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, and

$Q(x) = b_0 + b_1x + b_2x^2 + \dots + b_mx^m$

be two polynomial functions, then the function f is

defined by $f(x) = \frac{P(x)}{Q(x)}$ is a rational function of x .

- Explicit function:** A function $y = f(x)$ is said to be an explicit function of x if the dependent variable y can be expressed in terms of independent variable x only. For example, (i) $y = x - \cos x$, (ii) $y = x + \log_e x - 2x^3$.
- Implicit function:** A function $y = f(x)$ is said to be an implicit function of x if y cannot be written in terms of x only. For example, (i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, (ii) $xy = \sin(x + y)$.
- Bounded functions:** A function is said to be bounded if $|f(x)| \leq M$, where M is a finite positive real number.
- Identity function:** The function $f: R \rightarrow R$ is called an identity function if $f(x) = x \forall x \in R$.

DIFFERENT TYPES OF FUNCTIONS

Quadratic Function

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$.

$$\text{We have } f(x) = a \left[x^2 + \frac{b}{a}x + \frac{c}{a} \right]$$

$$\begin{aligned}
 &= a \left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2} \right] \\
 &= a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right] \\
 &= a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a^2} \right] \Rightarrow \left(y + \frac{D}{4a} \right) = a \left(x + \frac{b}{2a} \right)^2.
 \end{aligned}$$

Thus, $y = f(x)$ represents a parabola whose axis is parallel to y -axis and vertex $A \left(-\frac{b}{2a}, -\frac{D}{4a} \right)$. For some values of x , $f(x)$ may be positive, negative or zero and for $a > 0$, the parabola opens upwards and for $a < 0$, the parabola opens downwards. This gives the following cases:

- a. $a > 0$ and $D < 0$, so $f(x) > 0 \forall x \in R$,
i.e., $f(x)$ is positive for all values of x .

Range of function is $\left[-\frac{D}{4a}, \infty \right)$

$x = -\frac{b}{2a}$ is a point of minima.

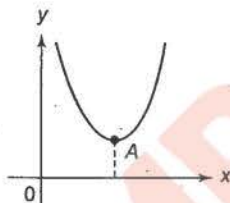


Fig. 1.15

- b. $a < 0$ and $D < 0$ so $f(x) < 0 \forall x \in R$,
i.e., $f(x)$ is negative for all values of x .

Range of function is $\left(-\infty, -\frac{D}{4a} \right]$

$x = -\frac{b}{2a}$ is a point of maxima.

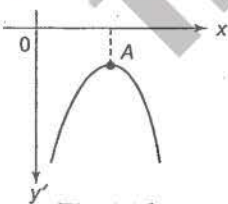


Fig. 1.16

- c. $a > 0$ and $D = 0$, so $f(x) \geq 0 \forall x \in R$, i.e.,
 $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.

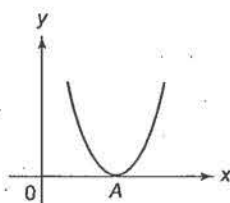


Fig. 1.17

- d. $a > 0$ and $D > 0$

Let $f(x) = 0$ have two real roots α and β (where $\alpha < \beta$) then.

$f(x) > 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0 \forall x \in (\alpha, \beta)$.

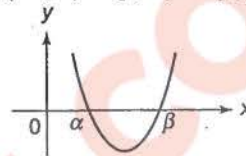


Fig. 1.18

- e. $a < 0$ and $D = 0$

so $f(x) \leq 0 \forall x \in R$, i.e., $f(x)$ is negative for all values of x except at vertex where $f(x) = 0$.

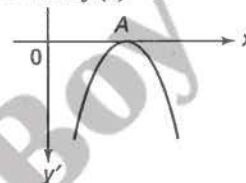


Fig. 1.19

- f. $a < 0$ and $D > 0$

Let $f(x) = 0$ have two roots α and β (where $\alpha < \beta$)

then $f(x) < 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0 \forall x \in (\alpha, \beta)$

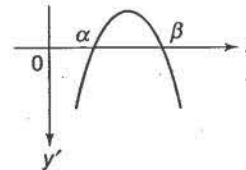


Fig. 1.20

Note: If $f(x) \geq 0, \forall x \in R \Rightarrow a > 0$ and $D \leq 0$
and if $f(x) \leq 0, \forall x \in R \Rightarrow a < 0$ and $D \leq 0$.

Example 1.9 Find the range of $f(x) = x^2 - x - 3$.

Sol. $f(x) = x^2 - x - 3 = \left(x - \frac{1}{2} \right)^2 - \frac{1}{4} - 3 = \left(x - \frac{1}{2} \right)^2 - \frac{13}{4}$

Now $\left(x - \frac{1}{2} \right)^2 \geq 0 \forall x \in R \Rightarrow \left(x - \frac{1}{2} \right)^2 - \frac{13}{4} \geq -\frac{13}{4}$,

$\forall x \in R$

Hence, range is $\left[-\frac{13}{4}, \infty \right)$.

Example 1.10 Find the domain and range of

$f(x) = \sqrt{x^2 - 3x + 2}$.

Sol. For domain $x^2 - 3x + 2 \geq 0$

$\Rightarrow (x - 1)(x - 2) \geq 0$

$\Rightarrow x \in (-\infty, 1] \cup [2, \infty)$

Now, $f(x) = \sqrt{x^2 - 3x + 2}$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 + 2 - \frac{9}{4}}$$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}$$

Now, the least permissible value of $\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$ is 0

when $\left(x - \frac{3}{2}\right) = \pm \frac{1}{2}$. Hence, the range is $[0, \infty)$.

Example 1.11 Find the range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.

Sol. $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$

$$= (\sqrt{6^x} - \sqrt{6^{-x}})^2 + (\sqrt{3^x} - \sqrt{3^{-x}})^2 + 6 \geq 6.$$

Hence, the range is $[6, \infty)$.

Example 1.12 Find the domain and range of

$$f(x) = \sqrt{x^2 - 4x + 6}.$$

Sol. $x^2 - 4x + 6 = (x - 2)^2 + 2$ which is always positive.

Hence, the domain is R .

$$\text{Now, } f(x) = \sqrt{(x - 2)^2 + 2}$$

The least value of $f(x)$ is $\sqrt{2}$ when $x - 2 = 0$.

Hence, the range is $[\sqrt{2}, \infty)$.

Example 1.13 Find the range of $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$.

Sol. Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$

$$\Rightarrow (1 - y)x^2 - (1 + y)x + 1 - y = 0$$

Now x is real, then $D \geq 0$

$$\Rightarrow (1 + y)^2 - 4(1 - y)^2 \geq 0$$

$$\Rightarrow (1 + y - 2 + 2y)(1 + y + 2 - 2y) \geq 0$$

$$\Rightarrow (3y - 1)(3 - y) \geq 0$$

$$\Rightarrow 3\left(y - \frac{1}{3}\right)(y - 3) \leq 0$$

$$\Rightarrow \frac{1}{3} \leq y \leq 3 \Rightarrow \text{The range is } \left[\frac{1}{3}, 3\right].$$

Example 1.14 Find the complete set of values of a such that

$$\frac{x^2 - x}{1 - ax} \text{ attains all real values.}$$

Sol. $y = \frac{x^2 - x}{1 - ax}$

$$\Rightarrow x^2 - x = y - axy$$

$$\Rightarrow x^2 + x(ay - 1) - y = 0$$

Since x is real $\Rightarrow (ay - 1)^2 + 4y \geq 0$

$$\Rightarrow a^2 y^2 + 2y(2 - a) + 1 \geq 0 \quad \forall y \in R$$

$$\Rightarrow \text{As } a^2 > 0, 4(2 - a)^2 - 4a^2 \leq 0 \Rightarrow 4 - 4a \leq 0 \Rightarrow a \in [1, \infty).$$

Concept Application Exercise 1.2

- Find the range of $f(x) = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$.
- Find the range of $f(x) = \sqrt{x - 1} + \sqrt{5 - x}$.
- If $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x , then find the values of a .
- Find the domain and range of $f(x) = \sqrt{3 - 2x - x^2}$.

Modulus Function

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} = \sqrt{x^2} = \max\{x, -x\}$$

Domain : R

Range : $[0, \infty)$

Nature : even function

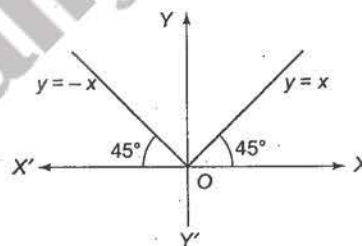


Fig. 1.21

$$y = |x - a| = \begin{cases} x - a, & x \geq a \\ a - x, & x < a \end{cases}, \text{ where } a > 0$$

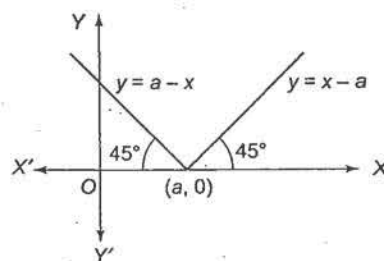
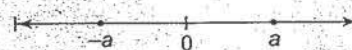


Fig. 1.21(a)

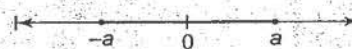
Properties of Modulus Function

a. $|x| = a \Rightarrow$ Points on the real number line whose distance from origin is a

$$\Rightarrow x = \pm a.$$



b. $|x| \leq a \Rightarrow x^2 \leq a^2$

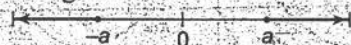


⇒ Points on the real number line whose distance from the origin is a or less than a

⇒ $-a \leq x \leq a$; ($a \geq 0$).

c. $|x| \geq a \Rightarrow x^2 \geq a^2$

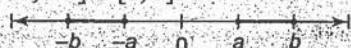
⇒ Points on the real number line whose distance from origin is a or greater than a



⇒ $x \leq -a$ or $x \geq a$; ($a \geq 0$).

d. $a \leq |x| \leq b \Rightarrow a^2 \leq x^2 \leq b^2$

⇒ $x \in [-b, -a] \cup [a, b]$.



e. $|x+y| = |x| + |y| \Leftrightarrow x$ and y have the same sign or at least one of x and y is zero or $xy \geq 0$.

f. $|x-y| = |x| - |y| \Rightarrow x \geq 0, y \geq 0$ and $|x| \geq |y|$ or $x \leq 0, y \leq 0$ and $|x| \geq |y|$.

g. $|x+y| \leq |x| + |y|$.

h. $|x+y| \geq ||x| - |y||$.

Example 1.15 Solve $|3x-2| \leq \frac{1}{2}$.

Sol. $|3x-2| \leq \frac{1}{2}$

$$\Rightarrow -\frac{1}{2} \leq 3x-2 \leq \frac{1}{2}$$

$$\Rightarrow \frac{3}{2} \leq 3x \leq \frac{5}{2}$$

$$\Rightarrow \frac{1}{2} \leq x \leq \frac{5}{6}$$

$$\Rightarrow x \in [1/2, 5/6].$$

Example 1.16 Solve $|x-1|-5 \geq 2$.

Sol. $|x-1|-5 \geq 2$

$$\Rightarrow |x-1|-5 \leq -2 \text{ or } |x-1|-5 \geq 2$$

$$\Rightarrow |x-1| \leq 3 \text{ or } |x-1| \geq 7$$

$$\Rightarrow -3 \leq x-1 \leq 3 \text{ or } x-1 \leq -7 \text{ or } x-1 \geq 7$$

$$\Rightarrow -2 \leq x \leq 4 \text{ or } x \leq -6 \text{ or } x \geq 8$$

$$\Rightarrow x \in (-\infty, -6] \cup [-2, 4] \cup [8, \infty).$$

Example 1.17 Solve $\frac{-1}{|x|-2} \geq 1$, where $x \in \mathbb{R}, x \neq \pm 2$ or find

$$\text{the domain of } f(x) = \frac{1-|x|}{|x|-2}.$$

Sol. Given $\frac{-1}{|x|-2} \geq 1$

$$\Rightarrow \frac{-1}{|x|-2} - 1 \geq 0$$

$$\Rightarrow \frac{-1-(|x|-2)}{|x|-2} \geq 0$$

$$\Rightarrow \frac{1-|x|}{|x|-2} \geq 0$$

$$\Rightarrow \frac{|x|-1}{|x|-2} \leq 0$$

$$\Rightarrow \frac{y-1}{y-2} \leq 0, \text{ where } y = |x|.$$



Fig. 1.22

$$\Rightarrow 1 \leq y < 2$$

$$\Rightarrow 1 \leq |x| < 2$$

$$\Rightarrow x \in (-2, -1] \cup [1, 2).$$

Example 1.18 Solve $\frac{|x+3|+x}{x+2} > 1$.

Sol. We have $\frac{|x+3|+x}{x+2} > 1$

Clearly, L.H.S. of this inequation is meaningful for $x \neq -2$.

$$\text{Given } \frac{|x+3|+x}{x+2} > 1$$

$$\Rightarrow \frac{|x+3|+x}{x+2} - 1 > 0$$

$$\Rightarrow \frac{|x+3|+x-x-2}{x+2} > 0$$

$$\Rightarrow \frac{|x+3|-2}{x+2} > 0$$

$$\text{If } |x+3|-2=0 \Rightarrow x+3=\pm 2 \Rightarrow x=-5, -1.$$

Hence, the sign scheme of the expression $\frac{|x+3|-2}{x+2}$ is as follows:

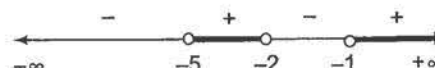


Fig. 1.23

From the above sign scheme, $x \in (-5, -2) \cup (-1, \infty)$.

Example 1.19 Solve $|x-1|+|x-2| \geq 4$.

Sol. Let $f(x) = |x-1|+|x-2|$

A.	B. $f(x)$	C. $f(x) \geq 4$	D. $A \cap C$
$x < 1$	$1-x+2-x$ $= 3-2x$	$3-2x \geq 4$ $\Rightarrow x \leq -1/2$	$x \leq -1/2$
$1 \leq x \leq 2$	$x-1+2$ $= x+1$	$1 \geq 4$, not possible	
$x > 2$	$x-1+x-2$ $= 2x-3$	$2x-3 \geq 4$ $\Rightarrow x \geq 7/2$	$x \geq 7/2$

Hence, the solution is $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Example 1.20 Solve $|\sin x + \cos x| = |\sin x| + |\cos x|$, $x \in [0, 2\pi]$.

Sol. The given relation holds only when $\sin x$ and $\cos x$ have same sign or at least one of them is zero.

Hence, $x \in [0, \pi/2] \cup [\pi, 3\pi/2] \cup \{2\pi\}$.

Concept Application Exercise 1.3

1. Solve the following:

- $1 \leq |x-2| \leq 3$
- $0 < |x-3| \leq 5$
- $|x-2| + |2x-3| = |x-1|$
- $\left| \frac{x-3}{x+1} \right| \leq 1$

2. Find the domain of

a. $f(x) = \frac{1}{\sqrt{x-|x|}}$

b. $f(x) = \frac{1}{\sqrt{x+|x|}}$

3. Find the set of real value(s) of a for which the equation $|2x+3| + |2x-3| = ax+6$ has more than two solutions.

4. If $a < b < c$, then find the range of $f(x) = |x-a| + |x-b| + |x-c|$.

5. Find the range of $f(x) = \sqrt{1-\sqrt{x^2-6x+9}}$.

$$\cos x \geq 0 \Rightarrow x \in \bigcup_{n \in I} [2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}]$$

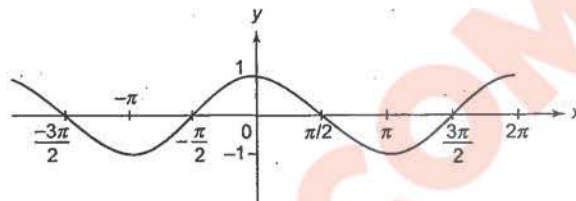


Fig. 1.25

3. $y = f(x) = \tan x$

Domain $\rightarrow \mathbb{R} \sim (2n+1)\pi/2, n \in I$;

Range $\rightarrow (-\infty, \infty)$

Period $\rightarrow \pi$

Nature \rightarrow odd, many-one in its actual domain

Discontinuous at $x = (2n+1)\pi/2, n \in I$

$\tan^2 x, |\tan x| \in [0, \infty)$

$\tan x = 0 \Rightarrow x = n\pi, n \in I$

$\tan x = \tan \alpha \Rightarrow x = n\pi + \alpha, n \in I$

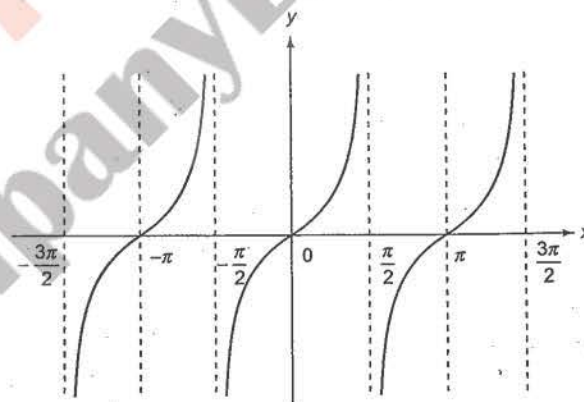


Fig. 1.26

4. $y = f(x) = \cot x$

Domain $\rightarrow \mathbb{R} \sim n\pi, n \in I$; Range $\rightarrow (-\infty, \infty)$;

Period $\rightarrow \pi$;

Nature \rightarrow odd, many-one in its actual domain

Discontinuous at $x = n\pi, n \in I$

$\cot^2 x, |\cot x| \in [0, \infty)$

$\cot x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

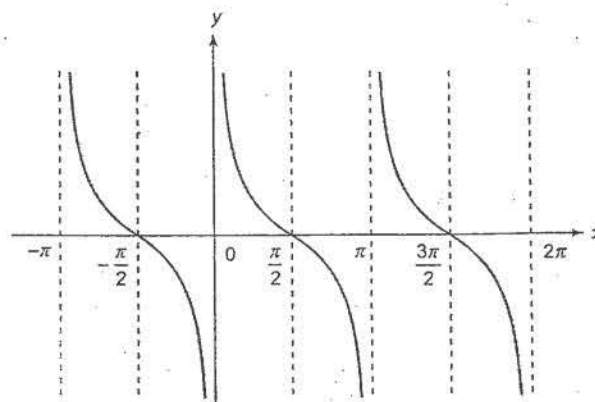


Fig. 1.27

Trigonometric Functions

1. $y = f(x) = \sin x$

Domain $\rightarrow \mathbb{R}$, Range $\rightarrow [-1, 1]$

Period $\rightarrow 2\pi$

Nature \rightarrow odd, many-one in its actual domain

$\sin^2 x, |\sin x| \in [0, 1]$

$\sin x = 0 \Rightarrow x = n\pi, n \in I$

$\sin x = 1 \Rightarrow x = (4n+1)\pi/2, n \in I$

$\sin x = -1 \Rightarrow x = (4n-1)\pi/2, n \in I$

$\sin x = \sin \alpha \Rightarrow x = n\pi + (-1)^n \alpha, n \in I$

$\sin x \geq 0 \Rightarrow x \in \bigcup_{n \in I} [2n\pi, \pi + 2n\pi]$

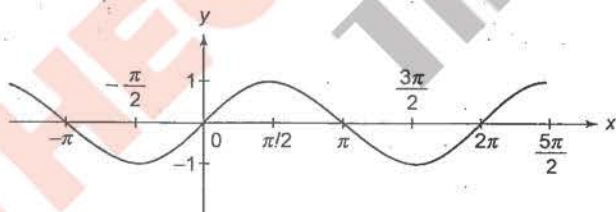


Fig. 1.24

2. $y = f(x) = \cos x$

Domain $\rightarrow \mathbb{R}$, Range $\rightarrow [-1, 1]$

Period $\rightarrow 2\pi$

Nature \rightarrow even, many-one in its actual domain

$\cos^2 x, |\cos x| \in [0, 1]$

$\cos x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

$\cos x = 1 \Rightarrow x = 2n\pi, n \in I$

$\cos x = -1 \Rightarrow x = (2n+1)\pi, n \in I$

$\cos x = \cos \alpha \Rightarrow x = 2n\pi \pm \alpha, n \in I$

5. $y = f(x) = \sec x$

Domain $\rightarrow \mathbb{R} - (2n+1)\pi/2, n \in \mathbb{I}$

Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Period $\rightarrow 2\pi$

$\sec^2 x, |\sec x| \in [1, \infty)$

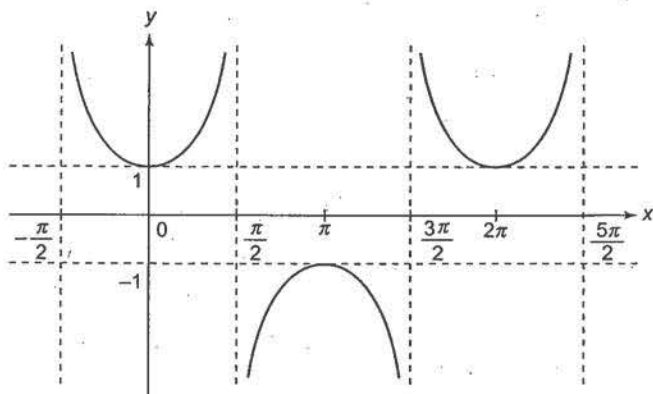
 Nature \rightarrow even, many-one in its actual domain


Fig. 1.28

6. $y = f(x) = \operatorname{cosec} x$

Domain $\rightarrow \mathbb{R} - n\pi, n \in \mathbb{I}$

Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Period $\rightarrow 2\pi$

$\operatorname{cosec}^2 x, |\operatorname{cosec} x| \in [1, \infty)$

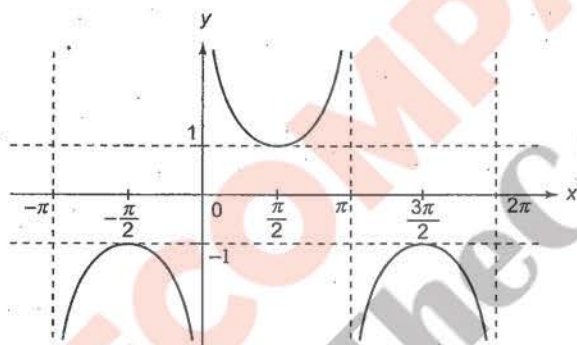
 Nature \rightarrow odd, many-one in its actual domain


Fig. 1.29

Important Result

$$\begin{aligned} f(x) &= a \cos x + b \sin x = \sqrt{a^2 + b^2} \sin \left(x + \tan^{-1} \frac{a}{b} \right) \\ &= \sqrt{a^2 + b^2} \cos \left(x - \tan^{-1} \frac{b}{a} \right) \end{aligned}$$

Proof: Let $a = r \sin \alpha, b = r \cos \alpha$

$$\Rightarrow a^2 + b^2 = r^2 \text{ and } \tan \alpha = \frac{a}{b}$$

Now, $f(x) = r(\cos x \sin \alpha + \sin x \cos \alpha)$

$$= r \sin(x + \alpha) = \sqrt{a^2 + b^2} \sin \left(x + \tan^{-1} \frac{a}{b} \right)$$

Since $-1 \leq \sin \left(x + \tan^{-1} \frac{a}{b} \right) \leq 1$

$$\Rightarrow -\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2} \sin \left(x + \tan^{-1} \frac{a}{b} \right) \leq \sqrt{a^2 + b^2}$$

 \Rightarrow The range of $f(x) = a \cos x + b \sin x$ is

$$\left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2} \right]$$

Example 1.21 Find the domain of the functions.

$$f(x) = \frac{1}{1 + 2 \sin x}$$

Sol. To define $f(x)$, we must have $1 + 2 \sin x \neq 0$

$$\Rightarrow \sin x \neq -\frac{1}{2} \Rightarrow x \neq n\pi + (-1)^n \frac{7\pi}{6}, n \in \mathbb{Z}$$

Hence, the domain of the function is

$$\mathbb{R} - \left\{ n\pi + (-1)^n \frac{7\pi}{6}, n \in \mathbb{Z} \right\}$$

Example 1.22 Solve $\sin x > -\frac{1}{2}$ or find the domain of

$$f(x) = \frac{1}{\sqrt{1 + 2 \sin x}}$$

Sol. To define $f(x)$, we must have $1 + 2 \sin x > 0$ or $\sin x > -\frac{1}{2}$.

 The function $\sin x$ has the least positive period 2π . That is why it is sufficient to solve inequality of the form $\sin x > a, \sin x \geq a, \sin x < a, \sin x \leq a$ first on the interval of length 2π , and then get the solution set by adding numbers of the form $2n\pi, n \in \mathbb{Z}$, to each of the solutions obtained on that interval.

Thus, let us solve this inequality on the interval

$$\left[-\frac{\pi}{2}, \frac{3\pi}{2} \right]$$

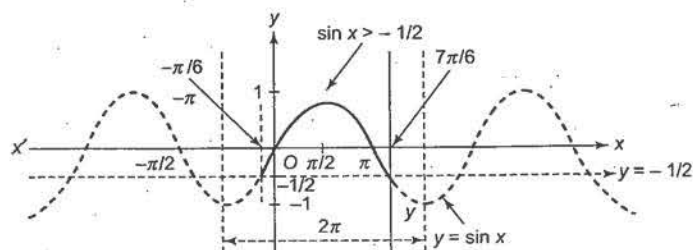


Fig. 1.30

 From Fig. 1.30, $\sin x > -\frac{1}{2}$ when $-\frac{\pi}{6} < x < \frac{7\pi}{6}$

Thus, on generalizing the above solution, we get

$$2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}; n \in \mathbb{Z}$$

Example 1.23 Find the number of solutions of $\sin x = \frac{x}{10}$.

Sol. Here, let $f(x) = \sin x$ and $g(x) = \frac{x}{10}$. Also we know that

$$-1 \leq \sin x \leq 1 \Rightarrow -1 \leq \frac{x}{10} \leq 1 \Rightarrow -10 \leq x \leq 10$$

 Thus, sketch both the curves when $x \in [-10, 10]$.

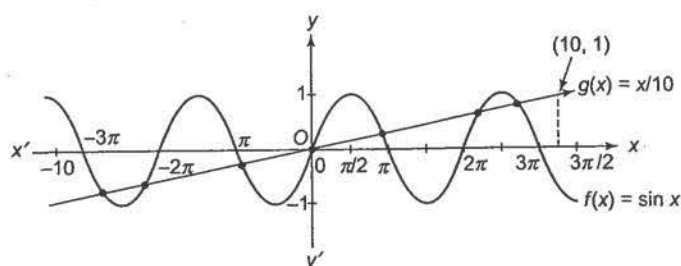


Fig. 1.31

From Fig. 1.31, $f(x) = \sin x$ and $g(x) = \frac{x}{10}$ intersect at 7 points. So, numbers of solutions are 7.

Example 1.24 Find the number of solutions of the equation $\sin x = x^2 + x + 1$.

Sol. Let $f(x) = \sin x$ and $g(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$, as shown in Fig. 1.32, which do not intersect at any point, therefore, there is no solution.

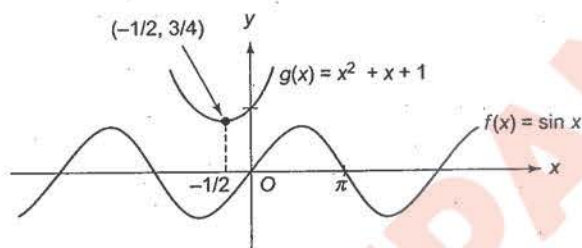


Fig. 1.32

Example 1.25 Find the range of $f(x) = \sin^2 x - \sin x + 1$

Sol. $f(x) = \sin^2 x - \sin x + 1 = \left(\sin x - \frac{1}{2}\right)^2 + \frac{3}{4}$

$$\text{Now, } -1 \leq \sin x \leq 1 \Rightarrow -\frac{3}{2} \leq \sin x - \frac{1}{2} \leq \frac{1}{2}$$

$$\Rightarrow 0 \leq \left(\sin x - \frac{1}{2}\right)^2 \leq \frac{9}{4} \Rightarrow \frac{3}{4} \leq \left(\sin x - \frac{1}{2}\right)^2 + \frac{3}{4} \leq 3$$

Hence, the range is $\left[\frac{3}{4}, 3\right]$.

Example 1.26 Find the range of $f(x) = \frac{1}{2\cos x - 1}$.

Sol. $-1 \leq \cos x \leq 1$

$$\Rightarrow -2 \leq 2\cos x \leq 2$$

$$\Rightarrow -3 \leq 2\cos x - 1 \leq 1$$

For $\frac{1}{2\cos x - 1}$, $-3 \leq 2\cos x - 1 < 0$ or $0 < 2\cos x - 1 \leq 1$

$$\Rightarrow -\infty < \frac{1}{2\cos x - 1} \leq \frac{-1}{3} \text{ or } 1 \leq \frac{1}{2\cos x - 1} < \infty$$

Hence, the range is $\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$.

Example 1.27 Find the domain for $f(x) = \sqrt{\cos(\sin x)}$.

Sol. $f(x) = \sqrt{\cos(\sin x)}$ is defined if $\cos(\sin x) \geq 0$ (1)

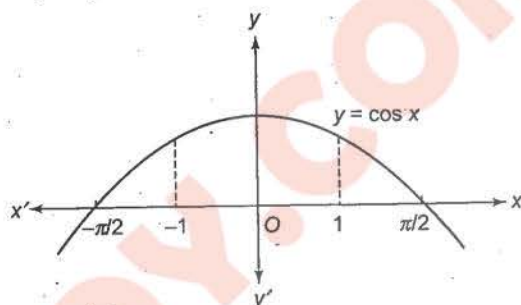


Fig. 1.33

As we know, $-1 \leq \sin x \leq 1$ for all x
 $\cos \theta \geq 0$

{Here, $\theta = \sin x$ lies in the first and fourth quadrant}
 i.e., $\cos(\sin x) \geq 0$ for all x
 i.e., $x \in R$

Thus, the domain $f(x)$ is R .

Example 1.28 If $f(x) = \frac{\sin x}{\sqrt{1+\tan^2 x}} - \frac{\cos x}{\sqrt{1+\cot^2 x}}$, then find the range of $f(x)$.

$$\text{Sol. } f(x) = \frac{\sin x}{|\sec x|} - \frac{\cos x}{|\csc x|} = \sin x |\cos x| - \cos x |\sin x|$$

Clearly, the domain of $f(x)$ is $R \sim \left\{n\pi, (2n+1)\frac{\pi}{2} / n \in I\right\}$

and period of $f(x)$ is 2π .

$$f(x) = \begin{cases} 0, & x \in (0, \pi/2) \\ -\sin 2x, & x \in (\pi/2, \pi) \\ 0, & x \in (\pi, 3\pi/2) \\ \sin 2x, & x \in (3\pi/2, 2\pi) \end{cases}$$

\Rightarrow The range of $f(x)$ is $(-1, 1)$.

Example 1.29 Find the range of $f(x) = |\sin x| + |\cos x|$, $x \in R$.

Sol. $f(x) = |\sin x| + |\cos x| \forall x \in R$.

Clearly $f(x) > 0$.

$$\text{Also, } f^2(x) = \sin^2 x + \cos^2 x + 2\sin x \cos x = 1 + |\sin 2x|$$

$$\Rightarrow 1 \leq f^2(x) \leq 2$$

$$\Rightarrow 1 \leq f(x) \leq \sqrt{2}$$

Example 1.30 Find the range of $f(\theta) = 5 \cos \theta$

$$+ 3 \cos \left(\theta + \frac{\pi}{3}\right) + 3.$$

Sol. $f(\theta) = 5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3}\right) + 3$

$$= 5 \cos \theta + \frac{3}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3$$

$$= \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3 = \sqrt{\left(\frac{169}{4} + \frac{27}{4}\right)} \sin(\theta - \alpha) + 3$$

Thus, the range of $f(\theta)$ is $[-4, 10]$.

Concept Application Exercise 1.4

- Find the domain of $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$.
- Solve (a) $\tan x < 2$, (b) $\cos x \leq -\frac{1}{2}$.
- Prove that the least positive value of x , satisfying $\tan x = x + 1$, lies in the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
- Find the range of $f(x) = \sec\left(\frac{\pi}{4} \cos^2 x\right)$, where $-\infty < x < \infty$.
- If $x \in [1, 2]$, then find the range of $f(x) = \tan x$.
- Find the range of $f(x) = \frac{1}{1 - 3\sqrt{1 - \sin^2 x}}$.

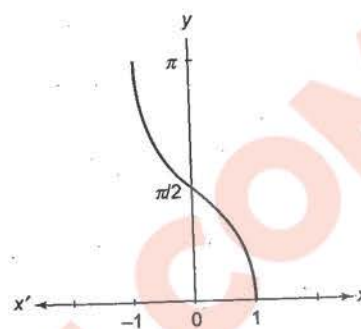


Fig. 1.35

$$f(x) = \tan^{-1} x$$

Domain: R

$$\text{Range: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\begin{aligned} \tan^{-1}(\tan x) &= x, & \text{for all } x \in (-\pi/2, \pi/2) \\ \tan(\tan^{-1} x) &= x, & \text{for all } x \in R \\ \tan^{-1}(-x) &= -\tan^{-1} x, & \text{for all } x \in R \end{aligned}$$

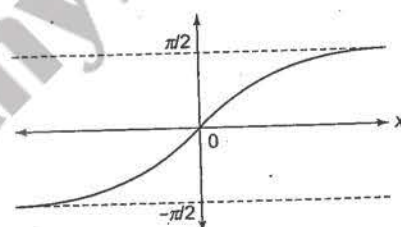


Fig. 1.36

$$f(x) = \cot^{-1} x$$

Domain: R

$$\text{Range: } (0, \pi)$$

$$\begin{aligned} \cot^{-1}(\cot x) &= x, & \text{for all } x \in (0, \pi) \\ \cot(\cot^{-1} x) &= x, & \text{for all } x \in R \\ \cot^{-1}(-x) &= \pi - \cot^{-1} x, & \text{for all } x \in R \end{aligned}$$

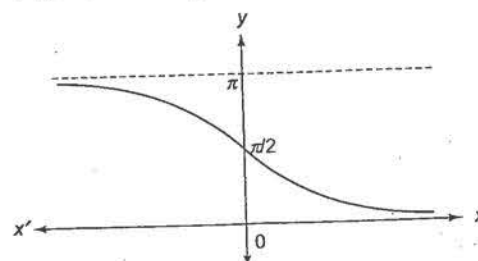


Fig. 1.37

$$f(x) = \sec^{-1} x$$

Domain: $(-\infty, -1] \cup [1, \infty)$

$$\text{Range: } [0, \pi] - \{\pi/2\}$$

$$\begin{aligned} \sec^{-1}(\sec x) &= x, & \text{for all } x \in [0, \pi] - \{\pi/2\} \\ \sec(\sec^{-1} x) &= x, & \text{for all } x \in (-\infty, -1] \cup [1, \infty) \\ \sec^{-1}(-x) &= \pi - \sec^{-1} x, & \text{for all } x \in (-\infty, -1] \cup [1, \infty) \end{aligned}$$

Inverse Trigonometric Functions

$$f(x) = \sin^{-1} x$$

Domain: $[-1, 1]$

$$\text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\begin{aligned} \sin^{-1}(\sin x) &= x, & \text{for all } x \in [-\pi/2, \pi/2] \\ \sin(\sin^{-1} x) &= x, & \text{for all } x \in [-1, 1] \\ \sin^{-1}(-x) &= -\sin^{-1}(x), & \text{for all } x \in [-1, 1] \end{aligned}$$

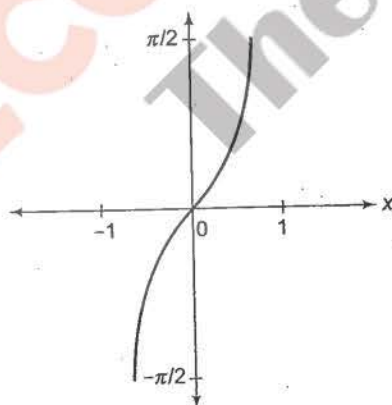


Fig. 1.34

$$f(x) = \cos^{-1} x$$

Domain: $[-1, 1]$

$$\text{Range: } [0, \pi]$$

$$\begin{aligned} \cos^{-1}(\cos x) &= x, & \text{for all } x \in [0, \pi] \\ \cos(\cos^{-1} x) &= x, & \text{for all } x \in [-1, 1] \\ \cos^{-1}(-x) &= \pi - \cos^{-1} x, & \text{for all } x \in [-1, 1] \end{aligned}$$

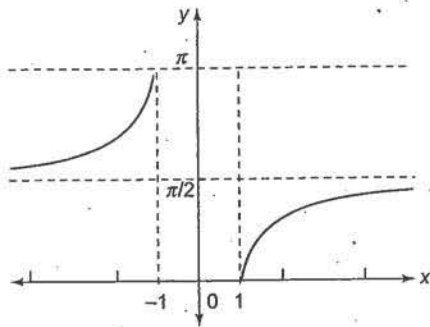


Fig. 1.38

$$f(x) = \operatorname{cosec}^{-1} x$$

$$\text{Domain: } (-\infty, -1] \cup [1, \infty)$$

$$\text{Range: } [-\pi/2, \pi/2] - \{0\}$$

$$\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x, \quad \text{for all } x \in [-\pi/2, \pi/2] - \{0\}$$

$$\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x, \quad \text{for all } x \in (-\infty, -1] \cup [1, \infty)$$

$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, \quad \text{for all } x \in (-\infty, -1] \cup [1, \infty)$$

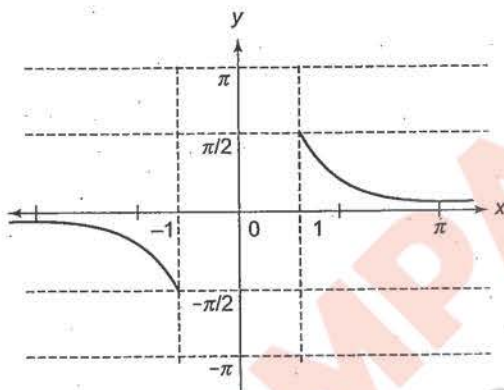


Fig. 1.39

Example 1.31 Find the domain of $f(x) = \sin^{-1}\left(\frac{x^2}{2}\right)$.

Sol. $f(x) = \sin^{-1}\left(\frac{x^2}{2}\right)$ is defined, if $-1 \leq \frac{x^2}{2} \leq 1$ or $-2 \leq x^2 \leq 2$
 $\Rightarrow 0 \leq x^2 \leq 2$ (as x^2 cannot be negative)
 $\Rightarrow -\sqrt{2} \leq x \leq \sqrt{2}$

Therefore, the domain of $f(x)$ is $[-\sqrt{2}, \sqrt{2}]$.

Example 1.32 Find the range of $f(x) = \sin^{-1} x + \tan^{-1} x + \cos^{-1} x$.

Sol. Clearly, the domain of the function is $[-1, 1]$.

Also $\tan^{-1} x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ for $x \in [-1, 1]$.

Now, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $x \in [-1, 1]$.

Thus, $f(x) = \tan^{-1} x + \frac{\pi}{2}$, where $x \in [-1, 1]$.

Hence, the range is $\left[-\frac{\pi}{4} + \frac{\pi}{2}, \frac{\pi}{4} + \frac{\pi}{2}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Example 1.33 Find the domain of $f(x) = \sqrt{\cos^{-1} x - \sin^{-1} x}$.

Sol. We must have $\cos^{-1} x \geq \sin^{-1} x$

$$\Rightarrow \frac{\pi}{2} - \sin^{-1} x \geq \sin^{-1} x$$

$$\Rightarrow \frac{\pi}{2} \geq 2\sin^{-1} x$$

$$\Rightarrow \sin^{-1} x \leq \frac{\pi}{4}, \text{ but } -\frac{\pi}{2} \leq \sin^{-1} x$$

$$\Rightarrow -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{4}$$

$$\Rightarrow \sin\left(-\frac{\pi}{2}\right) \leq x \leq \sin\frac{\pi}{4}$$

$$\left(\because \sin x \text{ is increasing function in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

$$\Rightarrow x \in \left[-1, \frac{1}{\sqrt{2}}\right]$$

Example 1.34 Find the range of $\tan^{-1}\left(\frac{2x}{1+x^2}\right)$.

Sol. First, we must get the range of $\frac{2x}{1+x^2} = y$

We have $yx^2 - 2x + y = 0$

Since x is real, $D \geq 0 \Rightarrow 4 - 4y^2 \geq 0 \Rightarrow -1 \leq y \leq 1$

$$\Rightarrow \tan^{-1}(y) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \text{ (as } \tan x \text{ is an increasing function).}$$

Example 1.35 Find the domain for $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$.

Sol. $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined for $-1 \leq \frac{1+x^2}{2x} \leq 1$, or

$$\left|\frac{1+x^2}{2x}\right| \leq 1$$

$$\Rightarrow |1+x^2| \leq |2x|, \text{ for all } x.$$

$$\Rightarrow 1+x^2 \leq |2x|, \text{ for all } x \quad (\text{as } 1+x^2 > 0)$$

$$\Rightarrow x^2 - 2|x| + 1 \leq 0$$

$$\Rightarrow |x|^2 - 2|x| + 1 \leq 0 \quad (\text{as } x^2 = |x|^2)$$

$$\Rightarrow (|x| - 1)^2 \leq 0$$

But $(|x| - 1)^2$ is always either positive or zero. Thus, $(|x| - 1)^2 = 0 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$.

Thus, the domain for $f(x)$ is $\{-1, 1\}$.

Example 1.36 Find the range of $f(x) = \cot^{-1}(2x - x^2)$.

Sol. Let $\theta = \cot^{-1}(2x - x^2)$, where $\theta \in (0, \pi)$

$$\Rightarrow \cot \theta = 2x - x^2, \text{ where } \theta \in (0, \pi)$$

$$\Rightarrow \cot \theta = 1 - (1 - 2x + x^2), \text{ where } \theta \in (0, \pi)$$

$$\Rightarrow \cot \theta = 1 - (1 - x)^2, \text{ where } \theta \in (0, \pi)$$

$$\Rightarrow \cot \theta \leq 1, \text{ where } \theta \in (0, \pi)$$

$$\Rightarrow \frac{\pi}{4} \leq \theta < \pi.$$

$$\Rightarrow \text{The range of } f(x) \in \left[\frac{\pi}{4}, \pi \right).$$

Concept Application Exercise 1.5

1. Find the domain of the following functions:

a. $f(x) = \frac{\sin^{-1} x}{x}$

b. $f(x) = \sin^{-1}(|x-1|-2)$

c. $f(x) = \cos^{-1}(1+3x+2x^2)$

d. $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$

e. $f(x) = \cos^{-1}\left(\frac{6-3x}{4}\right) + \operatorname{cosec}^{-1}\left(\frac{x-1}{2}\right)$

f. $f(x) = \sqrt{\sec^{-1}\left(\frac{2-|x|}{4}\right)}$

2. Find the range of $f(x) = \tan^{-1}(\sqrt{x^2-2x+2})$.

3. Find the range of $f(x) = \sqrt{\cos^{-1}\sqrt{1-x^2}} - \sin^{-1} x$

4. Find the range of the function,

$$f(x) = \cot^{-1} \log_{0.5}(x^4 - 2x^2 + 3)$$

Exponential and Logarithmic Functions

Exponential Function

$$y = a^x, a > 1$$

$$\text{Domain} \rightarrow \mathbb{R}$$

$$\text{Range} \rightarrow (0, \infty)$$

$$\text{Nature} \rightarrow$$

$$\text{Non-periodic}$$

$$\text{One - one}$$

$$\text{Neither odd nor even}$$

$$\text{Monotonically increasing, } (a > 1)$$

$$\text{Monotonically decreasing, } (0 < a < 1)$$

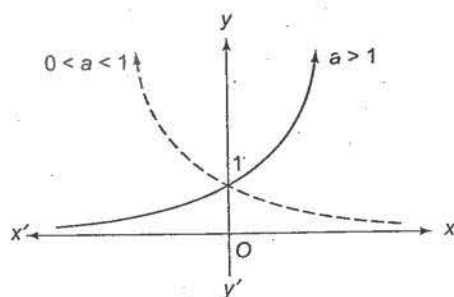


Fig. 1.40

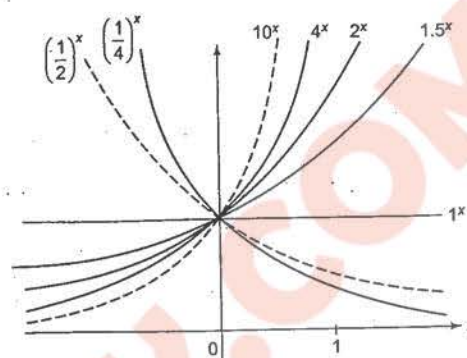


Fig. 1.41

Logarithmic Function

Logarithm function is the inverse of exponential function.

Hence, the domain and range of the logarithmic functions are range and domain of exponential function, respectively.

Also, the graph of function can be obtained by taking the mirror image of the graph of the exponential function in the line $y = x$.

$$y = \log_a x, a > 0 \text{ and } a \neq 1$$

$$\text{Domain} \rightarrow (0, \infty)$$

$$\text{Range} \rightarrow (-\infty, \infty)$$

$$\text{Period} \rightarrow \text{non-periodic}$$

$$\text{Nature} \rightarrow \text{neither odd nor even}$$

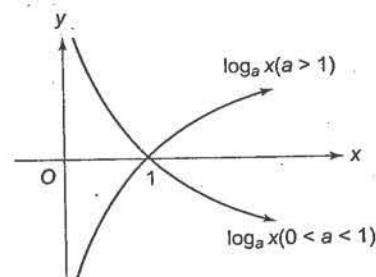


Fig. 1.42

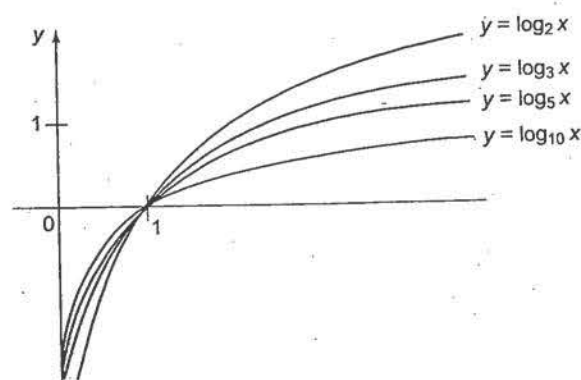


Fig. 1.43

Properties of Logarithmic Function

For $x, y > 0$ and $a > 0, a \neq 2$

$$1. \log_a(xy) = \log_a x + \log_a y$$

$$2. \log_a(x/y) = \log_a x - \log_a y$$

$$3. \log_a(x^b) = b \log_a x$$

$$4. \log_a x^b = \frac{b}{a} \log_a x$$

$$5. \text{ If } \log_a x > \log_a y \Rightarrow \begin{cases} x > y, & \text{if } a > 1 \\ x < y, & \text{if } 0 < a < 1 \end{cases}$$

$$6. \text{ If } \log_a x = y \Rightarrow x = a^y$$

$$7. \text{ If } \log_a x > y \Rightarrow \begin{cases} x > a^y, & \text{if } a > 1 \\ x < a^y, & \text{if } 0 < a < 1 \end{cases}$$

$$8. a^{\log_a x} = x$$

$$9. \log_a x = \frac{\log_a x}{\log_a a}$$

$$10. \log_a x > 0 \Rightarrow x > 1 \text{ and } a > 1 \text{ or } 0 < x < 1 \text{ and } 0 < a < 1$$

Example 1.37 Find the domain of $f(x) = \sqrt{\left(\frac{1-5^x}{7^x-7}\right)}$.

Sol. We must have $g(x) = \left(\frac{1-5^x}{7^x-7}\right) \geq 0 \Rightarrow \frac{5^x-1}{7^x-7} \leq 0$

Now $5^x-1=0 \Rightarrow x=0$ and $7^x-7=0 \Rightarrow x=-1$

The sign scheme of $g(x)$ is

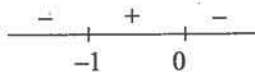


Fig. 1.44

Hence, from the sign scheme of $g(x)$
 $x \in (-\infty, -1) \cup [0, \infty)$.

Example 1.38 Find the domain of

$$f(x) = \sqrt{(0.625)^{4-3x} - (1.6)^{x(x+8)}}$$

Sol. Clearly, $(0.625)^{4-3x} \geq (1.6)^{x(x+8)}$

$$\Rightarrow \left(\frac{5}{8}\right)^{4-3x} \geq \left(\frac{8}{5}\right)^{x(x+8)}$$

$$\Rightarrow \left(\frac{8}{5}\right)^{3x-4} \geq \left(\frac{8}{5}\right)^{x(x+8)}$$

$$\Rightarrow 3x-4 \geq x^2+8x \Rightarrow x^2+5x+4 \leq 0$$

$$\Rightarrow -4 \leq x \leq -1$$

Hence, the domain of function $f(x)$ is $x \in [-4, -1]$.

Example 1.39 Find the range of

a. $f(x) = \log_e \sin x$

b. $f(x) = \log_3(5-4x-x^2)$

Sol. a. $f(x) = \log_e \sin x$ is defined if $\sin x \in (0, 1]$

for which $\log_e \sin x \in (-\infty, 0]$.

b. $f(x) = \log_3(5-4x-x^2)$

$$= \log_3(9-(x-2)^2)$$

$f(x)$ is defined if $9-(x-2)^2 > 0$, but $9-(x-2)^2 \leq 9$

$$\Rightarrow 0 < 9-(x-2)^2 \leq 9$$

$$\Rightarrow -\infty < \log_3(9-(x-2)^2) \leq \log_3 9$$

Hence, the range is $(-\infty, 2]$.

Example 1.40 Find the domain of
 $f(x) = \log_{10} \log_2 \log_{2/\pi} (\tan^{-1} x)^{-1}$.

Sol. We must have $\log_2 \log_{2/\pi} (\tan^{-1} x)^{-1} > 0$

$$\Rightarrow \log_{2/\pi} (\tan^{-1} x)^{-1} > 1$$

$$\Rightarrow 0 < (\tan^{-1} x)^{-1} < 2/\pi$$

$$\Rightarrow \pi/2 < \tan^{-1} x < \infty, \text{ which is not possible.}$$

Hence, the domain is ϕ .

Example 1.41 Find the domain and range of

$$f(x) = \sqrt{\log_3(\cos(\sin x))}$$

Sol. $f(x) = \sqrt{\log_3(\cos(\sin x))}$

$f(x)$ is defined only if $\log_3(\cos(\sin x)) \geq 0$

$$\Rightarrow \cos(\sin x) \geq 1$$

$$\Rightarrow \cos(\sin x) = 1 \text{ as } -1 \leq \cos \theta \leq 1$$

$$\Rightarrow \sin x = 0 \Rightarrow x = n\pi, n \in I$$

Hence, the domain consists of the multiples of π , i.e., Domain
 $= \{n\pi, n \in I\}$.

Also, the range is $\{0\}$.

Example 1.42 Solve $\log_x(x^2-1) \leq 0$.

Sol. Given $\log_x(x^2-1) \leq 0$

If $x > 1$, then

$$\Rightarrow 0 < x^2-1 \leq 1$$

$$\Rightarrow 1 < x^2 \leq 2$$

$$\Rightarrow x \in [-\sqrt{2}, -1) \cup (1, \sqrt{2}]$$

$$\Rightarrow x \in (1, \sqrt{2}]$$

$$\text{If } 0 < x < 1 \Rightarrow x^2-1 \geq 1 \Rightarrow x^2 \geq 2$$

$$\Rightarrow x \in (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$$

$$\Rightarrow x = \phi$$

$$\text{Thus, } x \in (1, \sqrt{2}].$$

Example 1.43 Find the number of solutions of $2^x + 3^x + 4^x - 5^x = 0$.

Sol. $2^x + 3^x + 4^x - 5^x = 0$

$$\Rightarrow 2^x + 3^x + 4^x = 5^x$$

$$\Rightarrow \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1$$

Now, the number of solution of the equation is equal to number of times

$$y = \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x \text{ and } y = 1 \text{ intersect.}$$

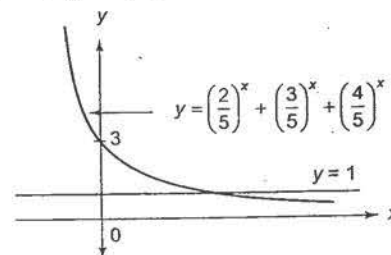


Fig. 1.45

From the graph, the equation has only one solution.

Example 1.44 Find the domain of $f(x) = \sin^{-1}(\log_9(x^2/4))$.

Sol. We have $f(x) = \sin^{-1}\left\{\log_9\left(\frac{x^2}{4}\right)\right\}$.

Since the domain of $\sin^{-1} x$ is $[-1, 1]$.

Therefore, $f(x) = \sin^{-1}\left\{\log_9\left(\frac{x^2}{4}\right)\right\}$ is defined,

$$\text{if } -1 \leq \log_9\left(\frac{x^2}{4}\right) \leq 1$$

$$\Rightarrow 9^{-1} \leq \frac{x^2}{4} \leq 9^1$$

$$\Rightarrow \frac{4}{9} \leq x^2 \leq 36$$

$$\Rightarrow \frac{2}{3} \leq |x| \leq 6$$

$$\Rightarrow x \in \left[-6, -\frac{2}{3}\right] \cup \left[\frac{2}{3}, 6\right] \quad (\because a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b])$$

Hence, the domain of $f(x)$ is $\left[-6, -\frac{2}{3}\right] \cup \left[\frac{2}{3}, 6\right]$.

Example 1.45 Find the domain of function

$$f(x) = \log_4 \{ \log_5 (\log_3 (18x - x^2 - 77)) \}$$

Sol. We have $f(x) = \log_4 \{ \log_5 (\log_3 (18x - x^2 - 77)) \}$

Since $\log_a x$ is defined for all $x > 0$. Therefore, $f(x)$ is defined if

$$\log_5 \{ \log_3 (18x - x^2 - 77) \} > 0 \text{ and } 18x - x^2 - 77 > 0$$

$$\Rightarrow \log_3 (18x - x^2 - 77) > 5^0 \text{ and } x^2 - 18x + 77 < 0$$

$$\Rightarrow \log_3 (18x - x^2 - 77) > 1 \text{ and } (x - 11)(x - 7) < 0$$

$$\Rightarrow 18x - x^2 - 77 > 3^1 \text{ and } 7 < x < 11$$

$$\Rightarrow 18x - x^2 - 80 > 0 \text{ and } 7 < x < 11$$

$$\Rightarrow x^2 - 18x + 80 < 0 \text{ and } 7 < x < 11$$

$$\Rightarrow (x - 10)(x - 8) < 0 \text{ and } 7 < x < 11$$

$$\Rightarrow 8 < x < 10 \text{ and } 7 < x < 11$$

$$\Rightarrow 8 < x < 10$$

$$\Rightarrow x \in (8, 10).$$

Hence, the domain of $f(x)$ is $(8, 10)$.

Example 1.46 Find the domain of $f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$.

Sol. $f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$ exists if $\log_{0.4}\left(\frac{x-1}{x+5}\right) \geq 0$ and

$$\left(\frac{x-1}{x+5}\right) > 0.$$

$$\Rightarrow \frac{x-1}{x+5} \leq (0.4)^0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\Rightarrow \frac{x-1}{x+5} \leq 1 \text{ and } \frac{x-1}{x+5} > 0$$

$$\Rightarrow \frac{x-1}{x+5} - 1 \leq 0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\Rightarrow \frac{-6}{x+5} \leq 0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\Rightarrow x+5 > 0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\Rightarrow x > -5 \text{ and } x-1 > 0$$

$$\Rightarrow x > -5 \text{ and } x > 1$$

Thus, the domain $f(x) \in (1, \infty)$.

(as $x+5 > 0$)

Concept Application Exercise 1.6

Find the domain of the following functions: (1 to 7)

1. $f(x) = \sqrt{4^x + 8^{\frac{2}{3}(x-2)}} - 13 - 2^{2(x-1)}$

2. $f(x) = \sin^{-1}(\log_2 x)$

3. $f(x) = \log_{(x-4)}(x^2 - 11x + 24)$

4. $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$

5. $f(x) = \sqrt{\frac{\log_{0.3}|x-2|}{|x|}}$

6. $f(x) = \sqrt{\log_{10} \left\{ \frac{\log_{10} x}{2(3 - \log_{10} x)} \right\}}$

7. $f(x) = \frac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$

8. Find the range of $f(x) = \log_2 \left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} \right)$.

Greatest Integer and Fractional Part Function

Greatest Integer Function (Floor Value Function)

$$y = f(x) = [x] \text{ (Greatest integer } \leq x)$$

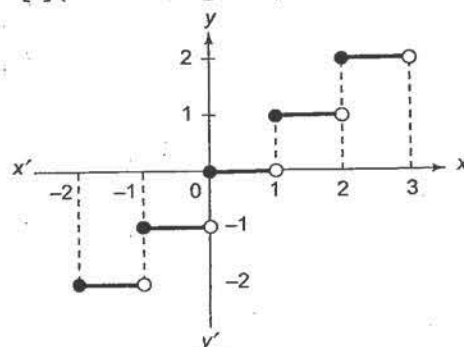


Fig. 1.46

Graph of $y = [x]$

Properties

- Domain $\rightarrow \mathbb{R}$; Range $\rightarrow \mathbb{Z}$;
- $[x] = n (n \in \mathbb{I}) \Rightarrow x \in [n, n+1)$

- $x-1 < [x] \leq x$
- $[-x] + [x] = 0$, if $x \in \mathbb{Z}$
- $[-x] + [x] = -1$, if $x \notin \mathbb{Z}$
- $[x] \geq n \Rightarrow x \geq n, n \in \mathbb{Z}$
- $[x] \leq n \Rightarrow x < n+1, n \in \mathbb{Z}$
- $[x] > n \Rightarrow x \geq n+1, n \in \mathbb{Z}$

e.g.,

$$[x] \geq 2 \Rightarrow x \in [2, \infty)$$

$$[x] > 3 \Rightarrow [x] \geq 4 \Rightarrow x \in [4, \infty)$$

$$[x] \leq 3 \Rightarrow x \in (-\infty, 4)$$

$$\bullet \left[\frac{x}{n} \right] + \left[\frac{x+1}{n} \right] + \left[\frac{x+2}{n} \right] + \dots + \left[\frac{x+n-1}{n} \right] = [x], n \in \mathbb{N}$$

$$\text{or } [x] + \left[x + \frac{1}{n} \right] + \left[x + \frac{2}{n} \right] + \dots + \left[x + \frac{n-1}{n} \right] = [nx]$$

Fractional Part Function

$$y = f(x) = \{x\} = x - [x]$$

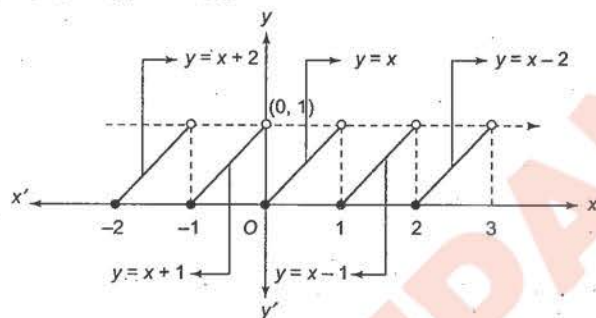


Fig. 1.47

Graph of $y = \{x\}$ **Properties**

- Domain $\rightarrow \mathbb{R}$; Range $\rightarrow [0, 1)$; Period $\rightarrow 1$.
- $[x+y] = [x] + [y]$, if $0 \leq \{x\} + \{y\} < 1$
- $[x+y] = [x] + [y] + 1$, $1 \leq \{x\} + \{y\} < 2$
- $\{x\} + \{-x\} = 0$ if $x \in \mathbb{I}$
- $\{x\} + \{-x\} = 1$ if $x \notin \mathbb{I}$

Example 1.47 Find the domain of

$$f(x) = \sqrt{([x]-1)} + \sqrt{(4-[x])}, \text{ (where } [] \text{ represents the greatest integer function).}$$

represents

$$\text{Sol. Given } f(x) = \sqrt{([x]-1)} + \sqrt{(4-[x])}$$

$$\therefore f(x) \text{ is defined when } [x]-1 \geq 0 \text{ and } 4-[x] \geq 0$$

$$\therefore 1 \leq [x] \leq 4 \text{ or } 1 \leq x < 5.$$

$$\text{Hence, the domain of } f(x) = D_f = [1, 5).$$

Example 1.48 Find the domain and range of $f(x) = \sin^{-1} [x]$ (where $[]$ represents the greatest integer function).

$$\text{Sol. } f(x) = \sin^{-1} [x] \text{ is defined if } -1 \leq [x] \leq 1$$

$$\Rightarrow [x] = -1, 0, 1$$

$$\Rightarrow x \in [-1, 2)$$

$$\Rightarrow \text{Range is } \{\sin^{-1}(-1), \sin^{-1} 0, \sin^{-1} 1\} \equiv \{-\pi/2, 0, \pi/2\}.$$

Example 1.49 Find the domain and range of $f(x) = \log \{x\}$, where $\{ \}$ represents the fractional part function).

$$\text{Sol. We know that } 0 \leq \{x\} < 1 \quad \forall x \in \mathbb{R}$$

Now when $\{x\} = 0$, $\log \{x\}$ is not defined. So x cannot be integer. Hence, the domain is $\mathbb{R} - \mathbb{I}$.

$$\text{Now for } 0 < \{x\} < 1, -\infty < \log \{x\} < 0 \Rightarrow \text{Range is } (-\infty, 0)$$

Example 1.50 Find the range of $f(x) = [\sin \{x\}]$ where $\{ \}$ represents the fractional part function, $[]$ represents greatest the integer function.

$$\text{Sol. } f(x) = [\sin \{x\}]$$

Here, $\{x\}$ can take all its possible values and sine function is defined for all real values.

$$\text{Hence, } 0 \leq \{x\} < 1$$

$$\Rightarrow 0 \leq \sin \{x\} < \sin 1$$

$$\Rightarrow [\sin \{x\}] = 0.$$

Hence, the range is $\{0\}$.**Example 1.51** Solve $2[x] = x + \{x\}$, where $[]$ and $\{ \}$ denote the greatest integer function and fractional part, respectively.

$$\text{Sol. Given } 2[x] = x + \{x\}$$

$$\Rightarrow 2[x] = [x] + 2\{x\}$$

$$\Rightarrow \{x\} = \frac{[x]}{2}$$

$$\Rightarrow 0 \leq \frac{[x]}{2} < 1$$

$$\Rightarrow 0 \leq [x] < 2$$

$$\Rightarrow [x] = 0, 1.$$

$$\text{For } [x] = 0, \text{ we get } \{x\} = 0 \Rightarrow x = 0$$

$$\text{For } [x] = 1, \text{ we get } \{x\} = \frac{1}{2} \Rightarrow x = \frac{3}{2}.$$

Example 1.52 Find the range of $f(x) = \frac{x-[x]}{1-[x]+x}$, where $[]$ represents the greatest integer function.

$$\text{Sol. } f(x) = \frac{x-[x]}{1-[x]+x} = \frac{\{x\}}{1+\{x\}} = 1 - \frac{1}{1+\{x\}}$$

$$\text{Now, } 0 \leq \{x\} < 1$$

$$\Rightarrow 1 \leq \{x\} + 1 < 2$$

$$\Rightarrow \frac{1}{2} < \frac{1}{1+\{x\}} \leq 1$$

$$\Rightarrow -1 \leq -\frac{1}{1+\{x\}} < -\frac{1}{2}$$

$$\Rightarrow 0 \leq 1 - \frac{1}{1+\{x\}} < \frac{1}{2}$$

Example 1.53 Solve the system of equation in x, y , and z satisfying the following equations

$$x + [y] + \{z\} = 3.1$$

$$\{x\} + y + [z] = 4.3$$

$$[x] + \{y\} + z = 5.4$$

(where $[]$ denotes the greatest integer function and $\{ \}$ denotes fractional part.)

Sol. Adding all the three equations $2(x + y + z) = 12.8$ or $x + y + z = 6.4$ (1)

Adding first two equations, we get $x + y + z + [y] + \{x\} = 7.4$ (2)

Adding 2nd and 3rd equations, we get $x + y + z + [z] + \{y\} = 9.7$ (3)

Adding 1st and 4th equations, we get $x + y + z + [x] + \{z\} = 8.5$ (4)

From (1) and (2), $[y] + \{x\} = 1$

From (1) and (3), $[z] + \{y\} = 3.3$

From (1) and (4), $[x] + \{z\} = 2.1$

$\Rightarrow [x] = 2, [y] = 1, [z] = 3, \{x\} = 0, \{y\} = 0.3$ and $\{z\} = 0.1$

$\Rightarrow x = 2, y = 1.3, z = 3.1$

Example 1.54 Solve $x^2 - 4 - [x] = 0$ is (where $[.]$ denotes the greatest integer function).

Sol. The best method to solve such system is graphical one.

Given equation is $x^2 - 4 = [x]$

Then, the solutions of the equation are values of x where graph $y = x^2 - 4$ and $y = [x]$ intersect.

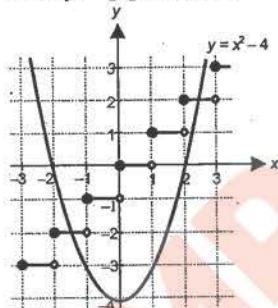


Fig. 1.48

From the graph, it is seen that graph intersects when

$$x^2 - 4 = 2 \text{ and } x^2 - 4 = -2$$

$$\Rightarrow x^2 = 6 \text{ or } x^2 = 2$$

$$\Rightarrow x = \sqrt{6} \text{ or } -\sqrt{2}$$

Example 1.55 If $f(x) = \begin{cases} [x], & 0 \leq \{x\} < 0.5 \\ [x] + 1, & 0.5 < \{x\} < 1 \end{cases}$ then prove that $f(x) = -f(-x)$ (where $[.]$ and $\{ \}$ represent the greatest integer function and fractional part function).

Sol. $f(-x) = \begin{cases} [-x], & 0 \leq \{-x\} < 0.5 \\ [-x] + 1, & 0.5 < \{-x\} < 1 \end{cases}$

$$= \begin{cases} [-x], & \{-x\} = 0 \\ [-x], & 0 < \{-x\} < 0.5 \\ [-x] + 1, & 0.5 < \{-x\} < 1 \end{cases}$$

$$= \begin{cases} -[x], & \{x\} = 0 \\ -1 - [x], & 0 < 1 - \{x\} < 0.5 \\ -1 - [x] + 1, & 0.5 < 1 - \{x\} < 1 \end{cases}$$

$$= \begin{cases} -[x], & \{x\} = 0 \\ -1 - [x], & 0.5 < \{x\} < 1 \\ -[x], & 0 < \{x\} < 0.5 \end{cases}$$

$$= \begin{cases} -[x], & 0 \leq \{x\} < 0.5 \\ -1 - [x], & 0.5 < \{x\} < 1 \end{cases} = -f(x)$$

Concept Application Exercise 1.7

In the following questions:

($[x]$ and $\{x\}$ represent the greatest integer function and fractional part function, respectively).

1. Solve $[x]^2 - 5[x] + 6 = 0$.

2. If $y = 3[x] + 1 = 4[x - 1] - 10$, then find the value of $[x + 2y]$.

3. Find the domain of

a. $f(x) = \frac{1}{\sqrt{x - [x]}}$ b. $f(x) = \frac{1}{\log[x]}$ c. $f(x) = \log\{x\}$.

4. Find the domain of $f(x) = \frac{1}{\sqrt{[|x| - 1] - 5}}$.

5. Find the domain of $f(x) = \frac{\sqrt{1 - \sin x}}{\log_5(1 - 4x^2)} + \cos^{-1}(1 - \{x\})$.

6. Find the range of $f(x) = \cos(\log_e \{x\})$.

7. Find the domain and range of $f(x) = \cos^{-1} \sqrt{\log_{[x]} \left(\frac{|x|}{x} \right)}$.

8. Find the range of $f(x) = \log_{[x-1]} \sin x$.

9. Solve: $(x - 2)[x] = \{x\} - 1$, (where $[x]$ and $\{x\}$ denotes the greatest integer function less than or equal to x and fractional part function respectively).

Signum Function

$$y = f(x) = \text{sgn}(x)$$

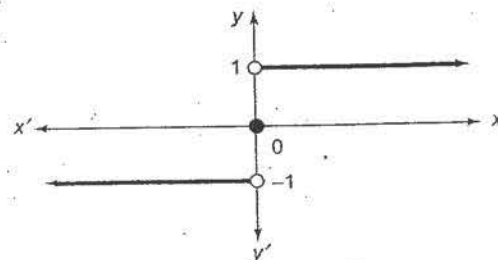


Fig. 1.49

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ or } \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Domain $\rightarrow \mathbb{R}$;

Range $\rightarrow \{-1, 0, 1\}$;

Nature : Many one, odd function

$$\text{In general, } \operatorname{sgn}(f(x)) = \begin{cases} \frac{|f(x)|}{f(x)}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

$$\text{or } \operatorname{sgn}(f(x)) = \begin{cases} -1, & f(x) < 0 \\ 0, & f(x) = 0 \\ 1, & f(x) > 0 \end{cases}$$

Example 1.56 Write the equivalent (piecewise) definition of $f(x) = \operatorname{sgn}(\sin x)$.

$$\text{Sol. } \operatorname{sgn}(\sin x) = \begin{cases} -1, & \sin x < 0 \\ 0, & \sin x = 0 \\ 1, & \sin x > 0 \end{cases}$$

$$= \begin{cases} -1, & x \in ((2n+1)\pi, (2n+2)\pi), n \in \mathbb{Z} \\ 0, & x = n\pi, n \in \mathbb{Z} \\ 1, & x \in (2n\pi, (2n+1)\pi), n \in \mathbb{Z} \end{cases}$$

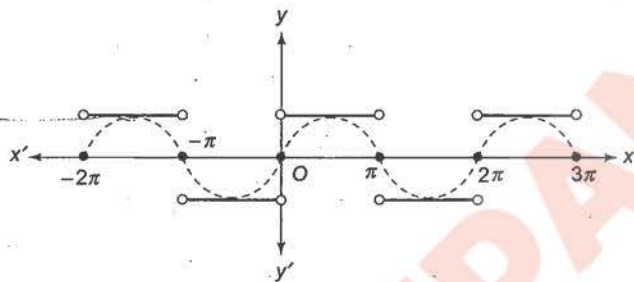


Fig. 1.50

Example 1.57 Find the range of $f(x) = \operatorname{sgn}(x^2 - 2x + 3)$.

$$\text{Sol. } x^2 - 2x + 3 = (x-1)^2 + 2 > 0 \quad \forall x \in \mathbb{R}.$$

$$\Rightarrow f(x) = \operatorname{sgn}(x^2 - 2x + 3) = 1$$

Hence, the range is $\{1\}$.

Functions of the Form $f(x) = \max\{g_1(x), g_2(x), \dots, g_n(x)\}$ or $f(x) = \min\{g_1(x), g_2(x), \dots, g_n(x)\}$

Let's consider the function $f(x) = \max\{x, x^2\}$

To write the equivalent definition of the function, first draw the graph of $y = x$ and $y = x^2$.

Now from the graph, we can see that

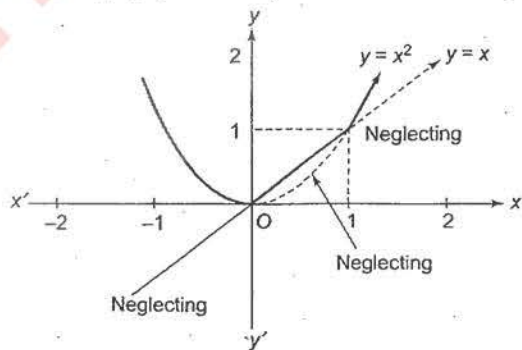


Fig. 1.51

For $x \in (-\infty, 0)$, the graph of $y = x^2$ lies above the graph of $y = x$, or $x^2 > x$.

For $x \in (0, 1)$, the graph of $y = x$ lies above the graph of $y = x^2$ or $x > x^2$.

For $x \in (1, \infty)$, the graph of $y = x^2$ lies above the graph of $y = x$ or $x^2 > x$.

$$\text{Hence, we have } f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \leq x \leq 1 \\ x^2, & x > 1 \end{cases}$$

$$\text{For } f(x) = \min\{x, x^2\}, \text{ we have } f(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ x, & x > 1 \end{cases}$$

Example 1.58 If $f: \mathbb{R} \rightarrow \mathbb{R}, g: \mathbb{R} \rightarrow \mathbb{R}$ be the two given functions, then prove that $2 \min\{f(x) - g(x), 0\} = f(x) - g(x) - |g(x) - f(x)|$.

$$\text{Sol. } h(x) = 2 \min\{f(x) - g(x), 0\}$$

$$= \begin{cases} 0 & f(x) > g(x) \\ 2(f(x) - g(x)), & f(x) \leq g(x) \end{cases}$$

$$= \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x) - |f(x) - g(x)|, & f(x) \leq g(x) \end{cases}$$

$$\therefore h(x) = f(x) - g(x) - |g(x) - f(x)|.$$

Example 1.59 Draw the graph of the function $f(x) = \max\{\sin x, \cos 2x\}$, $x \in [0, 2\pi]$. Write the equivalent definition of $f(x)$ and find the range of the function.

$$\text{Sol. } \sin x = \cos 2x$$

$$\Rightarrow \sin x = 1 - 2 \sin^2 x$$

$$\Rightarrow 2 \sin^2 x + \sin x - 1 = 0$$

$$\Rightarrow (2 \sin x - 1)(\sin x + 1) = 0$$

$$\Rightarrow \sin x = 1/2 \text{ or } \sin x = -1$$

$$\Rightarrow x = \pi/6, 5\pi/6 \text{ or } x = \pi$$

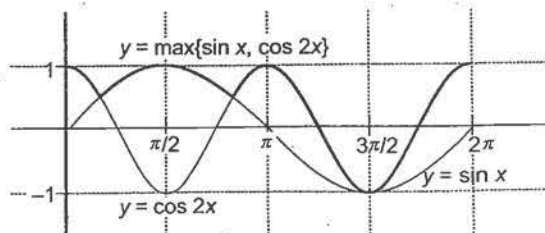


Fig. 1.52

$$\text{From the graph } f(x) = \begin{cases} \cos 2x, & 0 \leq x < \frac{\pi}{6} \\ \sin x, & \frac{\pi}{6} \leq x < \frac{5\pi}{6} \\ \cos 2x, & \frac{5\pi}{6} < x \leq 2\pi \end{cases}$$

Also range of the function is $[-1, 1]$.

Concept Application Exercise 1.8

- Consider the function $f(x) = \max\{1, |x-1|, \min\{4, |3x-1|\}\} \forall x \in R$, then find the value of $f(3)$.
- Find the equivalent definition of $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$ where $0 \leq x \leq 1$.
- Write the equivalent definition and draw the graphs of the following function.
 - $f(x) = \operatorname{sgn}(\log_e |x|)$
 - $f(x) = \operatorname{sgn}(x^3 - x)$

DIFFERENT TYPES OF MAPPINGS(FUNCTIONS)

One-One and Many-One Functions

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one-one. One-one functions are also called **injective** functions.

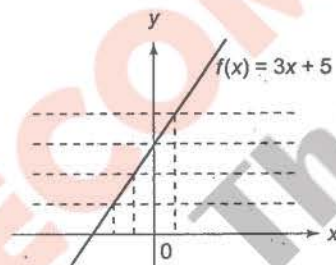
For example, $f: R \rightarrow R$ given by $f(x) = 3x + 5$ is one-one.

On the other hand, if there are at least two elements in the domain whose images are the same, the function is known as many-one.

For example, $f: R \rightarrow R$ given by $f(x) = x^2 + 1$ is many-one.

Note that a function will be either one-one or many-one.

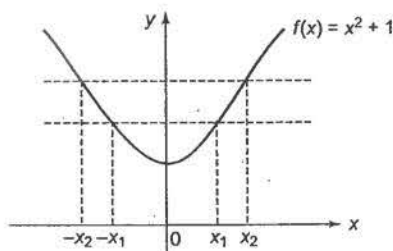
Lines drawn parallel to the x -axis from the each corresponding image point should intersect the graph of $y = f(x)$ at one (and only one) point if $f(x)$ is one-one and there will be at least one line parallel to x -axis that will cut the graph at least at two different points if $f(x)$ is many-one and vice versa.



Graph of $f(x) = 3x + 5$ (one-one function)

Fig. 1.53

Note that a many-one function can be made one-one by restricting the domain of the original function.



Graph of $f(x) = x^2 + 1$ (many-one function)

Fig. 1.54

Methods to Determine One-One and Many-One

- Let $x_1, x_2 \in \text{domain of } f$ and if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for every x_1, x_2 in the domain, then f is one-one else many-one.
- Conversely, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-one else many-one.
- If the function is entirely increasing or decreasing in the domain, then f is one-one else many-one.
- Any continuous function $f(x)$ that has at least one local maxima or local minima is many-one.
- All even functions are many-one.
- All polynomials of even degree defined in R have at least one local maxima or minima and hence are many-one in the domain R . Polynomials of odd degree can be one-one or many-one.
- If f is a rational function, then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain. Hence, we can write $f(x_1) - f(x_2) = (x_1 - x_2)g(x_1, x_2)$ where $g(x_1, x_2)$ is some function in x_1 and x_2 . Now, if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and lies in the domain, then f is many-one else one-one.
- Draw the graph of $y = f(x)$ and determine whether $f(x)$ is one-one or many-one.

Example 1.60 Let $f: R \rightarrow R$ where $f(x) = \frac{x^2 + 4x + 7}{x^2 + x + 1}$. Is $f(x)$ one-one?

Sol. Let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1^2 + 4x_1 + 7}{x_1^2 + x_1 + 1} = \frac{x_2^2 + 4x_2 + 7}{x_2^2 + x_2 + 1}$$

$$\Rightarrow (x_1 - x_2)(2x_1 + 2x_2 + x_1x_2 + 1) = 0$$

One solution is obviously $x_1 = x_2$

Let us consider $2x_1 + 2x_2 + x_1x_2 + 1 = 0$

Here, we have got a relation between x_1 and x_2 and for each value of x_1 in the domain we get a corresponding value of x_2 which may or may not be same as x_1 . Let us check this out:

If $x_1 = 0$, we get $x_2 = -1/2 \neq x_1$ and both lie in the domain of f . Hence, we have two different values $x_1 = 0$ and $x_2 = -1/2$ for which $f(x)$ has the same value.

That is, $f(0) = f(-1/2) = 7$ and hence f is many-one.

Example 1.61 If $f: X \rightarrow [1, \infty)$ be a function defined as $f(x) = 1 + 3x^3$. Find the super set of all the sets X such that $f(x)$ is one-one.

Sol. Note that $f(x) \geq 1$

$$\Rightarrow 1 + 3x^3 \geq 1$$

$$\Rightarrow x^3 \geq 0$$

$$\Rightarrow x \in [0, \infty).$$

Moreover, for $x_1, x_2 \in [0, \infty)$, $x_1 \neq x_2$

$$\Rightarrow 1 + 3x_1^3 \neq 1 + 3x_2^3$$

$$\Rightarrow f(x_1) \neq f(x_2)$$

Thus, $f: [1, \infty)$ is one-one for $x \in [0, \infty)$.

Onto and Into Functions

Let $f: X \rightarrow Y$ be a function. If each element in the co-domain Y has at least one pre-image in the domain X , that is, for every $y \in Y$ there exists at least one element $x \in X$ such that $f(x) = y$, then f is onto. In other words, the range of $f = Y$ for onto functions.

On the other hand, if there exists at least one element in the co-domain Y which is not an image of any element in the domain X , then f is into.

Onto function is also called *surjective function* and a function which is both one-one and onto is called *bijective function*.

For example, $f: R \rightarrow R$ where $f(x) = \sin x$ is into.

$f: R \rightarrow R$ where $f(x) = ax^3 + b$ is onto where $a \neq 0$, $b \in R$.

Note that a function will be either onto or into.

Methods to Determine Whether a Function is Onto or Into

- If range = co-domain, then f is onto. If range is a proper subset of co-domain, then f is into.
- Solve $f(x) = y$ for x , say $x = g(y)$.

Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain of f for all $y \in$ co-domain, then $f(x)$ is onto. If this requirement is not met by at least one value of y in the co-domain, then $f(x)$ is into.

Remark

- An into function can be made onto by redefining the co-domain as the range of the original function.
- Any polynomial function $f: R \rightarrow R$ is onto if degree is odd; into if degree of f is even.

Number of Functions (Mappings)

Consider set A has n different elements and set B has r different elements and function $f: A \rightarrow B$

Description	Equivalent to number of ways in which n different balls can be distributed among r persons if	Number of functions
1. Total number of functions	Any one can get any number of objects	r^n
2. Total number of one-to-one function	Each gets exactly 1 object or permutation of n different objects taken r at a time	$\begin{cases} {}^r C_n \cdot n!, & r \geq n \\ 0, & r < n \end{cases}$
3. Total number of many-one functions	At least one gets more than one ball	$\begin{cases} r^n - {}^n C_1 \cdot n!, & r \geq n \\ r^n, & r < n \end{cases}$
4. Total number of onto functions	Each gets at least one ball	$\begin{cases} r^n - {}^r C_1 (r-1)^n + {}^r C_2 (r-2)^n - {}^r C_3 (r-3)^n + \dots, & r < n \\ r!, & r = n \\ 0, & r > n \end{cases}$
5. Total number of into function		$\begin{cases} {}^r C_1 (r-1)^n - {}^n C_2 (r-2)^n + {}^r C_3 (r-3)^n - \dots, & r \leq n \\ r^n, & r > n \end{cases}$
6. Total number of constant functions	All the balls are received by any one person	r

Example 1.62 Let $f: R \rightarrow R$ where $f(x) = \sin x$. Show that f is into.

Sol. Since the co-domain of f is the set R , whereas the range of f is the interval $[-1, 1]$, hence f is into.

Can you make it onto?

The answer is 'yes', if you redefine the co-domain.

Let f be defined from R to another set $Y = [-1, 1]$, i.e.,

$f: R \rightarrow Y$ where $f(x) = \sin x$, then f is onto as range $f(x) = [-1, 1] = Y$.

Example 1.63 Let $f: N \rightarrow Z$ be a function defined as $f(x) = x - 1000$. Show that f is an into function.

Sol. Let $f(x) = y = x - 1000$

$$\Rightarrow x = y + 1000 = g(y) \text{ (say)}$$

Here, $g(y)$ is defined for each $y \in I$, but $g(y) \notin N$ for $y \leq -1000$. Hence, f is into.

Example 1.64 Let $A = \{x: -1 \leq x \leq 1\} = B$ be a mapping $f: A \rightarrow B$.

Then, match the following columns:

Column I (Function)	Column II (Type of mapping)
p. $f(x) = x $	a. one-one
q. $f(x) = x x $	b. many-one
r. $f(x) = x^3$	c. onto
s. $f(x) = [x]$ where $[\cdot]$ represents greatest integer function	d. into
t. $f(x) = \sin \frac{\pi x}{2}$	

p-(b, d) q-(a, c) r-(a, c) s-(b, d) t-(a, c)

Sol. p. $f(x) = |x|$

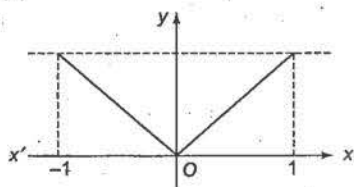


Fig. 1.55

The graph shows that $f(x)$ is many-one, as the straight line parallel to x -axis and cuts at two points. Here, the range for $f(x) \in [0, 1]$ which is clearly a subset of co-domain, i.e., $[0, 1] \subset [-1, 1]$. Thus, into.

Hence, function is many-one-into, therefore neither injective nor surjective.

$$q. f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

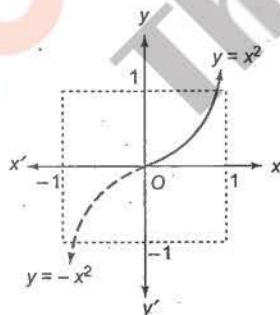


Fig. 1.56

The graph shows that $f(x)$ is one-one, as the straight line parallel to x -axis cuts only at one point.

Here, the range $f(x) \in [-1, 1]$.

Thus, range = co-domain.

Hence, onto.

Therefore, $f(x)$ is one-one and onto or (bijective).

r. $f(x) = x^3$,

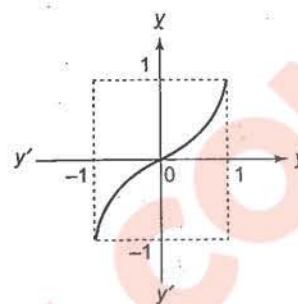


Fig. 1.57

The graph shows that $f(x)$ is one-one onto (i.e., bijective) (as explained in the above example).

s. $f(x) = [x]$,

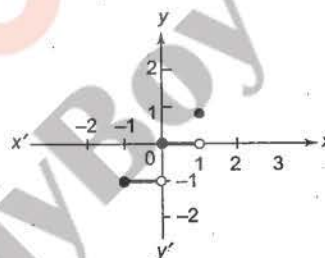


Fig. 1.58

The graph shows that $f(x)$ is many-one, as the straight line parallel to x -axis meets at more than one point.

Here, the range $f(x) \in \{-1, 0, 1\}$, which shows into as the range \subset co-domain.

Hence, many-one-into.

t. $f(x) = \sin \frac{\pi x}{2}$

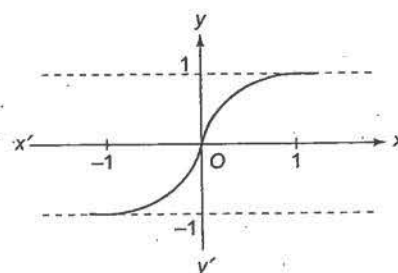


Fig. 1.59

The graph shows that $f(x)$ is one-one and onto as range = co-domain.

Therefore, $f(x)$ is bijective.

Example 1.65 Show $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = (x-1)(x-2)(x-3)$ is surjective but not injective.

Sol.

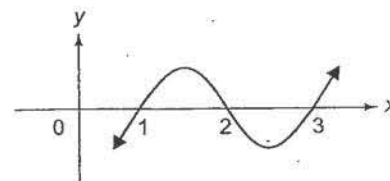


Fig. 1.60

Graphically, $y = f(x) = (x-1)(x-2)(x-3)$, which is clearly many-one and onto.

Example 1.66 If the function $f: R \rightarrow A$ given by $f(x) = \frac{x^2}{x^2+1}$ is surjection, then find A .

Sol. The domain of $f(x)$ is all real numbers.

Since, $f: R \rightarrow A$ is surjective, therefore A must be the range of $f(x)$.

Let $f(x) = y$

$$\Rightarrow y = \frac{x^2}{x^2+1}$$

$$\Rightarrow x^2 y + y = x^2$$

$$\Rightarrow x = \sqrt{\frac{y}{1-y}} \text{ exists, if } \frac{y}{1-y} \geq 0$$

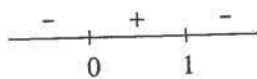


Fig. 1.61

$$\Rightarrow 0 \leq y < 1. \text{ Hence, } A \in [0, 1).$$

Example 1.67 If $f: R \rightarrow R$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then identify the type of function.

Sol. $f(x) = x^3 + x^2 + 3x + \sin x$

$$\Rightarrow f'(x)$$

$$= 3x^2 + 2x + 3 + \cos x$$

$$= 3 \left[\left(x + \frac{1}{3} \right)^2 + \frac{8}{9} \right] - (-\cos x) > 0 \text{ as } 3 \left[\left(x + \frac{1}{3} \right)^2 + \frac{8}{9} \right]_{\min}$$

$$= \frac{8}{3} \text{ and } -\cos x \text{ has a maximum value } 1.$$

$\Rightarrow f(x)$ is strictly increasing and hence one-one.

Also $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$. Thus, the range of $f(x)$ is R , hence onto.

Example 1.68 If $f: R \rightarrow R, f(x) = \begin{cases} x|x| - 4, & x \in Q \\ x|x| - \sqrt{3}, & x \notin Q \end{cases}$, then

identify the type of function.

Sol. $f(2) = f(3^{1/4}) \Rightarrow$ many-to-one function
and $f(x) \neq \sqrt{3} \forall x \in R \Rightarrow$ into function.

Concept Application Exercise 1.9

1. Which of the following functions from Z to itself are bijections?

a. $f(x) = x^3$

b. $f(x) = x + 2$

c. $f(x) = 2x + 1$

d. $f(x) = x^2 + x$

2. If $f: N \rightarrow Z$ $f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$ identify the

3. If $f: R \rightarrow R$ given by $f(x) = \frac{x^2 - 4}{x^2 + 1}$, identify the type of function.

4. If $f: R \rightarrow S$, defined by $f(x) = \sin x - \sqrt{3} \cos x + 1$, is onto, then find the set S .

5. Let $f: (-1, 1) \rightarrow B$ be a function defined by $f(x) = \tan^{-1} \frac{2x}{1-x^2}$, then f is both one-one and onto when B is the interval.

a. $\left[0, \frac{\pi}{2} \right)$

b. $\left(0, \frac{\pi}{2} \right)$

c. $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

d. $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

6. Let $g: R \rightarrow \left(0, \frac{\pi}{3} \right]$ is defined by $g(x) = \cos^{-1} \left(\frac{x^2 - k}{1 + x^2} \right)$.

Then find the possible values of ' k ' for which g is subjective function.

EVEN AND ODD FUNCTIONS

Even Function

A function $y = f(x)$ is said to be an even function if $f(-x) = f(x) \forall x \in D_f$.

Graph of an even function $y = f(x)$ is symmetrical about the y -axis, i.e., if point (x, y) lies on the graph then $(-x, y)$ also lies on the graph.

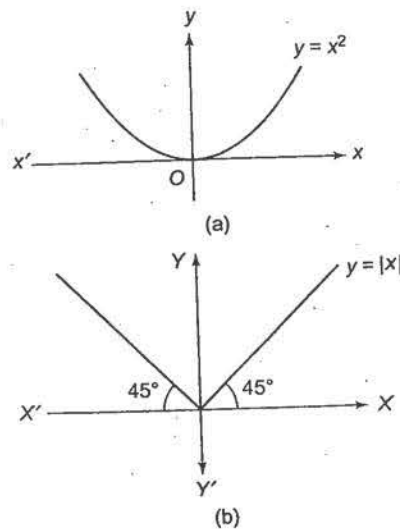


Fig. 1.62

Odd Function

A function $y = f(x)$ is said to be an odd function if $f(-x) = -f(x) \forall x \in D_f$.

Graph of an odd function $y = f(x)$ is symmetrical in opposite quadrants, i.e., if point (x, y) lies on the graph then $(-x, -y)$ also lies on the graph.

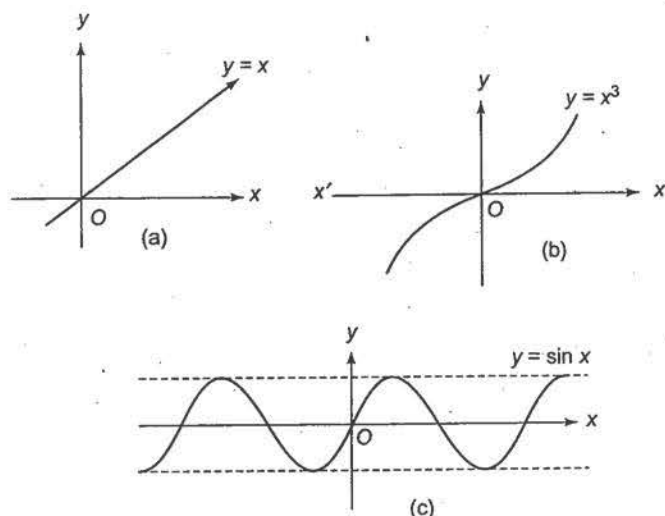


Fig. 1.63

Properties of Odd and Even Functions

- Sometimes, it is easy to prove that $f(x) - f(-x) = 0$ for even functions and $f(x) + f(-x) = 0$ for odd functions.

- A function can be either even or odd or neither.
- Function (not necessarily even or odd) can be expressed as a sum of an even and an odd function, i.e.,

$$f(x) = \left(\frac{f(x) + f(-x)}{2} \right) + \left(\frac{f(x) - f(-x)}{2} \right)$$

Let $h(x) = \left(\frac{f(x) + f(-x)}{2} \right)$ and $g(x) = \left(\frac{f(x) - f(-x)}{2} \right)$. It can now easily be shown that $h(x)$ is even and $g(x)$ is odd.

- The first derivative of an even function is an odd function and vice versa.
- If $x = 0 \in$ domain of f , then for odd function $f(x)$ which is continuous at $x = 0$, $f(0) = 0$, i.e., if for a function, $f(0) \neq 0$, then that function cannot be odd. It follows that for a differentiable even function $f'(0) = 0$, i.e., if for a differentiable function $f'(0) \neq 0$ then the function f cannot be even.
- $f(x) = 0$ is the only function which is defined on the entire number line is even and odd at the same time.
- Every even function $y = f(x)$ is many-one $\forall x \in D_f$.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$	$f(x)g(x)$	$f(x)/g(x)$	$f \circ g(x)$
Even	Even	Even	Even	Even	Even	Even
Even	Odd	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Even	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Odd	Odd	Odd	Even	Even	Odd

Example 1.69 Which of the following functions is (are) even, odd or neither

a. $f(x) = x^2 \sin x$.

b. $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$.

c. $f(x) = \log \left(\frac{1-x}{1+x} \right)$.

d. $f(x) = \log \left(x + \sqrt{1+x^2} \right)$.

e. $f(x) = \sin x - \cos x$.

f. $f(x) = \frac{e^x + e^{-x}}{2}$.

Sol. a. $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$, hence $f(x)$ is odd.

b. $f(-x) = \sqrt{1+(-x)+(-x)^2} - \sqrt{1-(-x)+(-x)^2}$
 $= \sqrt{1-x+x^2} - \sqrt{1+x+x^2}$
 $= -f(x)$, hence $f(x)$ is odd.

c. $f(-x) = \log \left(\frac{1-(-x)}{1+(-x)} \right) = \log \left(\frac{1+x}{1-x} \right)$

$= -f(x)$, hence $f(x)$ is odd.

d. $f(-x) = \log \left(-x + \sqrt{1+(-x)^2} \right)$

$= \log \left(\frac{(-x + \sqrt{1+x^2})(x + \sqrt{1+x^2})}{(x + \sqrt{1+x^2})} \right)$

$= \log \left(\frac{1}{x + \sqrt{1+x^2}} \right) = -f(x)$, hence $f(x)$ is odd.

e. $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$. Hence $f(x)$ is neither even nor odd.

f. $f(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = f(x)$, hence $f(x)$ is even.

Example 1.70 Prove that $f(x)$ given by $f(x+y) = f(x) + f(y) \forall x \in R$ is an odd function.

Sol. Given $f(x+y) = f(x) + f(y) \forall x \in R$ (1)
 Replace y by $-x$, we have $f(x-x) = f(x) + f(-x)$ (2)
 $\Rightarrow f(x) + f(-x) = f(0)$
 Now put $x = y = 0$ in (1), we have $f(0+0) = f(0) + f(0)$
 $\Rightarrow f(0) = 0$.

Then from (2), $f(x) + f(-x) = 0$. Hence, $f(x)$ is an odd function.

Extension of Domain

Let a function be defined on a certain domain which is entirely non-negative (or non-positive). The domain of $f(x)$ can be extended to the set $X = \{-x : x \in \text{domain of } f(x)\}$ in two ways:

- Even extension:** The even extension is obtained by defining a new function $f(-x)$ for $x \in X$, such that $f(-x) = f(x)$.
- Odd extension:** The odd extension is obtained by defining a new function $f(-x)$ for $x \in X$, such that $f(-x) = -f(x)$.

Example 1.71 If $f(x) = \begin{cases} x^3 + x^2 & \text{for } 0 \leq x \leq 2 \\ x + 2 & \text{for } 2 < x \leq 4 \end{cases}$. Then

find the even and odd extension of $f(x)$.

Sol. For even extension $f(x) = f(-x)$

$$f(x) = f(-x) = \begin{cases} (-x)^3 + (-x)^2, & 0 \leq -x \leq 2 \\ -x + 2, & 2 < -x \leq 4 \end{cases}$$

$$= \begin{cases} -x + 2, & -4 \leq x < -2 \\ -x^3 + x^2, & -2 \leq x \leq 0 \end{cases}$$

The odd extension of $f(x)$ is as follows:

$$h(x) = \begin{cases} x - 2, & -4 \leq x < -2 \\ x^3 - x^2, & -2 \leq x \leq 0. \end{cases}$$

Example 1.72 Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined on the interval $[0, 1]$. Define functions $g(x)$ and $h(x)$ in $[-1, 0]$ satisfying $g(-x) = -f(x)$ and $h(-x) = f(x)$ $\forall x \in [0, 1]$.

Sol. Clearly, $g(x)$ is the odd extension of the function $f(x)$ and $h(x)$ is the even extension.

Since x^2 , $\cos x$, $\log(1 + |x|)$ are even functions and x , $\sin x$ are odd functions.

$$g(x) = -x^2 + x + \sin x + \cos x - \log(1 + |x|)$$

$$\text{and } h(x) = x^2 - x - \sin x - \cos x + \log(1 + |x|)$$

Clearly, this function satisfies the restriction of the problem.

Concept Application Exercise 1.10

Identify the following functions whether odd or even or neither

- $f(x) = (g(x) - g(-x))^3$
- $f(x) = \log \left(\frac{x^4 + x^2 + 1}{x^2 + x + 1} \right)$
- $f(x) = xg(x)g(-x) + \tan(\sin x)$
- $f(x) = \cos |x| + \left\lfloor \frac{\sin x}{2} \right\rfloor$

where $[.]$ denotes the greatest integer function.

$$5. f(x) = \log \left(x + \sqrt{x^2 + 1} \right)$$

$$6. f(x) = \begin{cases} x|x|, & x \leq -1 \\ [x+1] + [1-x], & -1 < x < 1, \\ -x|x|, & x \geq 1 \end{cases}$$

where $[.]$ represents the greatest integer function.

PERIODIC FUNCTIONS

A function $f: X \rightarrow Y$ is said to be a periodic function if there exists a positive real number T such that $f(x + T) = f(x)$, for all $x \in X$. The least of all such positive numbers T is called the principal period or simply period of f . All periodic functions can be analyzed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

In other words, a function is said to be periodic function if its each value is repeated after a definite interval.

Here, the least positive value of T is called the fundamental period of the function. Clearly, $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \dots$

Important Facts about Periodic Functions

- If $f(x)$ is periodic with period T , then $af(x+b) \pm c$ where $a, b, c \in R (a \neq 0)$ is also periodic with period T .
- If $f(x)$ is periodic with period T , then $f(ax+b)$, where $a, b \in R (a \neq 0)$ is also period with period $\frac{T}{|a|}$.

Proof: Consider $a > 0$

$$\text{Let } f(x+T) = f(x) \text{ and } f[a(x+T') + b] = f(ax+b)$$

$$\Rightarrow f(ax+b+aT') = f(ax+b)$$

$$\Rightarrow f(y+aT') = f(y) = f(y+T)$$

$$\Rightarrow T = |aT'| \Rightarrow T' = T/|a| \quad (\because \text{period is always positive}).$$

- Let $f(x)$ has period $p = m/n$ ($m, n \in N$ and co-prime) and $g(x)$ has period $q = r/s$ ($r, s \in N$ and co-prime) and let t be the LCM of p and q , i.e., $t = \frac{\text{LCM of } (m, r)}{\text{HCF of } (n, s)}$

Then t will be the period of $f+g$, provided there does not exist a positive number $k (< t)$ for which

$f(x+k) + g(x+k) = f(x) + g(x)$, else k will be the period. The same rule is applicable for any other algebraic combination of $f(x)$ and $g(x)$.

LCM of p and q exists if p and q are rational quantities. If p and q are irrational, then LCM of p and q does not exist unless they have same irrational surd. LCM of rational and irrational is not possible.

- $\sin^n x$, $\cos^n x$, $\operatorname{cosec}^n x$ and $\sec^n x$ have period 2π if n is odd and π if n is even.
- $\tan^n x$ and $\cot^n x$ have period π whether n is odd or even.
- A constant function is periodic but does not have a well-defined period.

- If g is periodic, then $f \circ g$ will always be a periodic function. Period of $f \circ g$ may or may not be the period of g .
- A continuous periodic function is bounded.
- If $f(x)$, $g(x)$ are periodic functions with periods T_1 , T_2 , respectively, then, we have $h(x) = f(x) + g(x)$ has period as
 - LCM of $\{T_1, T_2\}$; if $f(x)$ and $g(x)$ cannot be interchanged by adding a least positive number less than the LCM of $\{T_1, T_2\}$.
 - k ; if $f(x)$ and $g(x)$ can be interchanged by adding a least positive number k ($k < \text{LCM of } \{T_1, T_2\}$).

For example, consider the function $f(x) = |\sin x| + |\cos x|$, $|\sin x|$ and $|\cos x|$ have period π , hence according to the rule of LCM period of $f(x)$ is π .

$$\text{But } f\left(x + \frac{\pi}{2}\right) = \left|\sin\left(x + \frac{\pi}{2}\right)\right| + \left|\cos\left(x + \frac{\pi}{2}\right)\right| \\ = |\cos x| + |\sin x|. \text{ Hence, period of } f(x) \text{ is } \pi/2.$$

Example 1.73 Find the periods (if periodic) of the following functions, $[.]$ denotes the greatest integer function.

- $f(x) = e^{\log(\sin x)} + \tan^3 x - \operatorname{cosec}(3x - 5)$
- $f(x) = x - [x - b]$, $b \in \mathbb{R}$
- $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$
- $f(x) = \tan \frac{\pi}{2}[x]$

Sol. a. $f(x) = e^{\log(\sin x)} + \tan^3 x - \operatorname{cosec}(3x - 5)$

Period of $e^{\log(\sin x)}$ is 2π , $\tan^3 x$ is π , $\operatorname{cosec}(3x - 5)$ is $\frac{2\pi}{3}$

$$\therefore \text{period} = \text{L.C.M of } \left\{2\pi, \pi, \frac{2\pi}{3}\right\} = 2\pi.$$

b. $f(x) = x - [x - b] = b + \{x - b\}$, (\therefore period of $\{.\}$ is 1) $\Rightarrow f(x)$ has period 1.

c. $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\pi/2$.

$$\text{Hence, period of } f(x) = \text{L.C.M. of } \left\{\frac{\pi}{2}, \pi\right\} = \pi$$

d. $f(x) = \tan \frac{\pi}{2}[x] \Rightarrow \tan \frac{\pi}{2}[x + T] = \tan \frac{\pi}{2}[x]$

$$\Rightarrow \frac{\pi}{2}[x + T] = n\pi + \frac{\pi}{2}[x]$$

$$\Rightarrow \text{Period} = 2 \text{ (least positive value).}$$

Example 1.74 Find the period if $f(x) = \sin x + \{x\}$, where $\{x\}$ is the fractional part of x .

Sol. Here $\sin x$ is periodic with period 2π , $\{x\}$ is periodic with 1. LCM of 2π (irrational) and 1 (rational) does not exist. Thus, $f(x)$ is not periodic.

Example 1.75 If $f(x) = \sin x + \cos ax$ is a periodic function, show that a is a rational number.

Sol. Period of $\sin x = 2\pi = \frac{2\pi}{1}$ and period of $\cos ax = \frac{2\pi}{|a|}$

$$\therefore \text{period of } \sin x + \cos ax = \text{LCM of } \frac{2\pi}{1} \text{ and } \frac{2\pi}{|a|}$$

$$= \frac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of } 1 \text{ and } a} = \frac{2\pi}{\lambda} \text{ where } \lambda \text{ is the HCF of } 1 \text{ and } a.$$

Since λ is the HCF of 1 and a , $\frac{1}{\lambda}$ and $\frac{|a|}{\lambda}$ should be both integers

Suppose $\frac{1}{\lambda} = p$ and $\frac{|a|}{\lambda} = q$, then $\frac{|a|}{\frac{1}{p}} = \frac{q}{p}$, where

$$p, q \in \mathbb{Z}$$

$$\text{i.e., } |a| = \frac{q}{p}.$$

Hence, a is the rational number.

Example 1.76 Discuss whether the function $f(x) = \sin(\cos x + x)$ is periodic or not, if yes then what is its period.

Sol. Clearly, $f(x + 2\pi) = \sin(\cos(2\pi + x) + 2\pi + x)$
 $= \sin(2\pi + (x + \cos x))$
 $= \sin(x + \cos x)$

Hence, period is 2π .

Example 1.77 Find the period of $\cos(\cos x) + \cos(\sin x)$.

Sol. Clearly, the domain of the function is \mathbb{R}

$$\text{Let } f(x) = f(x + T), \text{ for all } x$$

$$\Rightarrow f(0) = f(T)$$

$$\Rightarrow \cos 1 + 1 = \cos(\cos T) + \cos(\sin T)$$

$$\text{Clearly, } T = \frac{\pi}{2} \text{ satisfies the equation, hence period is } \frac{\pi}{2}.$$

Example 1.78 Find the period of the function satisfying the relation $f(x) + f(x + 3) = 0 \forall x \in \mathbb{R}$.

Sol. Given $f(x) + f(x + 3) = 0$ (1)

$$\text{Replace } x \text{ by } x + 3,$$

$$\text{We have } f(x + 3) + f(x + 6) = 0 \quad (2)$$

$$\text{From (1) and (2), } f(x) = f(x + 6).$$

Hence, the function has period 6.

Concept Application Exercise 1.11

1. Match the column

Column I (Function)	Column II (Period)
p. $f(x) = \sin^3 x + \cos^4 x$	a. $\pi/2$
q. $f(x) = \cos^4 x + \sin^4 x$	b. π
r. $f(x) = \sin^3 x + \cos^3 x$	c. 2π
s. $f(x) = \cos^4 x - \sin^4 x$	

2. Which of the following functions is not periodic?
- $|\sin 3x| + \sin^2 x$
 - $\cos \sqrt{x} + \cos^2 x$
 - $\cos 4x + \tan^2 x$
 - $\cos 2x + \sin x$
3. Let $[x]$ denote the greatest integer less than or equal to x . If the function $f(x) = \tan(\sqrt{[n]}x)$ has period $\frac{\pi}{3}$, then find the values of n .
4. Find the period of
- $\frac{|\sin 4x| + |\cos 4x|}{|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x|}$
 - $f(x) = \sin \frac{\pi x}{n!} - \cos \frac{\pi x}{(n+1)!}$
 - $f(x) = \sin x + \tan \frac{x}{2} + \sin \frac{x}{2^2} + \tan \frac{x}{2^3} + \dots + \sin \frac{x}{2^{n-1}} + \tan \frac{x}{2^n}$
5. If $f(x) = \lambda|\sin x| + \lambda^2|\cos x| + g(\lambda)$ has the period equal to $\pi/2$, then find the value of λ .
6. If $f(x)$ satisfying the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$ for all x , then prove that $f(x)$ is periodic and find its period.

COMPOSITE FUNCTION

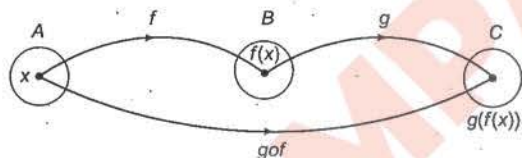


Fig. 1.64

Let A, B and C be three non-empty sets.

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions, then $gof: A \rightarrow C$. This function is called composition of f and g , given by

$$gof(x) = g(f(x)) \quad \forall x \in A.$$

Thus, the image of every $x \in A$ under the function gof is the g -image of the f -image of x .

The gof is defined only if $\forall x \in A, f(x)$ is an element of the domain of g so that we can take its g -image.

The range of f must be a subset of the domain of g in gof .

Properties of Composite Functions

- The composition of functions is not commutative in general, i.e., $fog \neq gof$.
- The composition of functions is associative i.e., if $h: A \rightarrow B, g: B \rightarrow C$ and $f: C \rightarrow D$ be three functions, then $(fog)h = fo(goh)$.
- The composition of any function with the identity function is the function itself, i.e., $f: A \rightarrow B$ then $foI_A = I_B \circ f = f$ where I_A and I_B are the identity functions of A and B , respectively.
- If $f: A \rightarrow B$ and $g: B \rightarrow C$ are one-one, then $gof: A \rightarrow C$ is also one-one.

Proof:

Suppose $gof(x_1) = gof(x_2)$

$$\Rightarrow g(f(x_1)) = g(f(x_2))$$

$$\Rightarrow f(x_1) = f(x_2), \text{ as } g \text{ is one-one}$$

$$\Rightarrow x_1 = x_2, \text{ as } f \text{ is one-one}$$

Hence, gof is one-one.

- e. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are onto, then $gof: A \rightarrow C$ is also onto.

Proof:

Given an arbitrary element $z \in C$, there exists a pre-image y of z under g such that $g(y) = z$, since g is onto. Further, for $y \in B$, there exists an element x in A with $f(x) = y$, since f is onto.

Therefore, $gof(x) = g(f(x)) = g(y) = z$, showing that gof is onto.

- f. If $gof(x)$ is one-one, then $f(x)$ is necessarily one-one but $g(x)$ may not be one-one.

Consider the function $f(x)$ and $g(x)$ as shown in the following figure.

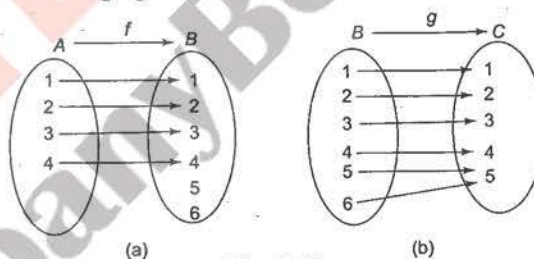


Fig. 1.65

Here f is one-one, but g is many-one. But $g(f(x)) = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ is one-one.

- g. If $gof(x)$ is onto, then $g(x)$ is necessarily onto but $f(x)$ may not be onto.

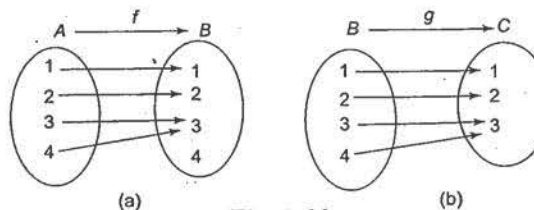


Fig. 1.66

Here, f is into and g is onto. But $gof(x) = \{(1, 1), (2, 2), (3, 3), (4, 3)\}$ is onto.

Thus, it can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.

Example 1.79 Let $f: \{2, 3, 4, 5\} \rightarrow \{3, 4, 5, 9\}$ and $g: \{3, 4, 5, 9\} \rightarrow \{7, 11, 15\}$ be functions defined as $f(2) = 3, f(3) = 4, f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof .

Sol. We have $gof(2) = g(f(2)) = g(3) = 7, gof(3) = g(f(3)) = g(4) = 7, gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(f(5)) = g(5) = 11$.

Example 1.80 Let $f(x)$ and $g(x)$ be bijective functions where $f: \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$ and $g: \{3, 4, 5, 6\} \rightarrow \{w, x, y, z\}$, respectively. Then, find the number of elements in the range set of $g(f(x))$.

Sol. The range of $f(x)$ for which $g(f(x))$ is defined is $\{3, 4\}$.
Hence, the domain of $g\{f(x)\}$ has two elements.
 \therefore The range of $g(f(x))$ also has two elements.

Example 1.81 Let $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0$, $c \neq 0$.
Assume $a = 1$, $b = 2$. If $(fog)(x) = (gof)(x)$ for all x , what can you say about c and d ?

Sol. $(fog)(x) = f(g(x)) = a(cx + d) + b$
and $(gof)(x) = g(f(x)) = c(ax + b) + d$
Given that $(fog)(x) = (gof)(x)$ and at $a = 1$, $b = 2$
 $\Rightarrow cx + d + 2 = cx + 2c + d \Rightarrow c = 1$ and d is arbitrary.

Example 1.82 Suppose that $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$, then find the function $f(x)$.

Sol. $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$ (1)
 $\Rightarrow f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$
Put $1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2$
Then $f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2$.
Therefore, $f(x) = 2 + x^2$.

Example 1.83 The function $f(x)$ is defined in $[0, 1]$. Find the domain of $f(\tan x)$.

Sol. Here, $f(x)$ is defined in $[0, 1]$.
 $\Rightarrow x \in [0, 1]$, i.e., the only value of x that we can substitute lies between $[0, 1]$
For $f(\tan x)$ to be defined, we must have $0 \leq \tan x \leq 1$
[as x replaced by $\tan x$]

i.e., $n\pi \leq x \leq n\pi + \frac{\pi}{4}$, $n \in \mathbb{Z}$ [in general]

Thus, the domain for $f(\tan x) \in \left[n\pi, n\pi + \frac{\pi}{4}\right]$, $n \in \mathbb{Z}$.

Example 1.84 $f(x) = \begin{cases} x+1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x^3, & x < 1 \\ 2x-1, & x \geq 1 \end{cases}$, then find $f(g(x))$ and find its domain and range.

Sol. $f(g(x)) = \begin{cases} g(x)+1, & g(x) < 0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$

$$= \begin{cases} x^3+1, & x^3 < 0, x < 1 \\ 2x-1+1, & 2x-1 < 0, x \geq 1 \\ (x^3)^2, & x^3 \geq 0, x < 1 \\ (2x-1)^2, & 2x-1 \geq 0, x \geq 1 \end{cases} = \begin{cases} x^3+1, & x < 0 \\ x^6, & 0 \leq x < 1 \\ (2x-1)^2, & x \geq 1 \end{cases}$$

For $x < 0$, $x^3 + 1 \in (-\infty, 1)$

For $0 \leq x < 1$, $x^6 \in [0, 1)$

For $x \geq 1$, $(2x-1)^2 \in [1, \infty)$

Hence, the range is \mathbb{R} and function is many-one.

Concept Application Exercise 1.12

- If f be the greatest integer function and g be the modulus function, then find the value of $(gof)\left(-\frac{5}{3}\right) - (fog)\left(-\frac{5}{3}\right)$.
- Let $f(x) = \begin{cases} 1+|x|, & x < -1 \\ [x], & x \geq -1 \end{cases}$, where $[.]$ denotes the greatest integer function. Then find the value of $f\{f(-2.3)\}$.
- If $f(x) = \log \left[\frac{1+x}{1-x} \right]$, then prove that $f\left[\frac{2x}{1+x^2} \right] = 2f(x)$.
- If the domain for $y = f(x)$ is $[-3, 2]$, find the domain of $g(x) = f([x])$, where $[.]$ denotes the greatest integer function.
- Let f be a function defined on $[0, 2]$, then the domain of function $g(x) = f(9x^2 - 1)$.
- $f(x) = \begin{cases} \log_e x, & 0 < x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$ and $g(x) = \begin{cases} x+1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$, then find $g(f(x))$.
- Let $f(x) = \tan x$, $g(f(x)) = f\left(x - \frac{\pi}{4}\right)$ where $f(x)$ and $g(x)$ are real-valued functions. Prove that $f(g(x)) = \tan\left(\frac{x-1}{x+1}\right)$.
- A function f has domain $[-1, 2]$ and range $[0, 1]$. Find the domain and range of the function g defined by $g(x) = 1 - f(x+1)$.

INVERSE FUNCTIONS

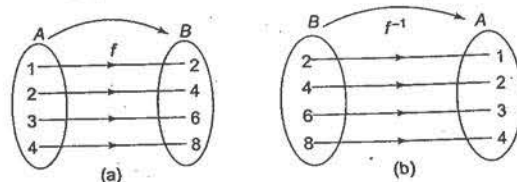


Fig. 1.67

If $f: A \rightarrow B$ be a function defined by $y = f(x)$ such that f is both one-one and onto, then there exists a unique function $g: B \rightarrow A$ such that for each $y \in B$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of f and denoted by f^{-1} . Also if g is the inverse of f , then f is the inverse of g and the two functions f and g are said to be inverses of each other.

The condition for the existence of inverse of a function is that the function must be one-one and onto. Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and the domain of the original function becomes the range of the inverse function.

Properties of Inverse Functions

- The inverse of bijective function is unique and bijective.
- Let $f: A \rightarrow B$ be a function such that f is bijective and $g: B \rightarrow A$ is inverse of f , then $fog = I_B$ = identity function of set B . Then $gof = I_A$ = identity function of set A .
- If $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ and $fog(x) = gof(x) = x$.

- If f and g are two bijective functions such that $f: A \rightarrow B$ and $g: B \rightarrow C$, then $g \circ f: A \rightarrow C$ is bijective. Also $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- Graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about $y = x$ line and intersect on line $y = x$ or $f(x) = f^{-1}(x) = x$ whenever graphs intersect.

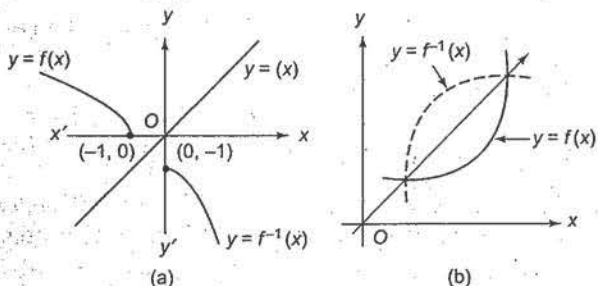


Fig. 1.68

But in the case of the function $f(x) = \begin{cases} x+4, & x \in [1, 2] \\ -x+7, & x \in [5, 6] \end{cases}$,

$$f^{-1}(x) = \begin{cases} x-4, & x \in [5, 6] \\ 7-x, & x \in [1, 2] \end{cases}$$

$y = f(x)$ and $y = f^{-1}(x)$ intersect at $(3/2, 11/2)$ and $(11/2, 3/2)$ which do not lie on the line $y = x$.

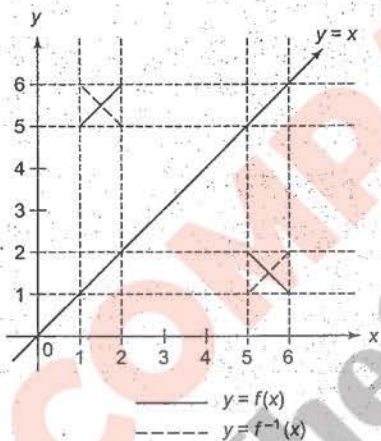


Fig. 1.69

Example 1.87

Which of the following functions has inverse function?

- $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = x + 2$
- $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = 2x$
- $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = x$
- $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = |x|$

Sol. Functions in options a and c are both one-one and have range \mathbb{Z} , i.e., onto, hence invertible. $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = 2x$ is one-one but has only even integers in the range, hence not onto.

$f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = |x|$ is many-one and has range $\mathbb{N} \cup \{0\}$.

Thus, both the functions are not invertible.

Example 1.88

Let $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$ and let $f: A \rightarrow B$

defined by $f(x) = \frac{x-2}{x-3}$. Is f invertible? Explain.

Sol. Let $x_1, x_2 \in A$
and let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 3x_1 + 6$$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow f \text{ is one-one.}$$

To find whether f is onto or not, first let us find the range of f .

$$\text{Let } y = f(x) = \frac{x-2}{x-3}$$

$$\Rightarrow xy - 3y = x - 2$$

$$\Rightarrow x(y-1) = 3y-2$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

x is defined if $y \neq 1$, i.e., the range of f is $\mathbb{R} - \{1\}$ which is also the co-domain of f .

Also, for no value of y , x can be 3, i.e., if we put

$$3 = x = \frac{3y-2}{y-1}$$

$$\Rightarrow 3y - 3 = 3y - 2 \Rightarrow -3 = -2 \text{ not possible. Hence, } f \text{ is onto.}$$

Example 1.89

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = (e^x - e^{-x})/2$. Is $f(x)$ invertible? If so, find its inverse.

Sol. Let us check for invertibility of $f(x)$

(a) One-one

Let $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$

$$\Rightarrow e^{x_1} < e^{x_2} \quad (\because e > 1) \quad (1)$$

Also $x_1 < x_2 \Rightarrow -x_2 < -x_1$

$$\Rightarrow e^{-x_2} < e^{-x_1} \quad (\because e > 1) \quad (2)$$

$$(1) + (2) \Rightarrow \frac{1}{2}(e^{x_1} - e^{-x_1}) < \frac{1}{2}(e^{x_2} - e^{-x_2}) \Rightarrow f(x_1) < f(x_2).$$

i.e. f is one-one.

(b) Onto

As $x \rightarrow \infty, f(x) \rightarrow \infty$

Similarly, as $x \rightarrow -\infty, f(x) \rightarrow -\infty$, i.e., $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$.

Hence, the range of f is same as the set R . Therefore, $f(x)$ is onto.

Since $f(x)$ is both one-one and onto, $f(x)$ is invertible.

(c) To find f^{-1}

$$\begin{aligned} y &= f(x) = (e^x - e^{-x})/2 \\ \Rightarrow e^x - e^{-x} &= 2y \\ \Rightarrow e^{2x} - 2ye^x - 1 &= 0 \\ \Rightarrow e^x &= \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1} \\ \Rightarrow e^x &= y + \sqrt{y^2 + 1} \\ (\text{as } y - \sqrt{y^2 + 1} < 0 \text{ for all } y \text{ and } e^x \text{ is always positive}) \\ \Rightarrow x &= \log_e (y + \sqrt{y^2 + 1}) \\ \Rightarrow f^{-1}(x) &= \log_e (x + \sqrt{x^2 + 1}) \end{aligned}$$

Example 1.90 If $f(x) = (ax^2 + b)^3$, then find the function g such that $f(g(x)) = g(f(x))$.

$$\begin{aligned} \text{Sol. } f(g(x)) &= g(f(x)) \\ f(x) &= (ax^2 + b)^3 \\ \text{If } g(x) &= f^{-1}(x) \\ y &= (ax^2 + b)^3 \Rightarrow \sqrt[3]{\frac{y}{a} - b} = x \\ \Rightarrow g(x) &= \sqrt[3]{\frac{x^{1/3} - b}{a}} \end{aligned}$$

Example 1.91 If $f(x) = 3x - 2$ and $(g \circ f)^{-1}(x) = x - 2$, then find the function $g(x)$.

$$\begin{aligned} \text{Sol. } f(x) &= 3x - 2 \\ \Rightarrow f^{-1}(x) &= \frac{x+2}{3} \\ \text{Now } (g \circ f)^{-1}(x) &= x - 2 \\ \Rightarrow f^{-1} \circ g^{-1}(x) &= x - 2 \\ \Rightarrow f^{-1}(g^{-1}(x)) &= x - 2 \\ \Rightarrow \frac{g^{-1}(x) + 2}{3} &= x - 2 \\ \Rightarrow g^{-1}(x) &= 3x - 8 \\ \Rightarrow g(x) &= \frac{x+8}{3} \end{aligned}$$

Example 1.92 Find the inverse of $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \leq x \leq 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

$$\text{Sol. Given } f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \leq x \leq 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$$

Let $f(x) = y$

$$\Rightarrow x = f^{-1}(y) \quad (1)$$

$$\therefore x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \leq \sqrt{y} \leq 4 \\ y^2/64, & y^2/64 > 4 \end{cases} = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \leq y \leq 16 \\ y^2/64, & y > 16 \end{cases}$$

$$f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \leq y \leq 16 \\ y^2/64, & y > 16 \end{cases} \quad [\text{From (1)}]$$

$$\text{Hence, } f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \leq x \leq 16 \\ x^2/64, & x > 16 \end{cases}$$

Example 1.93 Solve the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$, where $x \geq \frac{3}{4}$.

$$\text{Sol. } f(x) = x^2 - x + 1$$

$$\text{and } g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

are inverse of one another so $f(x) = g(x)$.

When $f(x) = x$

$$\begin{aligned} \Rightarrow x^2 - x + 1 &= x \\ \Rightarrow x &= 1 \end{aligned}$$

Concept Application Exercise 1.13

Find the inverse of the following functions:

- $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$
- $f: R \rightarrow (-\infty, 1)$ is given by $f(x) = 1 - 2^{-x}$
- Let $f: (2, 3) \rightarrow (0, 1)$ be defined by $f(x) = x - [x]$, where $[.]$ represents greatest integer function.
- $f: Z \rightarrow Z$ be defined by $f(x) = [x + 1]$, where $[.]$ denotes the greatest integer function.
- $f(x) = \begin{cases} x^3 - 1, & x < 2 \\ x^2 + 3, & x \geq 2 \end{cases}$
- $f: [-1, 1] \rightarrow [-1, 1]$ defined by $f(x) = x|x|$
- $f: (-\infty, 1] \rightarrow \left[\frac{1}{2}, \infty\right)$, where $f(x) = 2^{x(x-2)}$

IDENTICAL FUNCTION

Two functions f and g are said to be identical if

- The domain of f = the domain of g , i.e., $D_f = D_g$.

- b. The range of f = the range of g .
- c. $f(x) = g(x)$, $\forall x \in D_f$ or $x \in D_g$, e.g., $f(x) = x$ and $g(x) = \sqrt{x^2}$ are not identical functions as $D_f = D_g$ but $R_f = R$, $R_g = [0, \infty)$.

Example 1.94 Find the values of x for which the following functions are identical.

- a. $f(x) = x$ and $g(x) = \frac{1}{1/x}$
- b. $f(x) = \cos x$ and $g(x) = \frac{1}{\sqrt{1+\tan^2 x}}$
- c. $f(x) = \frac{\sqrt{9-x^2}}{\sqrt{x-2}}$ and $g(x) = \sqrt{\frac{9-x^2}{x-2}}$
- d. $f(x) = \tan^{-1}x + \tan^{-1}\frac{1}{x}$ and $g(x) = \sin^{-1}x + \cos^{-1}x$

Sol. a. $f(x) = x$ is defined for all x .

But $g(x) = \frac{1}{1/x} = x$ is not defined for $x = 0$ as $1/x$ is not defined at $x = 0$.

Hence, both the functions are identical for $x \in R - \{0\}$.

b. $f(x) = \cos x$ has the domain R and the range $[-1, 1]$.

But $g(x) = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{\sec^2 x}} = |\cos x|$, has

domain $R - \{(2n+1)\pi/2, n \in Z\}$ as $\tan x$ is not defined for $x = (2n+1)\pi/2, n \in Z$.

Also, the range of $g(x) = |\cos x|$ is $[0, 1]$.

Hence, $f(x)$ and $g(x)$ are identical if x lies in 1st and 4th quadrant.

$$\Rightarrow x \in \left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right), n \in Z.$$

c. $f(x) = \frac{\sqrt{9-x^2}}{\sqrt{x-2}}$ is defined if $9-x^2 \geq 0$ and $x-2 > 0$

$$\Rightarrow x \in [-3, 3] \text{ and } x > 2 \Rightarrow x \in (2, 3]$$

$g(x) = \sqrt{\frac{9-x^2}{x-2}}$ is defined if $\frac{9-x^2}{x-2} \geq 0$

$$\Rightarrow \frac{x^2-9}{x-2} \leq 0$$

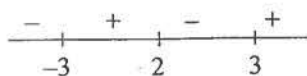


Fig. 1.70

From the sign scheme, $x \in (-\infty, -3] \cup (2, 3]$.
Hence, $f(x)$ and $g(x)$ are identical if $x \in (2, 3]$.

$$d. f(x) = \tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$$

$$\text{and } g(x) = \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \text{ for } x \in [-1, 1].$$

Hence, the functions are identical if $x \in (0, 1]$

TRANSFORMATION OF GRAPHS

a. $f(x)$ transforms to $f(x) \pm a$ (where a is +ve)

That is, $f(x) \rightarrow f(x) + a$ shift the given graph of $f(x)$ upward through a units.

$f(x) \rightarrow f(x) - a$, shift the given graph of $f(x)$ downward through a units.

Graphically, it could be stated as

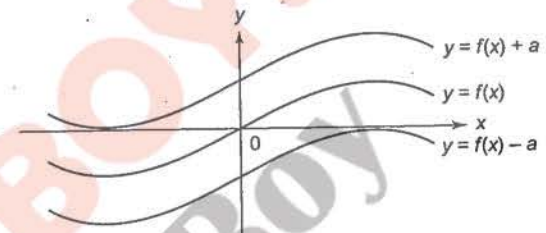


Fig. 1.71

b. $f(x)$ transforms to $f(x-a)$

That is, $f(x) \rightarrow f(x-a)$; a is positive. Shift the graph of $f(x)$ through a unit towards right.

That is, $f(x) \rightarrow f(x+a)$; a is positive. Shift the graph of $f(x)$ through a units towards left.

Graphically, it could be stated as

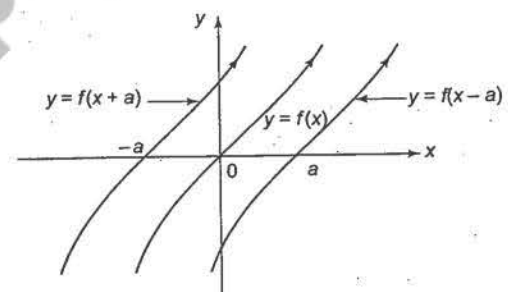


Fig. 1.72

Example 1.95 Plot $y = |x|$, $y = |x-2|$ and $y = |x+2|$.

Sol. As discussed $f(x) \rightarrow f(x-a)$; shift towards right.

$\Rightarrow y = |x-2|$ is shifted 2 units towards right.

Also $y = |x+2|$ is shifted 2 units towards left.

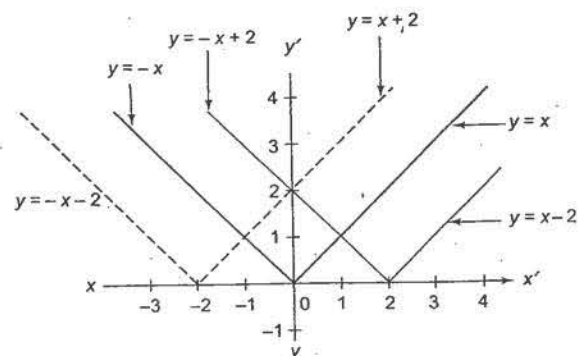


Fig. 1.73

c. $f(x)$ transforms to $f(ax)$

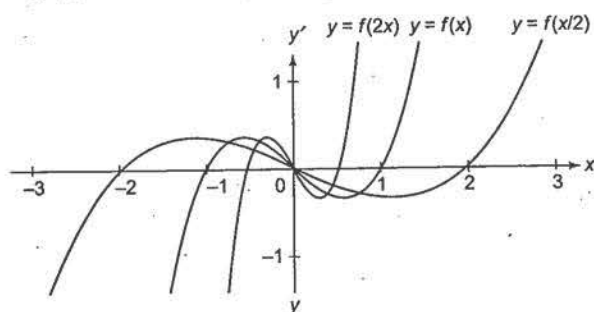


Fig. 1.74

That is, $f(x) \rightarrow f(ax)$; $a > 1$.

Shrink (or contract) the graph of $f(x)$ a times along the x -axis.

Again $f(x) \rightarrow f\left(\frac{1}{a}x\right)$; $a > 1$, stretch (or expand) the graph

of $f(x)$ a times along the x -axis.

Graphically, it could be stated as shown in Fig. 1.74.

Example 1.96 Plot $y = \sin x$ and $y = \sin 2x$.

Sol. Here $y = \sin 2x$, shrink (or contract) the graph of $\sin x$ by factor of 2 along the x -axis.

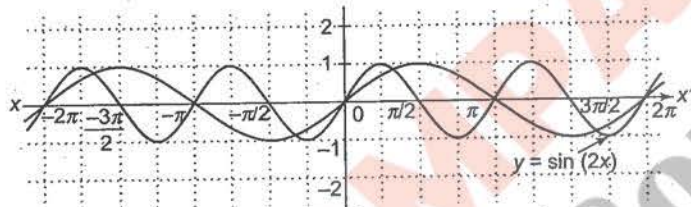


Fig. 1.75

From Fig. 1.75, $\sin x$ is periodic with period 2π and $\sin 2x$ with period π .

Example 1.97 Plot $y = \sin x$ and $y = \sin \frac{x}{2}$.

Sol. Here $y = \sin \left(\frac{x}{2}\right)$; stretch (or expand) the graph of $\sin x$, 2 times along the x -axis.

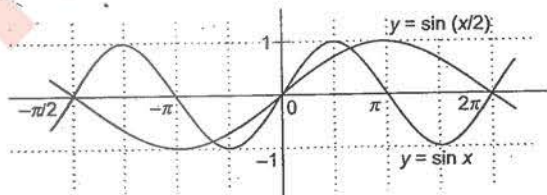


Fig. 1.76

From Fig. 1.76, $\sin x$ is periodic with period 2π and $\sin \left(\frac{x}{2}\right)$ is periodic with period 4π .

d. $f(x)$ transforms to $y = af(x)$

It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a .



Fig. 1.77

Example 1.98

Consider the function $f(x) = \begin{cases} 2x+3, & x \leq 1 \\ -x^2+6, & x > 1 \end{cases}$

Then draw the graph of the function $y = f(x)$, $y = f(|x|)$, $y = |f(x)|$ and $y = |f(|x|)$.

Sol.

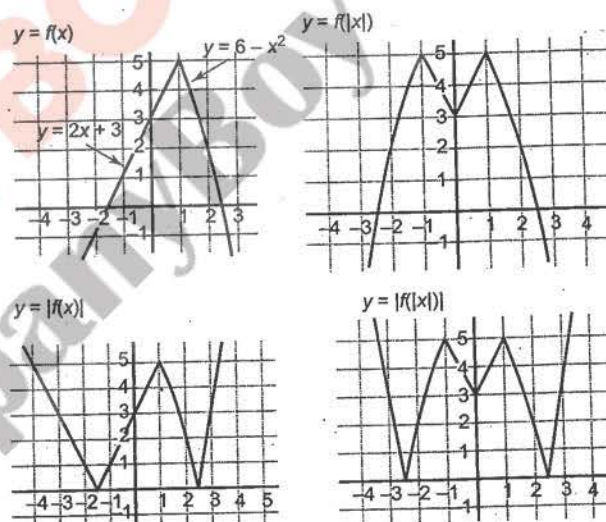


Fig. 1.78

Example 1.99 Plot $y = \sin x$ and $y = 2 \sin x$.

Sol. We know $y = \sin x$ and $f(x) \rightarrow af(x)$

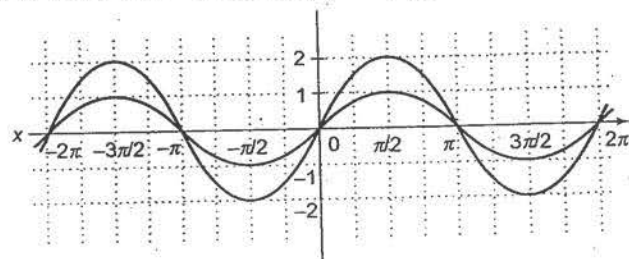


Fig. 1.79

\Rightarrow Stretch the graph of $f(x)$ a times along the y -axis.

$\therefore y = 2 \sin x$.

\Rightarrow Stretch the graph of $\sin x$, 2 times along y -axis.

e. $f(x)$ transforms to $f(-x)$

That is, $f(x) \rightarrow f(-x)$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the y -axis as plane mirror.

Or

Turn the graph of $f(x)$ by 180° about the y -axis.

Graphically, it is shown as

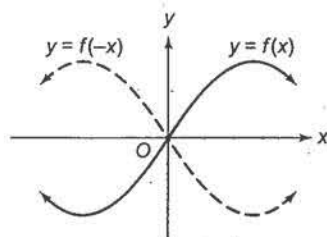


Fig. 1.80

Example 1.100 Plot the curve $y = \log_e(-x)$.

Sol. Here $y = \log_e(-x)$; take mirror image of $y = \log_e x$ about y -axis. Graphically, it is shown as

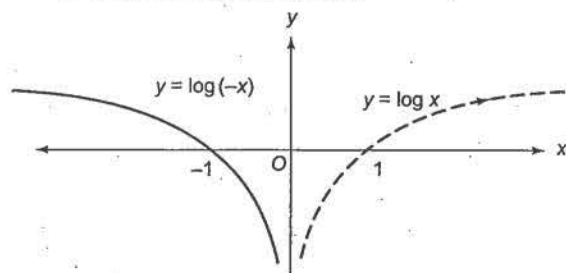


Fig. 1.81

f. $f(x)$ transforms to $-f(x)$

That is, $f(x) \rightarrow -f(x)$

To draw $y = -f(x)$, take image of $y = f(x)$ in the x -axis as plane mirror.

Or

Turn the graph of $f(x)$ by 180° about x -axis.

g. $f(x)$ transforms to $-f(-x)$

That is, $f(x) \rightarrow -f(-x)$

To draw $y = -f(-x)$, take image of $f(x)$ about y -axis to obtain $f(-x)$ and then the image of $f(-x)$ about x -axis to obtain $-f(-x)$.

$\therefore f(x) \rightarrow -f(-x)$

\Rightarrow a. Image about y -axis

b. Image about x -axis

Graphically, it is shown as

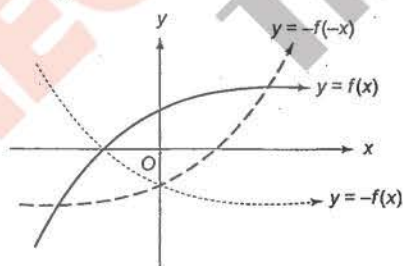


Fig. 1.82

h. $f(x)$ transform to $y = |f(x)|$

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$. It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \geq 0$ and the parts where $f(x) < 0$ would get inverted in the upward direction.

Figure 1.82 would make the procedure clear.

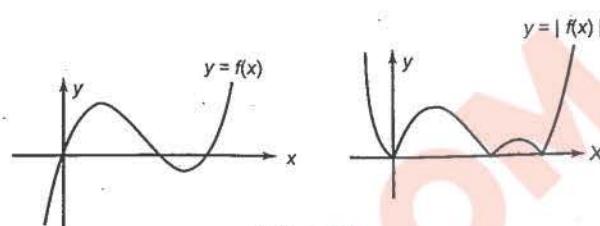


Fig. 1.83

Example 1.101 Draw the graph for $y = |\log x|$.

Sol. To draw graph for $y = |\log x|$, we have to follow two steps:

a. Leave the (+ve) part of $y = \log x$ as it is.

b. Take images of (-ve) part of $y = \log x$, i.e., the part below x -axis in the x -axis as plane mirror. Graphically, it is shown as

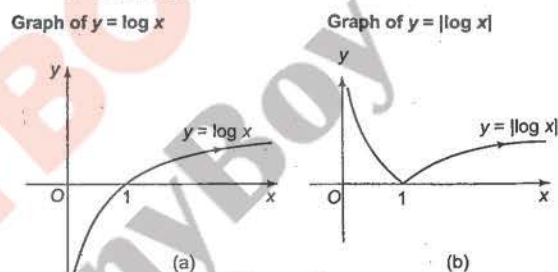


Fig. 1.84

$y = \log_e x$ is differentiable for all $x \in (0, \infty)$ (Fig. 1.84(a))

$y = |\log_e x|$ is clearly differentiable for all $x \in (0, \infty) - \{1\}$ as at $x = 1$ there is a sharp edge (Fig. 1.84(b)).

Example 1.102 Sketch the graph for $y = |\sin x|$.

Sol. Here $y = \sin x$ is known.

\therefore To draw $y = |\sin x|$, we take the mirror image (in x -axis) of the part of the graph of $\sin x$ which lies below x -axis.

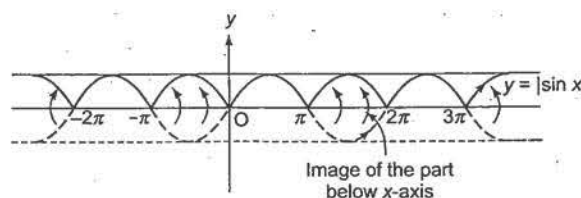


Fig. 1.85

From the above figure, it is clear

$y = |\sin x|$ is differentiable for all $x \in \mathbb{R} - \{n\pi, n \in \text{integer}\}$.

i. $f(x)$ transforms to $f(|x|)$

That is, $f(x) \rightarrow f(|x|)$

If we know $y = f(x)$, then to plot $y = f(|x|)$, we would follow two steps:

a. Leave the graph lying right side of the y -axis as it is.

b. Take the image of $f(x)$ in the right of y -axis with y -axis as the plane mirror and the graph of $f(x)$ lying left-side of the y -axis (if it exists) is omitted.

Or

Neglect the curve for $x < 0$ and take the images of curves for $x \geq 0$ about y -axis.

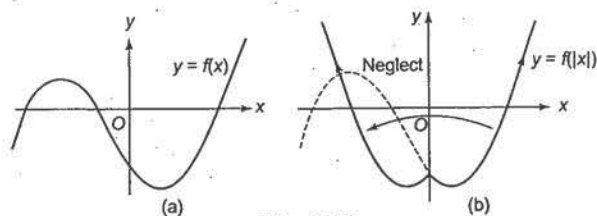


Fig. 1.86

Example 1.103 Sketch the curve $y = \log |x|$.

Sol. As we know, the curve $y = \log x$.

$\therefore y = \log |x|$ could be drawn in two steps:

- Leave the graph lying right side of y -axis as it is.
- Take the image of $f(x)$ in the y -axis as plane mirror.

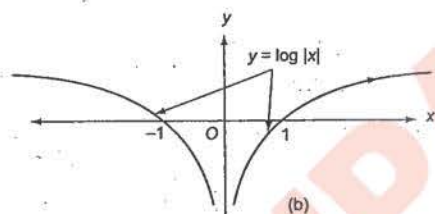
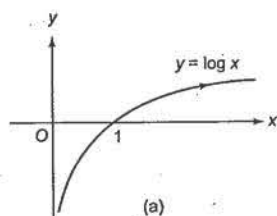


Fig. 1.87

j. Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly, $|y| \geq 0$, if $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence, the graph of $|y| = f(x)$ would exist only in the regions where $f(x)$ is non-negative and will be reflected about x -axis only in those regions. Regions where $f(x) < 0$ will be neglected.



Fig. 1.88

k. Drawing the graph of $y = [f(x)]$ from the known graph of $y = f(x)$

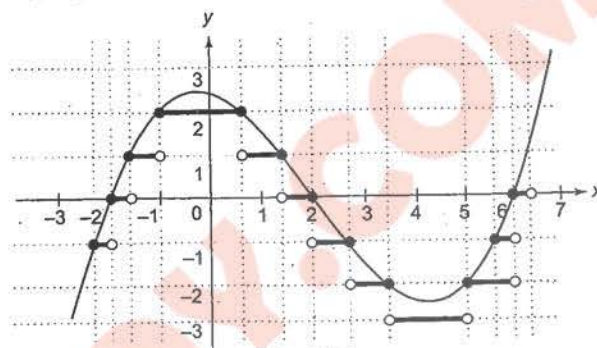


Fig. 1.90

It is clear that if $n \leq f(x) < n+1$, $n \in I$ then $[f(x)] = n$. Thus, we would draw lines parallel to the x -axis passing through different integral points. Hence, the values of x can be obtained so that $f(x)$ lies between two successive integers.

This procedure can be clearly understood from Fig. 1.90.

Concept Application Exercise 1.14

Draw the graph of the following functions: (1 to 5)

- $f(x) = \sin |x|$
- $f(x) = ||x-2|-3|$
- $|f(x)| = \tan x$
- $f(x) = |x^2 - 3|x| + 2|$
- $f(x) = -|x-1|^{1/2}$
- Find the total number of solutions of $\sin \pi x = |\ln |x||$.
- Solve $\left| \frac{x^2}{x-1} \right| \leq 1$ using the graphical method.

8. Which of the following pair(s) of function have same graphs?

(a) $f(x) = \frac{\sec x}{\cos x} - \frac{\tan x}{\cot x}$, $g(x) = \frac{\cos x}{\sec x} + \frac{\sin x}{\csc x}$

(b) $f(x) = \operatorname{sgn}(x^2 - 6x + 10)$,
 $g(x) = \operatorname{sgn}\left(\cos^2 x + \sin^2\left(x + \frac{\pi}{3}\right)\right)$, where sgn denotes signum function.

(c) $f(x) = e^{\ln(x^2 + 3x + 3)}$, $g(x) = x^2 + 3x + 3$

(d) $f(x) = \frac{\sin x}{\sec x} + \frac{\cos x}{\csc x}$, $g(x) = \frac{2\cos^2 x}{\cot x}$

MISCELLANEOUS SOLVED PROBLEMS

1. Let $f: X \rightarrow Y$ be a function defined by $f(x)$

$$= a \sin \left(x + \frac{\pi}{4} \right) + b \cos x + c. \text{ If } f \text{ is both one-one and onto, find sets } X \text{ and } Y.$$

Sol. $f(x) = a \sin \left(x + \frac{\pi}{4} \right) + b \cos x + c$

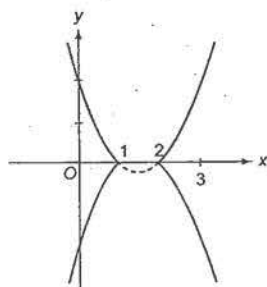


Fig. 1.90

$$\Rightarrow f(x) = a \left\{ \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} \right\} + b \cos x + c$$

$$\Rightarrow f(x) = \frac{a}{\sqrt{2}} \sin x + \left(\frac{a}{\sqrt{2}} + b \right) \cos x + c \quad (1)$$

$$\text{Let } \left(\frac{a}{\sqrt{2}} \right) = r \cos \alpha, \left(\frac{a}{\sqrt{2}} + b \right) = r \sin \alpha$$

$$\Rightarrow f(x) = r [\cos \alpha \sin x + \sin \alpha \cos x] + c$$

$$\Rightarrow f(x) = r [\sin(x + \alpha)] + c$$

$$\text{where } r = \sqrt{a^2 + \sqrt{2}ab + b^2}$$

$$\text{and } \alpha = \tan^{-1} \left(\frac{a + b\sqrt{2}}{a} \right) \quad (2)$$

For f to be one-one, we must have $-\pi/2 \leq x + \alpha \leq \pi/2$. Thus,

$$\text{domain} \in \left[-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha \right] \text{ and range} \in [c - r, c + r].$$

$$\text{Or } X = \left[-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha \right] \text{ and } Y = [c - r, c + r].$$

2. Find the set of all solutions of the equation $2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$.

$$\text{Sol. Here, } 2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1.$$

Case I: $y < 0$

$$\Rightarrow 2^{-y} + (2^{y-1} - 1) = 2^{y-1} + 1; \begin{cases} \text{as when } y < 0 \\ |y| = -y \\ \text{and } |2^{y-1} - 1| = -(2^{y-1} - 1) \end{cases}$$

$$\Rightarrow 2^{-y} = 2^1$$

Hence, $y = -1$, which is true when $y < 0$. (1)

Case II: $0 \leq y < 1$

$$\Rightarrow 2^y + (2^{y-1} - 1) = 2^{y-1} + 1; \begin{cases} \text{as when } 0 \leq y < 1 \\ |y| = y \\ \text{and } |2^{y-1} - 1| = -(2^{y-1} - 1) \end{cases}$$

$$\Rightarrow 2^y = 2$$

$\Rightarrow y = 1$, which shows no solution as $0 \leq y < 1$. (2)

Case III: $y \geq 1$

$$\Rightarrow 2^y - (2^{y-1} - 1) = 2^{y-1} + 1$$

$$\Rightarrow 2^y = 2^{y-1} + 2^{y-1}; \begin{cases} \text{as when } y \geq 0 \\ |y| = y \\ \text{and } |2^{y-1} - 1| = (2^{y-1} - 1) \end{cases}$$

$\Rightarrow 2^y = 2^y$, which is an identity, therefore it is true $\forall y \geq 1$ (3)

Hence, from (1), (2) and (3) the solution of the set is $\{y : y \geq 1 \cup y = -1\}$.

3. Let $x \in \left(0, \frac{\pi}{2} \right)$, then find the domain of the function

$$f(x) = \frac{1}{\sqrt{-\log_{\sin x} \tan x}}$$

$$\text{Sol. Here } x \in \left(0, \frac{\pi}{2} \right)$$

$$\Rightarrow 0 < \sin x < 1 \quad (1)$$

$$\text{and we know } \begin{cases} \log_a x < b \Rightarrow x > a^b, & \text{if } 0 < a < 1 \\ x < a^b, & \text{if } a > 1 \end{cases} \quad (2)$$

$$\text{Thus, } f(x) = \frac{1}{\sqrt{-\log_{\sin x} \tan x}} \text{ exists, if } -\log_{\sin x} (\tan x) > 0$$

$$\Rightarrow \log_{\sin x} \tan x < 0$$

[as inequality sign changes on multiplying by $-ve$]

$$\Rightarrow \tan x > (\sin x)^0 \quad [\text{using (1) and (2)}]$$

$$\Rightarrow \tan x > 1$$

$$\Rightarrow x \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right) \quad [\text{as } x \in (0, \pi/2)]$$

4. Find whether the given function is even or odd function,

$$\text{where } f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x+\pi}{\pi} \right] - \frac{1}{2}}, \text{ where } [] \text{ denotes the}$$

greatest integer function.

$$\text{Sol. } f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x+\pi}{\pi} \right] - \frac{1}{2}} = \frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi} \right] + 1 - \frac{1}{2}}$$

$$= \frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi} \right] + 0.5}$$

$$\Rightarrow f(-x) = \frac{-x(\sin(-x) + \tan(-x))}{\left[-\frac{x}{\pi} \right] + 0.5}$$

$$\Rightarrow f(-x) = \begin{cases} \frac{x(\sin x + \tan x)}{-1 - \left[\frac{x}{\pi} \right] + 0.5}, & x \neq n\pi \\ 0, & x = n\pi \end{cases}$$

$$\text{Hence, } f(-x) = - \left(\frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi} \right] + 0.5} \right) \text{ and } f(-x) = 0.$$

$f(-x) = -f(x)$. Hence, $f(x)$ is an odd function if $x \neq n\pi$ and $f(x) = 0$ if $x = n\pi$ is both even and odd function.

5. Let $f(x)$ be periodic and k be a positive real number such that $f(x+k) + f(x) = 0$ for all $x \in R$. Prove that $f(x)$ is a periodic with period $2k$.

Sol. We have $f(x+k) + f(x) = 0, \forall x \in R$

$$\Rightarrow f(x+k) = -f(x), \forall x \in R. \text{ Put } x = x+k$$

$$\Rightarrow f(x+2k) = -f(x+k), \forall x \in R \quad (\text{as } f(x+k) = -f(x))$$

$$\Rightarrow f(x+2k) = f(x), \forall x \in R$$

which clearly shows that $f(x)$ is periodic with period $2k$.

6. If $f(x)$ be a polynomial function satisfying $f(x) f\left(\frac{1}{x}\right)$

$$= f(x) + f\left(\frac{1}{x}\right) \text{ and } f(4) = 65. \text{ Then find } f(6).$$

Sol. Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$

$$\text{Then, } f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$$\Rightarrow (a_0x^n + a_1x^{n-1} + \dots + a_n) \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \dots + a_n \right)$$

$$= (a_0x^n + a_1x^{n-1} + \dots + a_n) + \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \dots + a_n \right)$$

On comparing the coefficient of x^n , we have $a_0a_n = a_0$

$$\Rightarrow a_n = 1 \quad (\text{as } a_0 \neq 0)$$

On comparing the coefficient of x^{n-1} , we have $a_0a_{n-1} + a_n a_1 = a_1$

$$\Rightarrow a_0a_{n-1} + a_1 = a_1 \quad (\text{as } a_n = 1)$$

$$\Rightarrow a_0a_{n-1} = 0$$

$$\Rightarrow a_{n-1} = 0 \quad (\text{as } a_0 \neq 0)$$

Similarly, $a_{n-1} = a_{n-2} = \dots = a_1 = 0$

and $a_0 = \pm 1$

$$\therefore f(x) = \pm x^n + 1, f(4) = \pm 4^n + 1$$

$$\Rightarrow 4^n + 1 = 65 \quad (\text{as } f(4) = 65)$$

$$\Rightarrow 4^n = 64$$

$$\Rightarrow n = 3.$$

So, $f(x) = x^3 + 1$. Hence, $f(6) = 6^3 + 1 = 217$.

7. Consider a real-valued function $f(x)$ satisfying $2f(xy) = (f(x))^y + (f(y))^x \quad \forall x, y \in R$ and $f(1) = a$ where $a \neq 1$. Prove

$$\text{that } (a-1) \sum_{i=1}^n f(i) = a^{n+1} - a.$$

Sol. We have $2f(xy) = (f(x))^y + (f(y))^x$

$$\text{Replacing } y \text{ by } 1, \text{ we get } 2f(x) = f(x) + (f(1))^x \Rightarrow f(x) = a^x$$

$$\Rightarrow \sum_{i=1}^n f(i) = a + a^2 + \dots + a^n = \frac{a^{n+1} - a}{a - 1}$$

$$\Rightarrow (a-1) \sum_{i=1}^n f(i) = a^{n+1} - a.$$

8. If $f(x+y+1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$ and $f(0) = 1, \forall x, y \in R$. Determine $f(n), n \in N$.

Sol. Given $f(x+y+1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$

Putting $x = y = 0$,

$$\text{then } f(1) = (\sqrt{f(0)} + \sqrt{f(0)})^2 = (1+1)^2 = 2^2.$$

Again putting $x = 0, y = 1$

$$\text{Then } f(2) = (\sqrt{f(0)} + \sqrt{f(1)})^2 = (1+2)^2 = 3^2$$

and for $x = 1, y = 1$

$$f(3) = (\sqrt{f(1)} + \sqrt{f(1)})^2 = (2+2)^2 = 4^2.$$

$$\text{Hence, } f(n) = (n+1)^2.$$

9. Check whether the function defined by $f(x + \lambda)$

$= 1 + \sqrt{2f(x) - f^2(x)} \quad \forall x \in R$ is periodic or not. If yes, then find its period. ($\lambda > 0$)

Sol. For the function to be true, $2f(x) - f^2(x) \geq 0$

$$\Rightarrow f(x)[f(x) - 2] \leq 0 \Rightarrow 0 \leq f(x) \leq 2 \quad (1)$$

$$\text{and from the given function, } f(x + \lambda) \geq 1 \Rightarrow f(x) \geq 1 \quad (2)$$

From (1) and (2), we have $1 \leq f(x) \leq 2$

Again, we have $\{f(x + \lambda) - 1\}^2 = 2f(x) - f^2(x)$

$$\Rightarrow \{f(x + \lambda) - 1\}^2 = 1 + \{2f(x) - f^2(x) - 1\}$$

$$\Rightarrow \{f(x + \lambda) - 1\}^2 = 1 - \{f(x) - 1\}^2 \quad (3)$$

Replacing x by $x + \lambda$, we get

$$\{f(x + 2\lambda) - 1\}^2 = 1 - \{f(x + \lambda) - 1\}^2 \quad (4)$$

Subtracting (3) from (4), we get

$$\{f(x + 2\lambda) - 1\}^2 = \{f(x) - 1\}^2$$

$$\Rightarrow |f(x + 2\lambda) - 1| = |f(x) - 1| \quad (\because 1 \leq f(x) \leq 2)$$

$$\Rightarrow f \text{ is periodic with period } 2\lambda.$$

10. If for all real values of u and v , $2f(u) \cos v = f(u + v) + f(u - v)$, prove that for all real values of x

$$\text{a. } f(x) + f(-x) = 2a \cos x.$$

$$\text{b. } f(\pi - x) + f(-x) = 0.$$

$$\text{c. } f(\pi - x) + f(x) = 2b \sin x.$$

Deduce that $f(x) = a \cos x + b \sin x$, where a, b are arbitrary constants.

Sol. Given $2f(u) \cos v = f(u + v) + f(u - v) \quad (1)$

Putting $u = 0$ and $v = x$ in (1), we get

$$f(x) + f(-x) = 2f(0) \cos x = 2a \cos x \quad (2)$$

a is an arbitrary constant.

Now putting $u = \frac{\pi}{2} - x$ and $v = \frac{\pi}{2}$ in (1), we get

$$f(\pi - x) + f(-x) = 0 \quad (3)$$

Again putting $u = \pi/2$ and $v = \pi/2 - x$ in (1), we get

$$f(\pi - x) + f(x) = 2f(\pi/2) \sin x = 2b \sin x \quad (4)$$

b is an arbitrary constant.

Adding (2) and (4), we get

$$2f(x) + f(\pi - x) + f(-x) = 2a \cos x + 2b \sin x$$

$$\Rightarrow 2f(x) + 0 = 2a \cos x + 2b \sin x \quad [\text{From (3)}]$$

$$\therefore f(x) = a \cos x + b \sin x.$$

11. Let $f(x) = \frac{9^x}{9^x + 3}$. Show $f(x) + f(1-x) = 1$, and hence evaluate

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \dots + f\left(\frac{1995}{1996}\right).$$

Sol. $f(x) = \frac{9^x}{9^x + 3} \quad (1)$

$$\text{and } f(1-x) = \frac{9^{1-x}}{9^{1-x} + 3}$$

$$\Rightarrow f(1-x) = \frac{\frac{9}{9^x}}{\frac{9}{9^x} + 3} = \frac{9}{9 + 3 \cdot 9^x}$$

$$\Rightarrow f(1-x) = \frac{3}{(3+9^x)} \quad (2)$$

Adding (1) and (2), we get $f(x) + f(1-x)$

$$= \frac{9^x}{9^x + 3} + \frac{3}{(3+9^x)} = 1$$

$$\Rightarrow f(x) + f(1-x) = 1. \quad (3)$$

Now, putting $x = \frac{1}{1996}, \frac{2}{1996}, \frac{3}{1996}, \dots, \frac{998}{1996}$ in (3), we get

$$f\left(\frac{1}{1996}\right) + f\left(\frac{1995}{1996}\right) = 1, f\left(\frac{2}{1996}\right) + f\left(\frac{1994}{1996}\right) = 1,$$

$$f\left(\frac{3}{1996}\right) + f\left(\frac{1993}{1996}\right) = 1$$

$$f\left(\frac{997}{1996}\right) + f\left(\frac{999}{1996}\right) = 1, f\left(\frac{998}{1996}\right) + f\left(\frac{998}{1996}\right) = 1$$

$$\text{or } f\left(\frac{998}{1996}\right) = \frac{1}{2}$$

Adding all the above expression, we get

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + \dots + f\left(\frac{1995}{1996}\right)$$

$$= (1+1+1+\dots+997) + \frac{1}{2} = 997 + \frac{1}{2} = 997.5$$

12. Let $f(x)$ be defined on $[-2, 2]$ and is given by

$$f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}, \text{ and } g(x) = f(|x|) + |f(x)|.$$

Then find $g(x)$.

Sol. We have

$$f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}$$

$$\Rightarrow f(|x|) = \begin{cases} -1, & -2 \leq |x| \leq 0 \\ |x|-1, & 0 \leq |x| \leq 2 \end{cases}$$

$$\Rightarrow f(|x|) = |x|-1, \quad 0 \leq |x| \leq 2$$

(as $-2 \leq |x| < 0$ is not possible)

$$\Rightarrow f(|x|) = \begin{cases} -x-1, & -2 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases} \quad (1)$$

$$\text{again, } f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x-1, & 0 \leq x \leq 2 \end{cases}$$

$$\Rightarrow |f(x)| = \begin{cases} |-1|, & -2 \leq x \leq 0 \\ |x-1|, & 0 < x \leq 2 \end{cases}$$

$$\Rightarrow |f(x)| = \begin{cases} 1, & -2 \leq x \leq 0 \\ -(x-1), & 0 < x \leq 1 \\ +(x-1), & 1 < x \leq 2 \end{cases} \quad (2)$$

$\therefore g(x) = f(|x|) + |f(x)|$ can be expressed as

$$g(x) = \begin{cases} (-x-1)+1, & -2 \leq x \leq 0 \\ (x-1)+(1-x), & 0 \leq x \leq 1 \\ (x-1)+(x-1), & 1 \leq x \leq 2 \end{cases}$$

[using (1) and (2)]

$$\Rightarrow g(x) = \begin{cases} -x, & -2 \leq x \leq 0 \\ 0, & 0 < x \leq 1 \\ 2(x-1), & 1 < x \leq 2 \end{cases}$$

13. For what integral value of n , is 3π period of the function $\cos(nx) \sin\left(\frac{5x}{n}\right)$?

Sol. Let $f(x) = \cos nx \sin\left(\frac{5x}{n}\right)$

$f(x)$ is periodic

$$\Rightarrow f(x+\lambda) = f(x) \text{ where } \lambda \text{ is period.}$$

$$\Rightarrow \cos(nx+n\lambda) \sin\left(\frac{5x+5\lambda}{n}\right) = \cos nx \sin\left(\frac{5x}{n}\right)$$

$$\text{at } x=0, \cos n\lambda \sin\left(\frac{5\lambda}{n}\right) = 0$$

$$\text{If } \cos n\lambda = 0$$

$$\Rightarrow n\lambda = r\pi + \frac{\pi}{2}, r \in I$$

$$\Rightarrow n(3\pi) = r\pi + \frac{\pi}{2} \quad (\because \lambda = 3\pi)$$

$$\Rightarrow 3n - r = \frac{1}{2} \text{ (Impossible).}$$

$$\text{Again, let } \sin\left(\frac{5\lambda}{n}\right) = 0$$

$$\therefore \frac{5\lambda}{n} = p\pi \quad (p \in I)$$

$$\Rightarrow \frac{5(3\pi)}{n} = p\pi \quad (\because \lambda = 3\pi)$$

$$\Rightarrow n = \frac{15}{p}$$

$$\text{For } p = \pm 1, \pm 3, \pm 5, \pm 15$$

$$\therefore n = \pm 15, \pm 5, \pm 3, \pm 1 \quad (\because n \in I)$$

14. Let $f: R \rightarrow R$ and $g: R \rightarrow R$ be functions defined

$$\text{by } f(x) = \begin{cases} 2x, & x < 1, \\ 2x^2 - 1, & x \geq 1, \end{cases} \quad g(x) = \begin{cases} x+2, & x < 0 \\ 2x, & x \geq 0. \end{cases}$$

Find (a) $f+g$, (b) fg .

Sol. $(f+g): R \rightarrow R$ and $(fg): R \rightarrow R$ are functions defined by $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$. To find $(f+g)(x)$ and $(fg)(x)$, we rewrite $f(x)$ and $g(x)$ as:

$$f(x) = \begin{cases} 2x, & x < 0 \\ 2x, & 0 \leq x < 1 \\ 2x^2 - 1, & x \geq 1 \end{cases} \quad g(x) = \begin{cases} x+2, & x < 0 \\ 2x, & 0 \leq x < 1 \\ 2x, & x \geq 1 \end{cases}$$

$$\text{Hence, a. } (f+g)(x) = \begin{cases} 3x+2, & x < 0 \\ 4x, & 0 \leq x < 1 \\ 2x^2+2x-1, & x \geq 1 \end{cases}$$

$$\text{b. } (fg)(x) = \begin{cases} 2x^2+4x, & x < 0 \\ 4x^2, & 0 \leq x < 1 \\ 4x^3-2x, & x \geq 1 \end{cases}$$

15. If $f(x) = -1 + |x-2|$, $0 \leq x \leq 4$ and $g(x) = 2 - |x|$, $-1 \leq x \leq 3$. Then find $(fog)(x)$ and $(gof)(x)$.

Sol. We have

$$f(x) = \begin{cases} 1-x, & 0 \leq x \leq 2 \\ x-3, & 2 < x \leq 4 \end{cases}$$

$$\text{and } g(x) = \begin{cases} 2+x, & -1 \leq x \leq 0 \\ 2-x, & 0 < x \leq 3 \end{cases}$$

$$\therefore (fog)x = f\{g(x)\} = \begin{cases} 1-g(x), & 0 \leq g(x) \leq 2 \\ g(x)-3, & 2 < g(x) \leq 4 \end{cases}$$

$$= \begin{cases} 1-(2+x), & 0 \leq 2+x \leq 2 \text{ and } -1 \leq x \leq 0 \\ 2+x-3, & 2 < 2+x \leq 4 \text{ and } -1 \leq x \leq 0 \\ 1-(2-x), & 0 \leq 2-x \leq 2 \text{ and } 0 < x \leq 3 \\ 2-x-3, & 2 < 2-x \leq 4 \text{ and } 0 < x \leq 3 \end{cases}$$

$$= \begin{cases} -1-x, & -2 \leq x \leq 0 \text{ and } -1 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \text{ and } -1 \leq x \leq 0 \\ -1+x, & 0 \leq x \leq 2 \text{ and } 0 < x \leq 3 \\ -x-1, & -2 \leq x < 0 \text{ and } 0 < x \leq 3 \end{cases}$$

$$= \begin{cases} -1-x, & -1 \leq x \leq 0 \\ x-1, & x \in \phi \\ -1+x, & 0 < x \leq 2 \\ -x-1, & x \in \phi \end{cases}$$

$$= \begin{cases} -1-x, & -1 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}$$

$$\text{and } (gof)x = g\{f(x)\} = \begin{cases} 2+f(x), & -1 \leq f(x) \leq 0 \\ 2-f(x), & 0 < f(x) \leq 3 \end{cases}$$

$$= \begin{cases} 2+1-x, & -1 \leq 1-x \leq 0 \text{ and } 0 \leq x \leq 2 \\ 2-(1-x), & 0 < 1-x \leq 3 \text{ and } 0 \leq x \leq 2 \\ 2+x-3, & -1 \leq x-3 \leq 0 \text{ and } 2 < x \leq 4 \\ 2-(x-3), & 0 < x-3 \leq 3 \text{ and } 2 < x \leq 4 \end{cases}$$

$$= \begin{cases} 3-x, & 1 \leq x \leq 2 \text{ and } 0 \leq x \leq 2 \\ 1+x, & -2 \leq x < 1 \text{ and } 0 \leq x \leq 2 \\ x-1, & 2 \leq x \leq 3 \text{ and } 2 < x \leq 4 \\ -x+5, & 3 < x \leq 6 \text{ and } 2 < x \leq 4 \end{cases}$$

$$= \begin{cases} 3-x, & 1 \leq x \leq 2 \\ 1+x, & 0 \leq x < 1 \\ x-1, & 2 < x \leq 3 \\ 5-x, & 3 < x \leq 4 \end{cases} = \begin{cases} 1+x, & 0 \leq x < 1 \\ 3-x, & 1 \leq x \leq 2 \\ x-1, & 2 < x \leq 3 \\ 5-x, & 3 < x \leq 4 \end{cases}$$

EXERCISES

Subjective Type

Solutions on page 1.57

1. Write explicit functions of y defined by the following equations and also find domains of definitions of the given implicit functions:

a. $x + |y| = 2y$

b. $e^y - e^{-y} = 2x$

c. $10^x + 10^y = 10$

d. $x^2 - \sin^{-1} y = \frac{\pi}{2}$

2. Let $g(x) = \sqrt{x-2k}$, $\forall 2k \leq x < 2(k+1)$, where $k \in \text{integer}$, check whether $g(x)$ is periodic or not.

3. Let $f(x) = x^2 - 2x$, $x \in R$ and $g(x) = f(f(x) - 1) + f(5 - f(x))$. Show that $g(x) \geq 0 \forall x \in R$.

4. If f and g are two distinct linear functions defined on R such that they map $[-1, 1]$ onto $[0, 2]$ and $h: R \rightarrow \{-1, 0, 1\} \rightarrow R$ defined by $h(x) = \frac{f(x)}{g(x)}$, then show that $|h(h(x)) + h(h(1/x))| > 2$.

5. Let $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$ and $g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$

Describe the function f/g and find its domain.

6. Let $f(x) = \log_2 \log_3 \log_4 \log_5 (\sin x + a^2)$. Find the set of values of a for which domain of $f(x)$ is R .

7. A certain polynomial $P(x)$, $x \in R$ when divided by $x - a$, $x - b$, $x - c$ leaves remainders a , b , c , respectively. Then find the remainder when $P(x)$ is divided by $(x - a)(x - b)(x - c)$ (a, b, c are distinct).

8. Let $R = \{(x, y) : x, y \in R, x^2 + y^2 \leq 25\}$ and $R' = \{(x, y) : x, y \in R, y \geq \frac{4}{9}x^2\}$, then find the domain and range of $R \cap R'$.

9. If f is a polynomial function satisfying $2 + f(x)f(y) = f(x) + f(y) + f(xy)$, $\forall x, y \in R$ and if $f(2) = 5$, then find the value of $f(f(2))$.

10. If $f(a - x) = f(a + x)$ and $f(b - x) = f(b + x)$ for all real x , where a, b ($a > b$) are constants, then prove that $f(x)$ is a periodic function.

11. If p, q are positive integers, f is a function defined for positive numbers and attains only positive values, such that $f(xf(y)) = x^p y^q$, then prove that $p^2 = q$.

12. If $f: R \rightarrow [0, \infty)$ is a function such that $f(x-1) + f(x+1) = \sqrt{3} f(x)$, then prove that $f(x)$ is periodic and find its period.

13. If a, b be two fixed positive integers such that $f(a+x) = b + [b^3 + 1 - 3b^2 f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$ for all real x , then prove that $f(x)$ is a periodic and find its period.

14. Let $f(x, y)$ be a periodic function, satisfying the condition $f(x, y) = f(2x + 2y, 2y - 2x) \forall x, y \in R$ and let $g(x)$ be a function defined as $g(x) = f(2^x, 0)$. Prove that $g(x)$ is periodic function and find its period.

15. Let $f: R \rightarrow R, f(x) = \frac{x-a}{(x-b)(x-c)}, b > c$. If f is onto, then prove that $a \in (b, c)$.

16. Show that there exists no polynomial $f(x)$ with integral coefficients which satisfy $f(a) = b, f(b) = c, f(c) = a$, where a, b, c are distinct integers.

17. Consider the function $f(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$ where

[.] denotes the fractional integral function and I is the set of integers. Then find $g(x) = \max. \{x^2, f(x), |x|\}; -2 \leq x \leq 2$.

18. Determine all functions $f: R \rightarrow R$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 \forall x, y \in R$.

19. Let $f(x) = (2 \cos x - 1)(2 \cos 2x - 1)(2 \cos 2^2 x - 1) \dots (2 \cos 2^{n-1} x - 1)$ (where $n \geq 1$). Then prove that $f\left(\frac{2\pi k}{2^n \pm 1}\right) = 1 \forall k \in I$.

20. If $f(x) = \frac{a^x}{a^x + \sqrt{a}}$ ($a > 0$), then find the value of $\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$.

Objective Type

Solutions on page 1.60

Each question has four choices a, b, c, and d, out of which only one is correct.

1. The function $f: N \rightarrow N$ (N is the set of natural numbers) defined by $f(n) = 2n + 3$ is
a. surjective only b. injective only
c. bijective d. None of these

2. The function $f(x) = \sin(\log(x + \sqrt{1+x^2}))$ is
a. even function b. odd function
c. neither even nor odd d. periodic function

3. If x is real, then the value of the expression $\frac{x^2 + 14x + 9}{x^2 + 2x + 3}$

lies between

- a. 5 and 4 b. 5 and -4
c. -5 and 4 d. None of these
4. The function $f: R \rightarrow R$ is defined by $f(x) = \cos^2 x + \sin^4 x$ for $x \in R$, then the range of $f(x)$ is

- a. $\left[\frac{3}{4}, 1\right]$ b. $\left[\frac{3}{4}, 1\right)$
c. $\left[\frac{3}{4}, 1\right]$ d. $\left(\frac{3}{4}, 1\right)$

5. The domain of the function $f(x) = \log_{3+x}(x^2 - 1)$ is
a. $(-3, -1) \cup (1, \infty)$
b. $[-3, -1) \cup [1, \infty)$

- c. $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
 d. $[-3, -2) \cup (-2, -1) \cup [1, \infty)$

6. The domain of the function $f(x) = \left[\log_{10} \left(\frac{5x-x^2}{4} \right) \right]^{1/2}$ is

a. $-\infty < x < \infty$ b. $1 \leq x \leq 4$
 c. $4 \leq x \leq 16$ d. $-1 \leq x \leq 1$

7. The domain of the function $f(x) = \frac{\sin^{-1}(3-x)}{\ln(|x|-2)}$ is

a. $[2, 4]$ b. $(2, 3) \cup (3, 4]$
 c. $[2, \infty)$ d. $(-\infty, -3) \cup [2, \infty)$

8. The domain of $f(x) = \log|\log x|$ is

a. $(0, \infty)$ b. $(1, \infty)$
 c. $(0, 1) \cup (1, \infty)$ d. $(-\infty, 1)$

9. The domain of $f(x) = \frac{\log_2(x+3)}{x^2+3x+2}$ is

a. $R - \{-1, -2\}$ b. $(-2, \infty)$
 c. $R - \{-1, -2, -3\}$ d. $(-3, \infty) - \{-1, -2\}$

10. Let $f: \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right] \rightarrow [0, 4]$ be a function defined as $f(x) = \sqrt{3} \sin x - \cos x + 2$. Then $f^{-1}(x)$ is given by

a. $\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}$ b. $\sin^{-1}\left(\frac{x-2}{2}\right) + \frac{\pi}{6}$
 c. $\frac{2\pi}{3} + \cos^{-1}\left(\frac{x-2}{2}\right)$ d. None of these

11. If $F(n+1) = \frac{2F(n)+1}{2}$, $n = 1, 2, \dots$ and $F(1) = 2$, then $F(101)$ equals

a. 52 b. 49 c. 48 d. 51

12. The domain of the function $f(x) = \frac{1}{\sqrt{10}C_{x-1} - 3 \times 10^5 C_x}$ contains the points

a. 9, 10, 11 b. 9, 10, 12
 c. all natural numbers d. None of these

13. The domain of the function $f(x) = \frac{x}{\sqrt{\sin(\ln x) - \cos(\ln x)}}$ ($n \in \mathbb{Z}$) is

a. $(e^{2n\pi}, e^{(3n+1/2)\pi})$ b. $(e^{(2n+1/4)\pi}, e^{(2n+5/4)\pi})$
 c. $(e^{(2n+1/4)\pi}, e^{(3n-3/4)\pi})$ d. None of these

14. If f is a function such that $f(0) = 2$, $f(1) = 3$ and $f(x+2) = 2f(x) - f(x+1)$ for every real x , then $f(5)$ is

a. 7 b. 13
 c. 1 d. 5

15. The range of $f(x) = \sin^{-1}\left(\frac{x^2+1}{x^2+2}\right)$ is

a. $[0, \pi/2]$ b. $(0, \pi/6)$
 c. $[\pi/6, \pi/2]$ d. None of these

16. The function $f(x) = \frac{\sec^{-1} x}{\sqrt{x-[x]}}$, where $[x]$ denotes the greatest integer less than or equal to x , is defined for all $x \in$

- a. R b. $R - \{(-1, 1) \cup \{n | n \in \mathbb{Z}\}\}$
 c. $R^+ - (0, 1)$ d. $R^+ - \{n | n \in \mathbb{N}\}$

17. The domain of $f(x) = \cos^{-1}\left(\frac{2-|x|}{4}\right) + [\log(3-x)]^{-1}$ is

a. $[-2, 6]$ b. $[-6, 2) \cup (2, 3)$
 c. $[-6, 2]$ d. $[-2, 2] \cup (2, 3)$

18. The domain of the function $f(x) = \sqrt{\log\left(\frac{1}{|\sin x|}\right)}$ is

a. $R - \{-\pi, \pi\}$ b. $R - \{n\pi | n \in \mathbb{Z}\}$
 c. $R - \{2n\pi | n \in \mathbb{Z}\}$ d. $(-\infty, \infty)$

19. The domain of the function $f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{x^{1/4}}\right) - 1\right)$ is

a. $(0, 1)$ b. $(0, 1]$
 c. $[1, \infty)$ d. $(1, \infty)$

20. The range of $f(x) = \sin^{-1}(\sqrt{x^2+x+1})$ is

a. $\left(0, \frac{\pi}{2}\right]$ b. $\left(0, \frac{\pi}{3}\right]$
 c. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ d. $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

21. If $f(x) = \text{maximum}\left\{x^3, x^2, \frac{1}{64}\right\} \forall x \in [0, \infty)$, then

a. $f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ x^3, & x > 1 \end{cases}$ b. $f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{4} \\ x^2, & \frac{1}{4} < x \leq 1 \\ x^3, & x > 1 \end{cases}$

c. $f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\ x^2, & \frac{1}{8} < x \leq 1 \\ x^3, & x > 1 \end{cases}$ d. $f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\ x^2, & \frac{1}{8} < x \leq 1 \\ x^3, & x > 1/8 \end{cases}$

22. If the period of $\frac{\cos(\sin(nx))}{\tan(x/n)}$, $n \in \mathbb{N}$, is 6π , then n is equal to

a. 3 b. 2
 c. 6 d. 1

23. The number of real solutions of the equation $\log_{0.5}|x| = 2|x|$ is

a. 1 b. 2
 c. 0 d. None of these

24. The period of the function $\left|\sin^3 \frac{x}{2}\right| + \left|\cos^5 \frac{x}{5}\right|$ is

a. 2π b. 10π
 c. 8π d. 5π

25. If $f(x) = \sqrt[n]{x^m}$, $n \in \mathbb{N}$, is an even function, then m is

a. even integer b. odd integer
 c. any integer d. $f(x)$ -even is not possible

26. If f is periodic, g is polynomial function and $f(g(x))$ is periodic and $g(2) = 3, g(4) = 7$ then $g(6)$ is
 a. 13 b. 15
 c. 11 d. None of these
27. The period of function $2^{\{x\}} + \sin \pi x + 3^{\{x/2\}} + \cos 2\pi x$ (where $\{x\}$ denotes the fractional part of x) is
 a. 2 b. 1
 c. 3 d. None of these
28. The equation $||x - 2| + a| = 4$ can have four distinct real solutions for x if a belongs to the interval
 a. $(-\infty, -4)$ b. $(-\infty, 0]$
 c. $[4, \infty)$ d. None of these
29. Given the function $f(x) = \frac{a^x + a^{-x}}{2}$ (where $a > 2$). Then $f(x+y) + f(x-y) =$
 a. $2f(x) \cdot f(y)$ b. $f(x) \cdot f(y)$
 c. $\frac{f(x)}{f(y)}$ d. None of these
30. If $\log_3(x^2 - 6x + 11) \leq 1$, then exhaustive range of values of x is
 a. $(-\infty, 2) \cup (4, \infty)$
 b. $(2, 4)$
 c. $(-\infty, 1) \cup (1, 3) \cup (4, \infty)$
 d. None of these
31. The domain of the function $f(x) = \sqrt{x^2 - [x]^2}$, where $[x]$ = the greatest integer less than or equal to x , is
 a. R b. $[0, +\infty)$
 c. $(-\infty, 0]$ d. None of these
32. The range of the function $f(x) = |x - 1| + |x - 2|$, $-1 \leq x \leq 3$, is
 a. $[1, 3]$ b. $[1, 5]$
 c. $[3, 5]$ d. None of these
33. Which of the following functions is inverse to itself?
 a. $f(x) = \frac{1-x}{1+x}$ b. $f(x) = 5^{\log x}$
 c. $f(x) = 2^{x(x-1)}$ d. None of these
34. A function $F(x)$ satisfies the functional equation $x^2 F(x) + F(1-x) = 2x - x^4$ for all real x . $F(x)$ must be
 a. x^{2^x} b. $1 - x^2$
 c. $1 + x^2$ d. $x^2 + x + 1$
35. If $f(x) = \begin{cases} x^2 \sin \frac{\pi x}{2}, & |x| < 1 \\ x|x|, & |x| \geq 1 \end{cases}$ then $f(x)$ is
 a. an even function b. an odd function
 c. a periodic function d. None of these
36. Function $f: (-\infty, -1) \rightarrow (0, e^5]$ defined by $f(x) = e^{x^3 - 3x + 2}$ is
 a. many-one and onto b. many-one and into
 c. one-one and onto d. one-one and into
37. If $f(x) = \frac{1}{x}, g(x) = \frac{1}{x^2}$ and $h(x) = x^2$
 a. $f \circ g(x) = x^2, x \neq 0, h(g(x)) = \frac{1}{x^2}$
- b. $h(g(x)) = \frac{1}{x^2}, x \neq 0, f \circ g(x) = x^2$
 c. $f \circ g(x) = x^2, x \neq 0, h(g(x)) = (g(x))^2, x \neq 0$
 d. None of these
38. If $[x]$ and $\{x\}$ represent the integral and fractional parts of x , respectively, then the value of $\sum_{r=1}^{2000} \frac{\{x+r\}}{2000}$ is
 a. x b. $[x]$
 c. $\{x\}$ d. $x + 2001$
39. If $f(x)$ is a polynomial satisfying $f(x)f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then $f(4)$ is equal to
 a. 63 b. 65
 c. 17 d. None of these
40. The values of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are
 a. $b = 2, c = 1$ b. $b = 4, c = -1$
 c. $b = -1, c = 4$ d. $b = -1, c = 1$
41. Let $f: R \rightarrow R, g: R \rightarrow R$ be two given functions such that f is injective and g is surjective, then which of the following is injective?
 a. $g \circ f$ b. $f \circ g$
 c. $g \circ g$ d. None of these
42. $f: N \rightarrow N$ where $f(x) = x - (-1)^x$ then f is
 a. one-one and into
 b. many-one and into
 c. one-one and onto
 d. many-one and onto
43. If $g(x) = x^2 + x - 2$ and $\frac{1}{2} g \circ f(x) = 2x^2 - 5x + 2$, then which is not a possible $f(x)$?
 a. $2x - 3$ b. $-2x + 2$
 c. $x - 3$ d. None of these
44. If $f: R \rightarrow R$ is an invertible function such that $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = -x$, then
 a. $f(x)$ is odd
 b. $f(x)$ and $f^{-1}(x)$ may not be symmetric about the line $y = x$
 c. $f(x)$ may not be odd
 d. None of these
45. Let $f: N \rightarrow N$ defined by $f(x) = x^2 + x + 1, x \in N$, then f is
 a. One-one onto
 b. Many-one onto
 c. One-one but not onto
 d. None of these
46. Let $f: X \rightarrow Y, f(x) = \sin x + \cos x + 2\sqrt{2}$ is invertible. Then which $X \rightarrow Y$ is not possible?
 a. $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \rightarrow [\sqrt{2}, 3\sqrt{2}]$
 b. $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right] \rightarrow [\sqrt{2}, 3\sqrt{2}]$
 c. $\left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right] \rightarrow [\sqrt{2}, 3\sqrt{2}]$
 d. None of these

47. If $f(x) = ax^7 + bx^3 + cx - 5$, a, b, c are real constants and $f(-7) = 7$, then the range of $f(7) + 17 \cos x$ is

a. $[-34, 0]$ b. $[0, 34]$
c. $[-34, 34]$ d. None of these

48. If $f(x) = \frac{\sin([x]\pi)}{x^2 + x + 1}$, where $[.]$ denotes the greatest integer function, then

a. f is one-one
b. f is not one-one and non-constant
c. f is a constant function
d. None of these

49. Let S be the set of all triangles and R^+ be the set of positive real numbers. Then the function $f: S \rightarrow R^+$, $f(\Delta) = \text{area of } \Delta$, where $\Delta \in S$ is

a. injective but not surjective
b. surjective but not injective
c. injective as well as surjective
d. neither injective nor surjective

50. The graph of $(y-x)$ against $(y+x)$ is shown

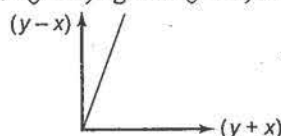
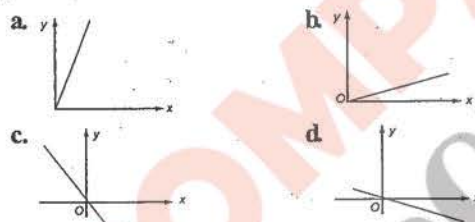


Fig. 1.91

Which one of the following shows the graph of y against x ?



51. If $g: [-2, 2] \rightarrow R$ where $f(x) = x^3 + \tan x + \left[\frac{x^2 + 1}{P} \right]$ is an odd function, then the value of parametric P where $[.]$ denotes the greatest integer function is

a. $-5 < P < 5$ b. $P < 5$
c. $P > 5$ d. None of these

52. If $f\left(2x + \frac{y}{8}, 2x - \frac{y}{8}\right) = xy$, then $f(m, n) + f(n, m) = 0$

a. only when $m = n$ b. only when $m \neq n$
c. only when $m = -n$ d. for all m and n

53. If $f(x+y) = f(x) + f(y) - xy - 1 \forall x, y \in R$ and $f(1) = 1$, then the number of solutions of $f(n) = n, n \in N$ is

a. 0 b. 1
c. 2 d. more than 2

54. The range of the function $f(x) = \frac{e^x - e^{|x|}}{e^x + e^{|x|}}$

a. $(-\infty, \infty)$ b. $[0, 1]$
c. $(-1, 0]$ d. $(-1, 1)$

55. If $f: R \rightarrow R$ is a function satisfying the property $f(2x+3) + f(2x+7) = 2, \forall x \in R$, then the fundamental period of $f(x)$ is

a. 2 b. 4 c. 8 d. 12

56. Let $f: R \rightarrow \left[0, \frac{\pi}{2}\right)$ defined by $f(x) = \tan^{-1}(x^2 + x + a)$, then

the set of values of a for which f is onto is

a. $[0, \infty)$ b. $[2, 1]$
c. $\left[\frac{1}{4}, \infty\right)$ d. None of these

57. The domain of the function $f(x) = \frac{1}{\sqrt{\{ \sin x \} + \{ \sin(\pi + x) \}}}$,

where $\{ \cdot \}$ denotes the fractional part, is

a. $[0, \pi]$ b. $(2n+1)\pi/2, n \in Z$
c. $(0, \pi)$ d. None of these

58. $f(x) = \frac{\cos x}{\left[\frac{2x}{\pi} \right] + \frac{1}{2}}$, where x is not an integral multiple of π

and $[.]$ denotes the greatest integer function is

a. An odd function b. Even function
c. Neither odd nor even d. None of these

59. Let $f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x - (\tan x) \times \operatorname{sgn} x$ be an even function for all $x \in R$, then the sum of all possible values of ' a ' is (where $[.]$ and $\{ \cdot \}$ denote greatest integer function and fractional part functions, respectively)

a. $\frac{17}{6}$ b. $\frac{53}{6}$ c. $\frac{31}{3}$ d. $\frac{35}{3}$

60. Let $f: [-10, 10] \rightarrow R$, where $f(x) = \sin x + [x^2/a]$ be an odd function. Then the set of values of parameter a is/are

a. $(-10, 10) \setminus \{0\}$ b. $(0, 10)$
c. $[100, \infty)$ d. $(100, \infty)$

61. The function f satisfies the functional equation $3f(x)$

$$+ 2f\left(\frac{x+59}{x-1}\right) = 10x + 30 \text{ for all real } x \neq 1. \text{ The value of } f(7) \text{ is}$$

a. 8 b. 4
c. -8 d. 11

62. The period of the function $f(x) = [6x + 7] + \cos \pi x - 6x$, where $[.]$ denotes the greatest integer function, is

a. 3 b. 2π
c. 2 d. None of these

63. If the graph of the function $f(x) = \frac{a^x - 1}{x^n(a^x + 1)}$ is

symmetrical about y -axis, then n equals

a. 2 b. $\frac{2}{3}$ c. $\frac{1}{4}$ d. $-\frac{1}{3}$

64. If $f(x)$ is an even function and satisfies the relation $x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$ where $g(x)$ is an odd function, then $f(5)$ equals

a. 0 b. $\frac{50}{75}$ c. $\frac{49}{75}$ d. None of these

65. If $f(x+y) = f(x)f(y)$ for all real x, y and $f(0) \neq 0$, then the

function $g(x) = \frac{f(x)}{1 + \{f(x)\}^2}$ is

a. Even function b. Odd function
c. Odd if $f(x) > 0$ d. Neither even nor odd

66. Possible values of a such that the equation $x^2 + 2ax + a = \sqrt{a^2 + x} - \frac{1}{16}$, $x \geq -a$, has two distinct real roots are given by
 a. $[0, 1]$ b. $[-\infty, 0)$
 c. $[0, \infty)$ d. $\left(\frac{3}{4}, \infty\right)$
67. Let $g(x) = f(x) - 1$. If $f(x) + f(1-x) = 2 \forall x \in R$, then $g(x)$ is symmetrical about
 a. Origin b. The line $x = \frac{1}{2}$
 c. The point $(1, 0)$ d. The point $\left(\frac{1}{2}, 0\right)$
68. Domain (D) and range (R) of $f(x) = \sin^{-1}(\cos^{-1}[x])$ where $[.]$ denotes the greatest integer function is
 a. $D \equiv x \in [1, 2], R \equiv \{0\}$
 b. $D \equiv x \in [0, 1], R \equiv \{-1, 0, 1\}$
 c. $D \equiv x \in [-1, 1], R \equiv \left\{0, \sin^{-1}\left(\frac{\pi}{2}\right), \sin^{-1}(\pi)\right\}$
 d. $D \equiv x \in [-1, 1], R \equiv \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$
69. If $f(x+1) + f(x-1) = 2f(x)$ and $f(0) = 0$, then $f(n)$, $n \in N$, is
 a. $n f(1)$ b. $\{f(1)\}^n$
 c. 0 d. None of these
70. The range of the function f defined by $f(x) = \left[\frac{1}{\sin\{x\}}\right]$ (where $[.]$ and $\{.\}$ respectively denote the greatest integer and the fractional part functions) is
 a. I , the set of integers
 b. N , the set of natural numbers
 c. W , the set of whole numbers
 d. $\{1, 2, 3, 4, \dots\}$
71. If $[\cos^{-1} x] + [\cot^{-1} x] = 0$, where $[.]$ denotes the greatest integer function, then the complete set of values of x is
 a. $(\cos 1, 1]$ b. $(\cos 1, \cot 1)$
 c. $(\cot 1, 1]$ d. $[0, \cot 1)$
72. If $f(x)$ and $g(x)$ are periodic functions with period 7 and 11, respectively. Then the period of $F(x) = f(x) g\left(\frac{x}{5}\right) - g(x)$ is
 a. 177 b. 222
 c. 433 d. 1155
73. The period of the function $f(x) = c \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ is (where c is constant)
 a. 1 b. $\frac{\pi}{2}$
 c. π d. Cannot be determined
74. If $f(x+f(y)) = f(x) + y \forall x, y \in R$ and $f(0) = 1$, then the value of $f(7)$ is
 a. 1 b. 7
 c. 6 d. 8
75. Let $f(x) = \sqrt{x|-\{x\}|}$ (where $\{.\}$ denotes the fractional part of x) and X, Y are its domain and range, respectively, then
 a. $x \in \left(-\infty, \frac{1}{2}\right]$ and $Y \in \left[\frac{1}{2}, \infty\right)$
 b. $x \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in \left[\frac{1}{2}, \infty\right)$
 c. $X \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in [0, \infty)$
 d. None of these
76. Let f be a function satisfying of x then $f(xy) = \frac{f(x)}{y}$ for all positive real numbers x and y if $f(30) = 20$, then the value of $f(40)$ is
 a. 15 b. 20
 c. 40 d. 60
77. The domain of the function $f(x) = \sqrt{\ln_{(x-1)}(x^2 + 4x + 4)}$ is
 a. $[-3, -1] \cup [1, 2]$
 b. $(-2, -1) \cup [2, \infty)$
 c. $(-\infty, -3] \cup (-2, -1) \cup (2, \infty)$
 d. None of these
78. The range of $f(x) = [1 + \sin x] + \left[2 + \sin \frac{x}{2}\right] + \left[3 + \sin \frac{x}{3}\right] + \dots + \left[n + \sin \frac{x}{n}\right]$, $\forall x \in [0, \pi]$, where $[.]$ denotes the greatest integer function, is
 a. $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}\right\}$
 b. $\left\{\frac{n(n+1)}{2}\right\}$
 c. $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$
 d. $\left\{\frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$
79. The total number of solutions of $[x]^2 = x + 2\{x\}$, where $[.]$ and $\{.\}$ denote the greatest integer function and fractional part, respectively, is equal to
 a. 2 b. 4
 c. 6 d. None of these
80. The domain of $f(x) = \sqrt{2\{x\}^2 - 3\{x\} + 1}$, where $\{.\}$ denotes the fractional part in $[-1, 1]$, is
 a. $[-1, 1] \sim \left(\frac{1}{2}, 1\right)$

b. $\left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right] \cup \{1\}$

c. $\left[-1, \frac{1}{2}\right]$

d. $\left[-\frac{1}{2}, 1\right]$

81. The range of $\sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$, where $[.]$ denotes the greatest integer function, is

a. $\left\{\frac{\pi}{2}, \pi\right\}$

b. $\{\pi\}$

c. $\left\{\frac{\pi}{2}\right\}$

d. None of these

82. If the period of $\frac{\cos(\sin nx)}{\tan\left(\frac{x}{n}\right)}$, $n \in N$ is 6π then $n =$

a. 3

b. 2

c. 6

d. 1

83. The domain of $f(x) = \ln(ax^3 + (a+b)x^2 + (b+c)x + c)$, where $a > 0$, $b^2 - 4ac = 0$, is (where $[.]$ represents greatest integer function).

a. $(-1, \infty) \sim \left\{-\frac{b}{2a}\right\}$

b. $(1, \infty) \sim \left\{-\frac{b}{2a}\right\}$

c. $(-1, 1) \sim \left\{-\frac{b}{2a}\right\}$

d. None of these

84. The period of $f(x) = [x] + [2x] + [3x] + [4x] + \dots + [nx] - \frac{n(n+1)}{2}x$, where $n \in N$, is (where $[.]$ represents greatest integer function)

a. n

b. 1

c. $\frac{1}{n}$

d. None of these

85. If $f\left(x + \frac{1}{2}\right) + f\left(x - \frac{1}{2}\right) = f(x)$ for all $x \in R$, then the period of $f(x)$ is

a. 1

b. 2

c. 3

d. 4

86. If $f: R^+ \rightarrow R$, $f(x) + 3xf\left(\frac{1}{x}\right) = 2(x+1)$, then $f(99)$ is equal to

a. 40

b. 30

c. 50

d. 60

87. If $f: X \rightarrow Y$, where X and Y are sets containing natural numbers, $f(x) = \frac{x+5}{x+2}$ then the number of elements in the domain and range of $f(x)$ are respectively

a. 1 and 1

b. 2 and 1

c. 2 and 2

d. 1 and 2

88. If $f(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ x & \text{for } x < 0 \end{cases}$ then $f \circ f(x)$ is given by

a. x^2 for $x \geq 0$, x for $x < 0$

b. x^4 for $x \geq 0$, x^2 for $x < 0$

c. x^4 for $x \geq 0$, $-x^2$ for $x < 0$

d. x^4 for $x \geq 0$, x for $x < 0$

89. If the graph of $y = f(x)$ is symmetrical about lines $x = 1$ and $x = 2$, then which of the following is true?

a. $f(x+1) = f(x)$

b. $f(x+3) = f(x)$

c. $f(x+2) = f(x)$

d. None of these

90. Let $f(x) = x + 2|x+1| + 2|x-1|$. If $f(x) = k$ has exactly one real solution, then the value of k is

a. 3

b. 0

c. 1

d. 2

91. The domain of $f(x) = \sin^{-1}[2x^2 - 3]$, where $[.]$ denotes the greatest integer function, is

a. $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$

b. $\left(-\sqrt{\frac{3}{2}}, -1\right] \cup \left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$

c. $\left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$

d. $\left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right)$

92. The range of $f(x) = \cos^{-1}\left(\frac{1+x^2}{2x}\right) + \sqrt{2-x^2}$ is

a. $\left\{0, 1 + \frac{\pi}{2}\right\}$

b. $\{0, 1 + \pi\}$

c. $\left\{1, 1 + \frac{\pi}{2}\right\}$

d. $\{1, 1 + \pi\}$

93. If $f(x) = \begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$ then $f(f(x))$ is

a. $x \forall x \in R$

b. $\begin{cases} x, & x \text{ is irrational} \\ 1-x, & x \text{ is rational} \end{cases}$

c. $\begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$

d. None of these

94. The range of $f(x) = [|\sin x| + |\cos x|]$, where $[.]$ denotes the greatest integer function, is

a. $\{0\}$

b. $\{0, 1\}$

c. $\{1\}$

d. None of these

95. If $f(x) = \log_e\left(\frac{x^2+e}{x^2+1}\right)$, then the range of $f(x)$ is

a. $(0, 1)$

b. $[0, 1]$

c. $[0, 1)$

d. $(0, 1]$

96. The domain of the function $f(x) = \frac{1}{\sqrt{4x - |x^2 - 10x + 9|}}$ is

- a. $(7 - \sqrt{40}, 7 + \sqrt{40})$ b. $(0, 7 + \sqrt{40})$
 c. $(7 - \sqrt{40}, \infty)$ d. None of these

97. If the function $f : [1, \infty) \rightarrow [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is

- a. $\left(\frac{1}{2}\right)^{x(x-1)}$ b. $\frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$
 c. $\frac{1}{2}(1 - \sqrt{1 + 4\log_2 x})$ d. Not defined

98. The number of roots of the equation $x \sin x = 1$, $x \in [-2\pi, 0) \cup (0, 2\pi]$, is

- a. 2 b. 3
 c. 4 d. 0

99. The number of solutions of $2 \cos x = |\sin x|$, $0 \leq x \leq 4\pi$, is

- a. 0 b. 2
 c. 4 d. Infinite

100. If $af(x+1) + bf\left(\frac{1}{x+1}\right) = x$, $x \neq -1$, $a \neq b$, then $f(2)$ is equal to

- a. $\frac{2a+b}{2(a^2-b^2)}$ b. $\frac{a}{a^2-b^2}$
 c. $\frac{a+2b}{a^2-b^2}$ d. None of these

101. The number of solutions of $\tan x - mx = 0$, $m > 1$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

- a. 1 b. 2
 c. 3 d. m

102. The range of $f(x) = [\sin x + [\cos x + [\tan x + [\sec x]]]]$, $x \in (0, \pi/4)$, where $[.]$ denotes the greatest integer function $\leq x$, is

- a. $\{0, 1\}$ b. $\{-1, 0, 1\}$
 c. $\{1\}$ d. None of these

103. If $f(3x+2) + f(3x+29) = 0 \forall x \in \mathbb{R}$, then the period of $f(x)$ is

- a. 7 b. 8
 c. 10 d. None of these

104. Let $f(x) = \begin{cases} \sin x + \cos x, & 0 < x < \frac{\pi}{2} \\ a, & x = \pi/2 \\ \tan^2 x + \operatorname{cosec} x, & \pi/2 < x < \pi \end{cases}$

then its odd extension is

- a. $\begin{cases} -\tan^2 x - \operatorname{cosec} x, & -\pi < x < -\frac{\pi}{2} \\ -a, & x = -\frac{\pi}{2} \\ -\sin x + \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

b. $\begin{cases} -\tan^2 x + \operatorname{cosec} x, & -\pi < x < -\frac{\pi}{2} \\ -a, & x = -\frac{\pi}{2} \\ \sin x - \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

c. $\begin{cases} -\tan^2 x + \operatorname{cosec} x, & -\pi < x < -\frac{\pi}{2} \\ a, & x = -\frac{\pi}{2} \\ \sin x - \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

d. $\begin{cases} \tan^2 x + \cos x, & -\pi < x < -\frac{\pi}{2} \\ -a, & x = -\frac{\pi}{2} \\ \sin x + \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

105. If f and g are one-one function, then

- a. $f+g$ is one-one b. fg is one-one
 c. fog is one-one d. None of these

106. The domain of $f(x)$ is $(0, 1)$, then, domain of $f(e^x) + f(\ln|x|)$ is

- a. $(-1, e)$ b. $(1, e)$
 c. $(-e, -1)$ d. $(-e, 1)$

107. The domain of $f(x) = \frac{1}{\sqrt{|\cos x| + \cos x}}$ is

- a. $[-2n\pi, 2n\pi]$, $n \in \mathbb{Z}$
 b. $(2n\pi, 2n\pi + \pi)$, $n \in \mathbb{Z}$
 c. $\left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2}\right)$, $n \in \mathbb{Z}$
 d. $\left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right)$, $n \in \mathbb{Z}$

108. If $f(2x+3y, 2x-7y) = 20x$, then $f(x, y)$ equals

- a. $7x-3y$ b. $7x+3y$
 c. $3x-7y$ d. $x-ky$

109. Let $X = \{a_1, a_2, \dots, a_6\}$ and $Y = \{b_1, b_2, b_3\}$. The number of functions f from x to y such that it is onto and there are exactly three elements x in X such that $f(x) = b_1$ is

- a. 75 b. 90
 c. 100 d. 120

110. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be two one-one and onto functions such that they are the mirror images of each other about the line $y = a$. If $h(x) = f(x) + g(x)$, then $h(x)$ is

- a. One-one and onto.
 b. Only one-one and not onto.
 c. Only onto but not one-one.
 d. Neither one-one nor onto.

111. If $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, $g(x) = |\sin x| - |\cos x|$ and $\phi(x) = f(x)g(x)$ (where $[.]$ denotes the greatest integer function) then the respective fundamental periods of $f(x)$, $g(x)$ and $\phi(x)$ are

a. π, π, π

b. $\pi, 2\pi, \pi$

c. $\pi, \pi, \frac{\pi}{2}$

d. $\pi, \frac{\pi}{2}, \pi$

112. Let $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, then $f(1) + f(2) + f(3) + \dots + f(n)$ is equal to

a. $nf(n) - 1$

b. $(n+1)f(n) - n$

c. $(n+1)f(n) + n$

d. $nf(n) + n$

113. Let $f(x) = e^{\{x\} \operatorname{sgn} x}$ and $g(x) = e^{[x] \operatorname{sgn} x}$, $x \in \mathbb{R}$ where $\{ \}$ and $[\]$ denotes the fractional and integral part functions, respectively. Also $h(x) = \log(f(x)) + \log(g(x))$ then for real x , $h(x)$ is

a. An odd function.

b. An even function.

c. Neither an odd nor an even function.

d. Both odd as well as even function.

114. Let $f_1(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & x > 1 \\ 0, & \text{otherwise} \end{cases}$

and $f_2(x) = f_1(-x)$ for all x

$f_3(x) = -f_2(x)$ for all x

$f_4(x) = f_3(-x)$ for all x

Which of the following is necessarily true?

a. $f_4(x) = f_1(x)$ for all x

b. $f_1(x) = -f_3(-x)$ for all x

c. $f_2(-x) = f_4(x)$ for all x

d. $f_1(x) + f_3(x) = 0$ for all x

115. The number of solutions of the equation $[y + [y]] = 2 \cos x$,

 where $y = \frac{1}{3} [\sin x + [\sin x + [\sin x]]]$ (where $[\]$ denotes the greatest integer function) is

a. 4

b. 2

c. 3

d. 53

116. The sum of roots of the equation $\cos^{-1}(\cos x) = [x]$, $[\]$ denotes the greatest integer function is

a. $2\pi + 3$

b. $\pi + 3$

c. $\pi - 3$

d. $2\pi - 3$

117. The range of

$$f(x) = \sqrt{(1 - \cos x)} \sqrt{(1 - \cos x)} \sqrt{(1 - \cos x)} \sqrt{\dots \infty}$$

is

a. $[0, 1]$

b. $[0, 1/2]$

c. $[0, 2]$

d. None of these

118. Let $h(x) = [kx + 5]$, the domain of $f(x)$ is $[-5, 7]$, the domain of $f(h(x))$ is $[-6, 1]$ and the range of $h(x)$ is the same as the domain of $f(x)$, then the value of k is

a. 1

b. 2

c. 3

d. 4

119. The range of $f(x) = (x+1)(x+2)(x+3)(x+4) + 5$ for $x \in [-6, 6]$ is

a. $[4, 5045]$

b. $[0, 5045]$

c. $[-20, 5045]$

d. None of these

120. The exhaustive domain of

$$f(x) = \sqrt{x^{12} - x^9 + x^4 - x + 1}$$

is

a. $[0, 1]$

b. $[1, \infty)$

c. $(-\infty, 1]$

d. \mathbb{R}

121. The range of $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$ is

a. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$

b. $\left[0, \frac{\pi}{2}\right)$

c. $\left(\frac{2\pi}{3}, \pi\right]$

d. None of these

122. The range of the function $f(x) = 7^{-x} P_{x-3}$ is

a. $\{1, 2, 3\}$

b. $\{1, 2, 3, 4, 5, 6\}$

c. $\{1, 2, 3, 4\}$

d. $\{1, 2, 3, 4, 5\}$

123. A real-valued function $f(x)$ satisfies the functional equation $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$, where a is a given constant and $f(0) = 1$. $f(2a-x)$ is equal to

a. $f(x)$

b. $-f(x)$

c. $f(-x)$

d. $f(a) + f(a-x)$

Multiple Correct Answers Type

Solutions on page 1.71

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}$, $x \in [0, 2\pi]$ and $g(x) = \max\{1, |x-1|\}$ $x \in \mathbb{R}$, then

a. $g(f(0)) = 1$

b. $g(f(1)) = 1$

c. $f(f(1)) = 1$

d. $f(g(0)) = 1 + \sin 1$

2. Which of the following functions are identical?

a. $f(x) = \ln x^2$ and $g(x) = 2 \ln x$

b. $f(x) = \log_e e$ and $g(x) = \frac{1}{\log_e x}$

c. $f(x) = \sin(\cos^{-1} x)$ and $g(x) = \cos(\sin^{-1} x)$

d. None of these

3. Which of the following function/functions have the graph symmetrical about the origin?

a. $f(x)$ given by $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

b. $f(x)$ given by $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

c. $f(x)$ given by $f(x+y) = f(x) + f(y) \forall x, y \in \mathbb{R}$

d. None of these

4. If the function f satisfies the relation $f(x+y) + f(x-y) = 2f(x)f(y) \forall x, y \in \mathbb{R}$ and $f(0) \neq 0$, then

 a. $f(x)$ is an even function

 b. $f(x)$ is an odd function

 c. If $f(2) = a$ then $f(-2) = a$

 d. If $f(4) = b$ then $f(-4) = -b$

5. Consider the function $y = f(x)$ satisfying the condition

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} \quad (x \neq 0), \text{ then}$$

 a. domain of $f(x)$ is \mathbb{R}

 b. domain of $f(x)$ is $\mathbb{R} - (-2, 2)$

c. range of $f(x)$ is $[-2, \infty)$ d. range of $f(x)$ is $[2, \infty)$

6. Let $f(x) + f(y) = f(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ ($f(x)$ is not identically zero). Then

a. $f(4x^3 - 3x) + 3f(x) = 0$

b. $f(4x^3 - 3x) = 3f(x)$

c. $f(2x\sqrt{1-x^2}) + 2f(x) = 0$

d. $f(2x\sqrt{1-x^2}) = 2f(x)$

7. Consider the real-valued function satisfying $2f(\sin x) + f(\cos x) = x$. Then

a. domain of $f(x)$ is R b. domain of $f(x)$ is $[-1, 1]$ c. range of $f(x)$ is $\left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$ d. range of $f(x)$ is R

8. If $f(x)$ satisfies the relation $f(x+y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = 5$, then

a. $f(x)$ is an odd function b. $f(x)$ is an even function

c. $\sum_{r=1}^m f(r) = 5^{m+1} C_2$

d. $\sum_{r=1}^m f(r) = \frac{5m(m+2)}{3}$

9. Let $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$ and

$$g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$$

then, which of the following is/are true?

a. $(f+g)(3.5) = 0$

b. $f(g(3)) = 3$

c. $(fg)(2) = 1$

d. $(f-g)(4) = 0$

10. $f(x) = x^2 - 2ax + a(a+1)$, $f: [a, \infty) \rightarrow [a, \infty)$. If one of the solutions of the equation $f(x) = f^{-1}(x)$ is 5049, then the other may be

a. 5051

b. 5048

c. 5052

d. 5050

11. Which of the following function is/are periodic

a. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

b. $f(x) = \begin{cases} x - [x], & 2n \leq x < 2n+1 \\ \frac{1}{2}, & 2n+1 \leq x < 2n+2 \end{cases}$, where $[.]$ denotes the greatest integer function, $n \in Z$

c. $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, where $[.]$ denotes the greatest integer function

d. $f(x) = x - [x+3] + \tan\left(\frac{\pi x}{2}\right)$, where $[.]$ denotes the greatest integer function, and a is a rational number

12. If $f: R^+ \rightarrow R^+$ is a polynomial function satisfying the functional equation $f(f(x)) = 6x - f(x)$, then $f(17)$ is equal to

a. 17

b. -51

c. 34

d. -34

13. Let $f: R \rightarrow R$ be a function defined by $f(x+1) = \frac{f(x)-5}{f(x)-3}$

$\forall x \in R$. Then which of the following statement(s) is/are true

a. $f(2008) = f(2004)$

b. $f(2006) = f(2010)$

c. $f(2006) = f(2002)$

d. $f(2006) = f(2018)$

14. Let $f(x) = \sec^{-1}[1 + \cos^2 x]$ where $[.]$ denotes the greatest integer function. Then

a. the domain of f is R b. the domain of f is $[1, 2]$ c. the domain of f is $[1, 2]$ d. the range of f is $\{\sec^{-1} 1, \sec^{-1} 2\}$

15. Which of the following pairs of functions is/are identical?

a. $f(x) = \tan(\tan^{-1} x)$ and $g(x) = \cot(\cot^{-1} x)$

b. $f(x) = \operatorname{sgn}(x)$ and $g(x) = \operatorname{sgn}(\operatorname{sgn}(x))$

c. $f(x) = \cot^2 x \cdot \cos^2 x$ and $g(x) = \cot^2 x - \cos^2 x$

d. $f(x) = e^{\ln \sec^{-1} x}$ and $g(x) = \sec^{-1} x$

16. $f: R \rightarrow [-1, \infty)$ and $f(x) = \ln([|\sin 2x| + |\cos 2x|])$ (where $[.]$ is the greatest integer function).

a. $f(x)$ has range Z b. $f(x)$ is periodic with fundamental period $\pi/4$ c. $f(x)$ is invertible in $\left[0, \frac{\pi}{4}\right]$ d. $f(x)$ is into function

17. Which of the following is/are not a function ($[.]$ and $\{.\}$ denotes the greatest integer and fractional part functions respectively)?

a. $\frac{1}{\ln[1-x]}$

b. $\frac{x!}{\{x\}}$

c. $x! \{x\}$

d. $\frac{\ln(x-1)}{\sqrt{1-x^2}}$

18. If the following functions are defined from $[-1, 1]$ to $[-1, 1]$, select those which are not objective

a. $\sin(\sin^{-1} x)$

b. $\frac{2}{\pi} \sin^{-1}(\sin x)$

c. $(\operatorname{sgn}(x)) \ln(e^x)$

d. $x^3 (\operatorname{sgn}(x))$

19. If $f: R \rightarrow N \cup \{0\}$, where f (area of triangle joining points $P(5, 0)$, $Q(8, 4)$ and $R(x, y)$ such that the angle PRQ is a right) = number of triangle. Then, which of the following is true?

a. $f(5) = 4$

b. $f(7) = 0$

c. $f(6.25) = 2$

d. $f(x)$ is into

20. If $f(x)$ is a polynomial of degree n such that $f(0) = 0$, $f(1) = \frac{1}{2}$, ..., $f(n) = \frac{n}{n+1}$, then the value of $f(n+1)$ is

a. 1 when n is odd b. $\frac{n}{n+2}$ when n is even

c. $-\frac{n}{n+1}$ when n is odd d. -1 when n is even

SA 21. Let $f(x) = \frac{3}{4}x + 1$, and $f^n(x)$ be defined as $f^2(x) = f(f(x))$, and for $n \geq 2$, $f^{n+1}(x) = f(f^n(x))$. If $\lambda = \lim_{n \rightarrow \infty} f^n(x)$, then

- λ is independent of x
- λ is a linear polynomial in x
- the line $y = \lambda$ has slope 0
- the line $4y = \lambda$ touches the unit circle with centre at the origin.

22. The domain of the function

$$f(x) = \log_e \left\{ \log_{|\sin x|} (x^2 - 8x + 23) - \frac{3}{\log_2 |\sin x|} \right\}$$

contains which of the following interval / intervals.

- $(3, \pi)$
- $\left(\pi, \frac{3}{2}\right)$
- $\left(\frac{3\pi}{2}, 5\right)$
- None of these

SA 23. Let $f(x) = \operatorname{sgn}(\cot^{-1} x) + \tan \left(\frac{\pi}{2} [x] \right)$, where $[x]$ is the greatest integer function less than or equal to x . Then which of the following alternatives is/are true?

- $f(x)$ is many one but not even function
- $f(x)$ is periodic function
- $f(x)$ is bounded function
- Graph of $f(x)$ remains above the x -axis

Reasoning Type

Solutions on page 1.75

Each question has four choices a, b, c and d, out of which only one is correct. Each question contains STATEMENT 1 and STATEMENT 2.

- if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
- if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
- if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
- if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. Statement 1: $f(x) = \log_e x$ cannot be expressed as a sum of odd and even function.

Statement 2: $f(x) = \log_e x$ is neither odd nor even function.

2. Statement 1: If $g(x) = f(x) - 1$. If $f(x) + f(1-x) = 2 \forall x \in R$, then $g(x)$ is symmetrical about the point $(1/2, 0)$.

Statement 2: If $g(a-x) = -g(a+x) \forall x \in R$, then $g(x)$ is symmetrical about the point $(a, 0)$.

LO 3. Consider the function satisfying the relation

$$\text{if } f\left(\frac{2 \tan x}{1 + \tan^2 x}\right) = \frac{(1 + \cos 2x)(\sec^2 x + 2 \tan x)}{2}$$

Statement 1: Range of $y = f(x)$ is R .

Statement 2: Linear function has range R if domain is R .

4. Consider the function if $f(x) = \sin(kx) + \{x\}$, where $\{x\}$ represents the fractional part function.

Statement 1: $f(x)$ is periodic for $k = m\pi$ where m is a rational number.

Statement 2: The sum of two periodic functions is always periodic.

LO 5. Statement 1: Function $f(x) = x^2 + \tan^{-1} x$ is a non-periodic function.

Statement 2: The sum of two non-periodic function is always non-periodic.

6. Statement 1: If $x \in [1, \sqrt{3}]$, then the range of $f(x) = \tan^{-1} x$ is $[\pi/4, \pi/3]$.

Statement 2: If $x \in [a, b]$, then the range of $f(x)$ is $[f(a), f(b)]$.

7. Statement 1: $f: N \rightarrow R$, $f(x) = \sin x$ is a one-one function.

Statement 2: The period of $\sin x$ is 2π and 2π is an irrational number.

LO 8. Statement 1: A continuous surjective function $f: R \rightarrow R$, $f(x)$ can never be a periodic function.

Statement 2: For a surjective function $f: R \rightarrow R$, $f(x)$ to be periodic, it should necessarily be a discontinuous function.

LO 9. Statement 1: The solution of equation $\|x^2 - 5x + 4\| - |2x - 3| = |x^2 - 3x + 1|$ is $x \in (-\infty, 1] \cup \left[\frac{3}{2}, 4\right]$.

Statement 2: If $|x+y| = |x| + |y|$, then $x, y \geq 0$.

LO 10. Consider f and g be real-valued functions such that $f(x+y) + f(x-y) = 2f(x) \cdot g(y) \forall x, y \in R$.

Statement 1: If $f(x)$ is not identically zero and $|f(x)| \leq 1 \forall x \in R$, then $|g(y)| \leq 1 \forall y \in R$.

Statement 2: For any two real numbers x and y , $|x+y| \leq |x| + |y|$.

LO 11. Statement 1: $f(x) = \cos(x^2 - \tan x)$ is a non-periodic function.

Statement 2: $x^2 - \tan x$ is a non-periodic function.

LO 12. Statement 1: The period of function $f(x) = \sin\{x\}$ is 1, where $\{.\}$ represents fractional part function.

Statement 2: $g(x) = \{x\}$ has period 1.

13. Statement 1: If $f: R \rightarrow R$, $y = f(x)$ is periodic and continuous function, then $y = f(x)$ cannot be onto.

Statement 2: A continuous periodic function is bounded.

LO 14. Consider the functions $f(x) = \log_e x$ and $g(x) = 2x + 3$.

Statement 1: $f(g(x))$ is a one-one function.

Statement 2: $g(x)$ is a one-one function.

LO 15. Consider the functions $f: R \rightarrow R$, $f(x) = x^3$ and $g: R \rightarrow R$, $g(x) = 3x + 4$.

Statement 1: $f(g(x))$ is an onto an function.

Statement 2: $g(x)$ is an onto function.

16. Statement 1: $f(x) = \sin x$ and $g(x) = \cos x$ are identical functions.

Statement 2: Both the functions have the same domain and range.

LO 17. Statement 1: The period of $f(x) = \sin x$ is $2\pi \Rightarrow$ the period of $g(x) = |\sin x|$ is π .

Statement 2: The period of $f(x) = \cos x$ is $2\pi \Rightarrow$ the period of $g(x) = |\cos x|$ is π .

18. **Statement 1:** $f(x) = \sqrt{ax^2 + bx + c}$ has a range $[0, \infty)$ if $b^2 - 4ac > 0$.
Statement 2: $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac = 0$.
19. **Statement 1:** If $f(x) = \cos x$ and $g(x) = x^2$, then $f(g(x))$ is an even function.
Statement 2: If $f(g(x))$ is an even function, then both $f(x)$ and $g(x)$ must be even function.
20. **Statement 1:** The graph of $y = \sec^2 x$ is symmetrical about y -axis.
Statement 2: The graph of $y = \tan x$ is symmetrical about origin.

Linked Comprehension Type

Solutions on page 1.76

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which **only one** is correct.

For Problems 1–3

Consider the functions

$$f(x) = \begin{cases} x+1, & x \leq 1 \\ 2x+1, & 1 < x \leq 2 \end{cases} \text{ and } g(x) = \begin{cases} x^2, & -1 \leq x < 2 \\ x+2, & 2 \leq x \leq 3 \end{cases}$$

- The domain of the function $f(g(x))$ is
 a. $[0, \sqrt{2}]$ b. $[-1, 2]$
 c. $[-1, \sqrt{2}]$ d. None of these
- The range of the function $f(g(x))$ is
 a. $[1, 5]$ b. $[2, 3]$
 c. $[1, 2] \cup (3, 5]$ d. None of these
- The number of roots of the equation $f(g(x)) = 2$ is
 a. 1 b. 2
 c. 4 d. None of these

For Problems 4–6

Consider the function $f(x)$ satisfying the identity $f(x) + f\left(\frac{x-1}{x}\right) = 1+x, \forall x \in \mathbb{R} - \{0, 1\}$ and $g(x) = 2f(x) - x + 1$.

4. The domain of $y = \sqrt{g(x)}$ is

- $\left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup \left[1, \frac{1+\sqrt{5}}{2}\right]$
- $\left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup (0, 1) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$
- $\left[\frac{-1-\sqrt{5}}{2}, 0\right] \cup \left[\frac{-1+\sqrt{5}}{2}, 1\right]$
- None of these

5. The range of $y = g(x)$ is

- $(-\infty, 5]$ b. $[1, \infty)$
- $(-\infty, 1] \cup [5, \infty)$ d. None of these

6. The number of roots of the equation $g(x) = 1$ is

- 2 b. 1 c. 3 d. 0

For Problems 7–9

Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a function satisfying the following conditions, $f(1) = 1/2$ and $f(1) + 2, f(2) + 3, f(3) + \dots + n f(n) = n(n+1), f(n)$ for $n \geq 2$.

7. The value of $f(1003) = \frac{1}{K}$, where K equals

- 1003 b. 2003 c. 2005 d. 2006

8. The value of $f(999)$ is $\frac{1}{K}$, where K equals

- 999 b. 1000 c. 1998 d. 2000

9. $f(1), f(2), f(3), f(4), \dots$ represents a series of

- an A.P. b. a G.P.
- a H.P. d. An arithmetico-geometric

For Problems 10–12

If $(f(x))^2 \times f\left(\frac{1-x}{1+x}\right) = 64x, \forall x \in D_f$, then

10. $f(x)$ is equal to

- $4x^{2/3} \left(\frac{1+x}{1-x}\right)^{1/3}$ b. $x^{1/3} \left(\frac{1-x}{1+x}\right)^{1/3}$
- $x^{2/3} \left(\frac{1-x}{1+x}\right)^{1/3}$ d. $x \left(\frac{1+x}{1-x}\right)^{1/3}$

11. The domain of $f(x)$ is

- $[0, \infty)$ b. $\mathbb{R} - \{1\}$
- $(-\infty, \infty)$ d. None of these

12. The value of $f(9/7)$ is

- $8(9/7)^{2/3}$ b. $4(9/7)^{1/3}$
- $-8(9/7)^{2/3}$ d. None of these

For Problems 13–15

$f(x) = \begin{cases} x-1, & -1 \leq x \leq 0 \\ x^2, & 0 \leq x \leq 1 \end{cases}$ and $g(x) = \sin x$. Consider the functions $h_1(x) = f(|g(x)|)$ and $h_2(x) = |f(g(x))|$.

13. Which of the following is not true about $h_1(x)$?

- It is periodic function with period π
- Range is $[0, 1]$
- Domain is \mathbb{R}
- None of these

14. Which of the following is not true about $h_2(x)$?

- Domain is \mathbb{R}
- It periodic function with period 2π
- Range is $[0, 1]$
- None of these

15. For $h_1(x)$ and $h_2(x)$ are identical function, then which of the following is not true?

- Domain of $h_1(x)$ and $h_2(x), x \in [2n\pi, (2n+1)\pi], n \in \mathbb{Z}$
- Range of $h_1(x)$ and $h_2(x)$ is $[0, 1]$
- Period of $h_1(x)$ and $h_2(x)$ is π
- None of these

For Problems 16–18

If $a_0 = x$, $a_{n+1} = f(a_n)$, where $n = 0, 1, 2, \dots$, then answer the following questions.

16. If $f(x) = \sqrt[m]{a-x^m}$, $x > 0$, $m \geq 2$, $m \in \mathbb{N}$. Then
- $a_n = x$, $n = 2k+1$, where k is integer
 - $a_n = f(x)$ if $n = 2k$, where k is integer
 - Inverse of a_n exists for any value of n and m
 - None of these

17. If $f(x) = \frac{1}{1-x}$, then which of the following is not true?

- $a_n = \frac{1}{1-x}$ if $n = 3k+1$
- $a_n = \frac{x-1}{x}$ if $n = 3k+2$
- $a_n = x$ if $n = 3k$
- None of these

18. If $f: R \rightarrow R$ be given by $f(x) = 3+4x$ and $a_n = A+Bx$, then which of the following is not true?

- $A+B+1 = 2^{2n+1}$
- $|A-B| = 1$
- $\lim_{n \rightarrow \infty} \frac{A}{B} = -1$
- None of these

For Problems 19–21

Let $f(x) = f_1(x) - 2f_2(x)$,

where $f_1(x) = \begin{cases} \min \{x^2, |x|\}, & |x| \leq 1 \\ \max \{x^2, |x|\}, & |x| > 1 \end{cases}$

and $f_2(x) = \begin{cases} \min \{x^2, |x|\}, & |x| > 1 \\ \max \{x^2, |x|\}, & |x| \leq 1 \end{cases}$

and $g(x) = \begin{cases} \min \{f(t) : -3 \leq t \leq x, & -3 \leq x < 0 \\ \max \{f(t) : 0 \leq t \leq x, & 0 \leq x \leq 3 \end{cases}$

19. For $-3 \leq x \leq -1$, the range of $g(x)$ is

- $[-1, 3]$
- $[-1, -15]$
- $[-1, 9]$
- None of these

20. For $x \in (-1, 0)$, $f(x) + g(x)$ is

- $x^2 - 2x + 1$
- $x^2 + 2x - 1$
- $x^2 + 2x + 1$
- $x^2 - 2x - 1$

21. The graph of $y = g(x)$ in its domain is broken at

- 1 point
- 2 points
- 3 points
- None of these

For Problems 22–24

Let $f(x) = \begin{cases} 2x + a, & x \geq -1 \\ bx^2 + 3, & x < -1 \end{cases}$

and $g(x) = \begin{cases} x + 4, & 0 \leq x \leq 4 \\ -3x - 2, & -2 < x < 0 \end{cases}$

22. $g(f(x))$ is not defined if

- $a \in (10, \infty)$, $b \in (5, \infty)$
- $a \in (4, 10)$, $b \in (5, \infty)$
- $a \in (10, \infty)$, $b \in (0, 1)$
- $a \in (4, 10)$, $b \in (1, 5)$

23. If the domain of $g(f(x))$ is $[-1, 4]$, then

- $a = 1$, $b > 5$
- $a = 2$, $b > 7$
- $a = 2$, $b > 10$
- $a = 0$, $b \in R$

24. If $a = 2$ and $b = 3$, then the range of $g(f(x))$ is

- $(-2, 8]$
- $(0, 8]$
- $[4, 8]$
- $[-1, 8]$

For Problems 25–27

Let $f: R \rightarrow R$ is a function satisfying $f(2-x) = f(2+x)$ and $f(20-x) = f(x)$, $\forall x \in R$. For this function f , answer the following.

25. If $f(0) = 5$, then the minimum possible number of values of x satisfying $f(x) = 5$, for $x \in [0, 170]$, is

- 21
- 12
- 11
- 22

26. The graph of $y = f(x)$ is not symmetrical about

- symmetrical about $x = 2$
- symmetrical about $x = 10$
- symmetrical about $x = 8$
- None of these

27. If $f(2) \neq f(6)$, then the

- fundamental period of $f(x)$ is 1
- fundamental period of $f(x)$ may be 1
- period of $f(x)$ cannot be 1
- fundamental period of $f(x)$ is 8

For Problems 28–30

Consider two functions $f(x) = \begin{cases} [x], & -2 \leq x \leq -1 \\ |x|+1, & -1 < x \leq 2 \end{cases}$ and

$g(x) = \begin{cases} [x], & -\pi \leq x < 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$, where $[.]$ denotes the greatest integer function.

28. The exhaustive domain of $g(f(x))$ is

- $[0, 2]$
- $[-2, 0]$
- $[-2, 2]$
- $[-1, 2]$

29. The range of $g(f(x))$ is

- $[\sin 3, \sin 1]$
- $[\sin 3, 1] \cup \{-2, -1, 0\}$
- $[\sin 1, 1] \cup \{-2, -1\}$
- $[\sin 1, 1]$

30. The number of integral points in the range of $g(f(x))$ is

- 2
- 4
- 3
- 5

Matrix-Match Type

Solutions on page 1.79

Each question contains statements given in two columns which have to be matched.

Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match is a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4×4 matrix should be as follows:

	p	q	r	s
a	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
b	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
c	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
d	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

1. The function $f(x)$ is defined on the interval $[0, 1]$.

Then match the following columns

Column I: Function	Column II: Domain
a. $f(\tan x)$	p. $\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right], n \in \mathbb{Z}$
b. $f(\sin x)$	q. $\left[2n\pi, 2n\pi + \frac{\pi}{6}\right] \cup \left[2n\pi + \frac{5\pi}{6}, (2n+1)\pi\right], n \in \mathbb{Z}$
c. $f(\cos x)$	r. $[2n\pi, (2n+1)\pi], n \in \mathbb{Z}$
d. $f(2\sin x)$	s. $\left[n\pi, n\pi + \frac{\pi}{4}\right], n \in \mathbb{Z}$

2.

Column I: Function	Column II: Type of function
a. $f(x) = \{(\operatorname{sgn} x)^{\operatorname{sgn} x}\}^n, x \neq 0, n$ is an odd integer	p. odd function
b. $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$	q. even function
c. $f(x) = \begin{cases} 0, & \text{If } x \text{ is rational} \\ 1, & \text{If } x \text{ is irrational} \end{cases}$	r. neither odd nor even function
d. $f(x) = \max\{\tan x, \cot x\}$	s. periodic

3.

Column I: Functions	Column II: Values of x for which both the functions in any option of the column I are identical
a. $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right), g(x) = 2\tan^{-1}x$	p. $x \in \{-1, 1\}$
b. $f(x) = \sin^{-1}(\sin x)$ and $g(x) = \sin(\sin^{-1}x)$	q. $x \in [-1, 1]$
c. $f(x) = \log_{x^2} 25$ and $g(x) = \log_x 5$	r. $x \in (-1, 1)$
d. $f(x) = \sec^{-1}x + \operatorname{cosec}^{-1}x, g(x) = \sin^{-1}x + \cos^{-1}x$	s. $x \in (0, 1)$

4.

Column I	Column II
a. $f: \mathbb{R} \rightarrow \left[\frac{3\pi}{4}, \pi\right)$ and $f(x) = \cot^{-1}(2x - x^2 - 2)$, then $f(x)$ is	p. one-one
b. $f: \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = e^{px} \sin qx$ where $p, q \in \mathbb{R}^+$, then $f(x)$ is	q. into
c. $f: \mathbb{R}^+ \rightarrow [4, \infty]$ and $f(x) = 4 + 3x^2$, then $f(x)$ is	r. many-one
d. $f: X \rightarrow X$ and $f(f(x)) = x \forall x \in X$, then $f(x)$ is	s. onto

5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions such that $f(g(x))$ is a one-one function.

Column I	Column II
a. Then $g(x)$	p. must be one-one
b. Then $f(x)$	q. may not be one-one
c. If $g(x)$ is onto then $f(x)$	r. may be many-one
d. If $g(x)$ is into then $f(x)$	s. must be many-one

6. Column I: Function	Column II: Period
a. $f(x) = \cos(\sin x - \cos x)$	p. π
b. $f(x) = \cos(\tan x + \cot x) \cdot \cos(\tan x - \cot x)$	q. $\pi/2$
c. $f(x) = \sin^{-1}(\sin x) + e^{\tan x}$	r. 4π
d. $f(x) = \sin^3 x \sin 3x$	s. 2π

7. $\{.\}$ denotes the fractional part function and $[.]$ denotes the greatest integer function:

Column I: (Function)	Column II: (Period)
a. $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$	p. $1/3$
b. $f(x) = \cos 2\pi\{2x\} + \sin 2\pi\{2x\}$	q. $1/4$
c. $f(x) = \sin 3\pi\{x\} + \tan \pi[x]$	r. $1/2$
d. $f(x) = 3x - [3x + a] - b$, where $a, b \in \mathbb{R}^+$	s. 1

8. Column I: (Function)	Column II: (Range)
a. $f(x) = \log_3(5 + 4x - x^2)$	p. function not defined
b. $f(x) = \log_3(x^2 - 4x - 5)$	q. $[0, \infty)$
c. $f(x) = \log_3(x^2 - 4x + 5)$	r. $(-\infty, 2]$
d. $f(x) = \log_3(4x - 5 - x^2)$	s. \mathbb{R}

9. Column I: Equation	Column II: Number of roots
a. $x^2 \tan x = 1, x \in [0, 2\pi]$	p. 5
b. $2^{\cos x} = \sin x , x \in [0, 2\pi]$	q. 2
c. If $f(x)$ is a polynomial of degree 5 with real coefficients such that $f(x) = 0$ has 8 real roots, then the number of roots of $f(x) = 0$	r. 3
d. $7^{ x }(5 - x) = 1$	s. 4

Integer Type

Solutions on page 1.81

1. Let f be a real-valued invertible function such that $f\left(\frac{2x-3}{x-2}\right) = 5x-2, x \neq 2$. Then the value of $f^{-1}(13)$ is
2. Number of values of x for which $||x^2 - x + 4| - 2| - 3| = x^2 + x - 12$ is
3. Let $f(x) = 3x^2 - 7x + c$, where ' c ' is a variable coefficient and $x > \frac{7}{6}$. Then the value of $[c]$ such that $f(x)$ touches $f^{-1}(x)$ is (where $[.]$ represents greatest integer function)
4. Number of integral values of x for which $\frac{\left(2^{\frac{\pi}{\tan^{-1} x}} - 4\right)(x-4)(x-10)}{x! - (x-1)!} < 0$ is
5. Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function which satisfies $f(x) \cdot f(y) = f(xy) + 2\left(\frac{1}{x} + \frac{1}{y} + 1\right)$ for $x, y > 0$, then possible value of $f(1/2)$ is

6. A continuous function $f(x)$ on $\mathbb{R} \rightarrow \mathbb{R}$ satisfies the relation $f(x) + f(2x+y) + 5xy = f(3x-y) + 2x^2 + 1$ for $\forall x, y \in \mathbb{R}$, then the value of $|f(4)|$ is
7. Let $a > 2$ be a constant. If there are just 18 positive integers satisfying the inequality $(x-a)(x-2a)(x-a^2) < 0$, then the value of a is
8. Number of integers in the domain of function, satisfying $f(x) + f(x^{-1}) = \frac{x^3 + 1}{x}$, is
9. $f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17 \forall x \in \mathbb{R}$, then the value of $f(5)$ is
10. If $f(x)$ is an odd function and $f(1) = 3$, and $f(x+2) = f(x) + f(2)$, then the value of $f(3)$ is
11. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous onto function satisfying $f(x) + f(-x) = 0, \forall x \in \mathbb{R}$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5, 5]$, then the minimum number of roots of the equation $f(x) = 0$ is
12. Number of integral values of x for which the function $\sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$ is defined is

13. Suppose that f is an even, periodic function with period 2, and that $f(x) = x$ for all x in the interval $[0, 1]$. The value of $[10 f(3.14)]$ is (where $[\cdot]$ represents the greatest integer function)
14. If $f(x) = \sqrt{4-x^2} + \sqrt{x^2-1}$, then the maximum value of $(f(x))^2$ is
15. The function $f(x) = \frac{x+1}{x^3+1}$ can be written as the sum of an even function $g(x)$ and an odd function $h(x)$. Then the value of $|g(0)|$ is
16. If T is the period of the function $f(x) = [8x+7] + |\tan 2\pi x + \cot 2\pi x| - 8x$ (where $[\cdot]$ denotes the greatest integer function), then the value of $1/T$ is
17. If a, b and c are non-zero rational numbers, then the sum of all the possible values of $\frac{|a|}{a} + \frac{|b|}{b} + \frac{|c|}{c}$ is
18. An even polynomial function $f(x)$ satisfies a relation $f(2x) \left(1 - f\left(\frac{1}{2x}\right)\right) + f(16x^2y) = f(-2) - f(4xy) \quad \forall x, y \in \mathbb{R} - \{0\}$ and $f(4) = -255, f(0) = 1$, then the value of $|(f(2)+1)/2|$ is
19. If $f(x) = \sin^2 x + \sin^2 \left(x + \frac{\pi}{3}\right) + \cos x \cos \left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$ then $(g \circ f)(x)$ is
20. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. If N is number of onto functions from E to F , then the value of $N/2$ is
21. The function f is continuous and has the property $f(f(x)) = 1-x$, then the value of $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)$ is
22. Number of integral values of x satisfying the inequality $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$
23. A function f from integers to integers is defined as $f(x) = \begin{cases} n+3, & n \in \text{odd} \\ n/2, & n \in \text{even} \end{cases}$. Suppose $k \in \text{odd}$ and $f(f(f(k))) = 27$, then the sum of digits of k is
24. If θ be the fundamental period of function $f(x) = \sin^{99} x + \sin^{99} \left(x + \frac{2\pi}{3}\right) + \sin^{99} \left(x + \frac{4\pi}{3}\right)$, then complex number $z = |z|(\cos \theta + i \sin \theta)$ lies in the quadrant number.
25. If $x = \frac{4}{9}$ satisfy the equation $\log_a(x^2 - x + 2) > \log_a(-x^2 + 2x + 3)$, then sum of all possible distinct values of $[x]$ is (where $[\cdot]$ represents greatest integer function)
26. If $4^x - 2^{x^2+2} + 5 + ||b-1|-3| = |\sin y|$, $x, y, b \in \mathbb{R}$, then the possible value of b is
27. If $f: N \rightarrow N$, and $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$, $\forall x_1, x_2 \in N$ and $f(f(n)) = 3n$, $\forall n \in N$, then $f(2) =$
28. Number of integral values of a for which $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ be defined for every real value of x

29. Let $f(x) = \sin^{23} x - \cos^{22} x$ and $g(x) = 1 + \frac{1}{2} \tan^{-1} |x|$, then the number of values of x in interval $[-10\pi, 8\pi]$ satisfying the equation $f(x) = \text{sgn}(g(x))$ is
30. Suppose that $f(x)$ is a function of the form $f(x) = \frac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}$ ($x \neq 0$). If $f(5) = 2$, then the value of $|f(-5)|/4$ is

Archives

Solutions on page 1.84

Subjective

1. Find the domain and range of the function $f(x) = \frac{x^2}{1+x^2}$.
Is the function one-to-one? (IIT-JEE, 1978)
2. Draw the graph of $y = |x|^{1/2}$ for $-1 \leq x \leq 1$ (IIT-JEE, 1978)
3. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find $f(6)$. (IIT-JEE, 1979)
4. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false $f(x) = 1$, $f(y) \neq 1$, $f(z) \neq 2$ determine $f^{-1}(1)$. (IIT-JEE, 1982)
5. Find the natural number a for which $\sum_{k=1}^n f(a+k) = 16(2^n - 1)$, where the function f satisfies the relation $f(x+y) = f(x)f(y)$ for all natural numbers x, y and further $f(1) = 2$. (IIT-JEE, 1992)
6. Let $\{x\}$ and $[x]$ denote the fractional and integral part of a real number x , respectively. Solve $4\{x\} = x + [x]$. (IIT-JEE, 1992)
7. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha = 3$? Justify your answer. (IIT-JEE, 1996)

Objective

Fill in the blanks

1. The values of $f(x) = 3 \sin \left(\sqrt{\frac{\pi^2}{16} - x^2} \right)$ lie in the interval, (IIT-JEE, 1983)
2. The domain of the function $f(x) = \sin^{-1} \left(\log_2 \frac{x^2}{2} \right)$ is given by (IIT-JEE, 1984)
3. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is _____ and out of these _____ are onto functions. (IIT-JEE, 1985)
4. If $f(x) = \sin \log_e \left(\frac{\sqrt{4-x^2}}{1-x} \right)$, then the domain of $f(x)$ is _____ and its range is _____. (IIT-JEE, 1985)

5. There are exactly two distinct linear functions, _____, and _____ which map $[-1, 1]$ onto $[0, 2]$.

6. If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are _____, _____, _____, and _____.

(IIT-JEE, 1985)

7. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then $(g \circ f)(x) =$ _____.

(IIT-JEE, 1996)

8. The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is _____.

(IIT-JEE, 2011)

True or false

1. If $f(x) = (a - x^n)^{1/n}$ where $a > 0$ and n is a positive integer, then $f[f(x)] = x$.

(IIT-JEE, 1983)

2. The function $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$ is not onto.

(IIT-JEE, 1983)

3. If $f_1(x)$ and $f_2(x)$ are defined on the domain D_1 and D_2 respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$.

(IIT-JEE, 1988)

Multiple choice questions with one correct answer

1. Let R be the set of real numbers. If $f: R \rightarrow R$ is a function defined by $f(x) = x^2$, then f is

- a. Injective but not surjective
b. Surjective but not injective
c. Bijective
d. None of these

(IIT-JEE, 1979)

2. The entire graph of the equation $y = x^2 + kx - x + 9$ is strictly above the x -axis if and only if

- a. $k < 7$
b. $-5 < k < 7$
c. $k > -5$
d. None of these

(IIT-JEE, 1979)

3. Let $f(x) = |x - 1|$. Then

- a. $f(x^2) = (f(x))^2$
b. $f(x+y) = f(x) + f(y)$
c. $f(|x|) = |f(x)|$
d. None of these

(IIT-JEE, 1983)

4. If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then

- a. $0 \leq x \leq 4$
b. $x \leq -2$ or $x \geq 4$
c. $x \leq 0$ or $x \geq 4$
d. None of these

(IIT-JEE, 1983)

5. If $f(x) = \cos(\log_e x)$, then $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

has the value

- a. -1
b. $1/2$
c. -2
d. None of these

(IIT-JEE, 1983)

6. The domain of definition of the function

$$y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2} =$$

- a. $(-3, -2)$ excluding -2.5 b. $[0, 1]$ excluding 0.5
c. $[-2, 1)$ excluding 0 d. None of these

(IIT-JEE, 1983)

7. Which of the following functions is periodic?
a. $f(x) = x - [x]$ where $[x]$ denotes the largest integer less than or equal to the real number x

b. $f(x) = \sin \frac{1}{x}$ for $x \neq 0$, $f(0) = 0$

c. $f(x) = x \cos x$

d. None of these

(IIT-JEE, 1983)

8. If the function $f: [1, \infty) \rightarrow [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is

a. $\left(\frac{1}{2}\right)^{x(x-1)}$

b. $\frac{1}{2}\left(1 + \sqrt{1 + 4 \log_2 x}\right)$

c. $\frac{1}{2}\left(1 - \sqrt{1 + 4 \log_2 x}\right)$ d. Not defined

(IIT-JEE, 1992)

9. Let $f(x) = \sin x$ and $g(x) = \log_e |x|$. If the ranges of the composition function $f \circ g$ and $g \circ f$ are R_1 and R_2 , respectively, then

a. $R_1 = \{u: -1 \leq u < 1\}$, $R_2 = \{v: -\infty < v < 0\}$

b. $R_1 = \{u: -\infty < u < 0\}$, $R_2 = \{v: -\infty < v < 0\}$

c. $R_1 = \{u: -1 < u < 1\}$, $R_2 = \{v: -\infty < v < 0\}$

d. $R_1 = \{u: -1 \leq u \leq 1\}$, $R_2 = \{v: -\infty < v \leq 0\}$

(IIT-JEE, 1994)

10. Let $f(x) = (x+1)^2 - 1$, $x \geq -1$. Then the set $\{x: f(x) = f^{-1}(x)\}$ is

a. $\left\{0, -1, \frac{-3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}$

b. $\{0, 1, -1\}$

c. $\{0, -1\}$

d. empty

(IIT-JEE, 1995)

11. Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then

a. $f(x)$ is bounded

b. $f\left(\frac{1}{x}\right) \rightarrow 0$ as $x \rightarrow 0$

c. $x f(x) \rightarrow 1$ as $x \rightarrow 0$

d. $f(x) = \log_e x$

(IIT-JEE, 1995)

12. The domain of definition of the function $f(x)$ given by the equation $2^x + 2^y = 2$ is

a. $0 < x \leq 1$

b. $0 \leq x \leq 1$

c. $-\infty < x \leq 0$

d. $-\infty < x < 1$

(IIT-JEE, 2000)

13. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x ,

$f(g(x))$ is equal to (where $[\cdot]$ represents greatest integer function)

a. x

b. 1

c. $f(x)$

d. $g(x)$

(IIT-JEE, 2001)

14. If $f: [1, \infty) \rightarrow [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals

(IIT-JEE, 2001)

$$\text{a. } \frac{(x + \sqrt{x^2 - 4})}{2}$$

$$\text{c. } \frac{(x - \sqrt{x^2 - 4})}{2}$$

$$\text{b. } \frac{x}{1+x^2}$$

$$\text{d. } 1 + \sqrt{x^2 - 4}$$

15. The domain of definition of $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$ is

- a. $R - \{-1, -2\}$ b. $(-2, \infty)$
 c. $R - \{-1, -2, -3\}$ d. $(-3, \infty) - \{-1, -2\}$

(IIT-JEE, 2001)

16. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then the number of onto functions from E to F is

- a. 14 b. 16 c. 12 d. 8

(IIT-JEE, 2001)

17. Let $f(x) = \frac{\alpha x}{x+1}, x \neq -1$. Then for what value of α is $f(f(x)) = x$?

- a. $\sqrt{2}$ b. $-\sqrt{2}$
 c. 1 d. -1

(IIT-JEE, 2001)

18. Suppose $f(x) = (x+1)^2$ for $x \geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals

- a. $-\sqrt{x} - 1, x \geq 0$ b. $\frac{1}{(x+1)^2}, x > -1$
 c. $\sqrt{x+1}, x \geq -1$ d. $\sqrt{x} - 1, x \geq 0$

(IIT-JEE, 2002)

19. Let function $f: R \rightarrow R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$, then f is

- a. one-to-one and onto
 b. one to one but NOT onto
 c. onto but NOT one-to-one
 d. neither one-to-one nor onto

(IIT-JEE, 2002)

20. If $f: [0, \infty) \rightarrow [0, \infty)$, and $f(x) = \frac{x}{1+x}$, then f is

- a. one-one and onto b. one-one but not onto
 c. onto but not one-one d. neither one-one nor onto

(IIT-JEE, 2003)

21. The domain of definition of the function

$$f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$$

for real-valued x is

- a. $\left[-\frac{1}{4}, \frac{1}{2}\right]$ b. $\left[-\frac{1}{2}, \frac{1}{2}\right]$
 c. $\left[-\frac{1}{2}, \frac{1}{9}\right]$ d. $\left[-\frac{1}{4}, \frac{1}{4}\right]$

(IIT-JEE, 2003)

22. The range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}, x \in R$, is

- a. $(1, \infty)$ b. $(1, 11/7)$
 c. $(1, 7/3]$ d. $(1, 7/5)$

(IIT-JEE, 2003)

23. If $f(x) = \sin x + \cos x, g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the domain

- a. $\left[0, \frac{\pi}{2}\right]$ b. $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
 c. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ d. $[0, \pi]$

(IIT-JEE, 2004)

24. If the functions $f(x)$ and $g(x)$ are defined on $R \rightarrow R$ such that

$$f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$$

$$\text{and } g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases} \text{ then } (f-g)(x) \text{ is}$$

- a. one-one and onto b. neither one-one nor onto
 c. one-one but not onto d. onto but not one-one

(IIT-JEE, 2005)

25. X and Y are two sets and $f: X \rightarrow Y$. If $\{f(c) = y; c \in X, y \in Y\}$ and $f^{-1}(d) = x; d \in Y, x \in X$, then the true statement is

- a. $f(f^{-1}(b)) = b$ b. $f^{-1}(f(a)) = a$
 c. $f(f^{-1}(b)) = b, b \in Y$ d. $f^{-1}(f(a)) = a, a \in X$

(IIT-JEE, 2005)

Multiple choice questions with one or more than one correct answer

1. If $y = f(x) = \frac{x+2}{x-1}$ then

- a. $x = f(y)$
 b. $f(1) = 3$
 c. y increases with x for $x < 1$
 d. f is a rational function of x

(IIT-JEE, 1984)

2. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $(x, g(x))$ is $\sqrt{3}/4$ then the function $g(x)$ is

- a. $g(x) = \pm \sqrt{1-x^2}$ b. $g(x) = \sqrt{1-x^2}$
 c. $g(x) = -\sqrt{1-x^2}$ d. $g(x) = \sqrt{1+x^2}$

(IIT-JEE, 1989)

3. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where $[x]$ stands for the greatest integer function, then

- a. $f\left(\frac{\pi}{2}\right) = -1$ b. $f(\pi) = 1$
 c. $f(-\pi) = 0$ d. $f\left(\frac{\pi}{4}\right) = 1$

(IIT-JEE, 1991)

4. If $f(x) = 3x - 5$, then $f^{-1}(x)$

- a. is given by $\frac{1}{3x-5}$
 b. is given by $\frac{x+5}{3}$
 c. does not exist because f is not one-one
 d. does not exist because f is not onto

(IIT-JEE, 1998)

Q 5. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin \sqrt{x})^2$, then

- a. $f(x) = \sin^2 x, g(x) = \sqrt{x}$
 b. $f(x) = \sin x, g(x) = |x|$
 c. $f(x) = x^2, g(x) = \sin \sqrt{x}$
 d. f and g cannot be determined (IIT-JEE, 1998)

Match the following type

This question contains statements given in two columns which have to be matched. Statements a, b, c, d in Column I have to be matched with statements p, q, r, s in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct match is a-p, a-s, b-q, b-r, c-q and d-s, then the correctly bubbled 4×4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. Let $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$

Match of expressions/statements in Column I with expressions/statements in Column II

Column I	Column II
a. If $-1 < x < 1$, then $f(x)$ satisfies	p. $0 < f(x) < 1$
b. If $1 < x < 2$, then $f(x)$ satisfies	q. $f(x) < 0$
c. If $3 < x < 5$, then $f(x)$ satisfies	r. $f(x) > 0$
d. If $x > 5$, then $f(x)$ satisfies	s. $f(x) < 1$

(IIT-JEE, 2007)

ANSWERS AND SOLUTIONS

Subjective Type

1. a. $x + |y| = 2y$

If $y \geq 0$, we have $x + y = 2y \Rightarrow y = x$
 $\Rightarrow y = x, x \geq 0$

If $y < 0$ $x - y = 2y \Rightarrow y = \frac{x}{3}$

$\Rightarrow y = \frac{x}{3}; x < 0$

$\Rightarrow y = \begin{cases} \frac{x}{3}, & x < 0 \\ x, & x \geq 0 \end{cases}$ $D_f \equiv R$

b. $e^y - e^{-y} = 2x$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0$ (Multiplying by e^y)

$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$

$\Rightarrow e^y = x + \sqrt{x^2 + 1}$ (as $\sqrt{x^2 + 1} > x$, then
 $x - \sqrt{x^2 + 1} < 0$, which is not possible)

$\Rightarrow y = \ln(x + \sqrt{x^2 + 1})$

$D_f \equiv R$

c. $10^x + 10^y = 10$

$\Rightarrow 10^y = 10 - 10^x$

$\Rightarrow y = \log_{10}(10 - 10^x)$

For domain $10 - 10^x > 0 \Rightarrow 10^x < 10 \Rightarrow x < 1$

$\Rightarrow D_f \equiv (-\infty, 1)$

d. $x^2 - \sin^{-1} y = \frac{\pi}{2}$

$\Rightarrow \sin^{-1} y = x^2 - \pi/2$

$\Rightarrow y = \sin(x^2 - \pi/2)$

$D_f \equiv R$

2. $g(x) = \sqrt{x - 2k}, \forall 2k \leq x < 2(k+1)$, where $k \in \mathbb{Z}$, integer

$$\Rightarrow g(x) = \begin{cases} \dots \\ \dots \\ \sqrt{x+2}, & -2 \leq x < 0 \\ \sqrt{x}, & 0 \leq x < 2 \\ \sqrt{x-2}, & 2 \leq x < 4 \\ \sqrt{x-4}, & 4 \leq x < 6 \\ \dots \\ \dots \end{cases}$$

$\Rightarrow g$ is periodic with period = 2

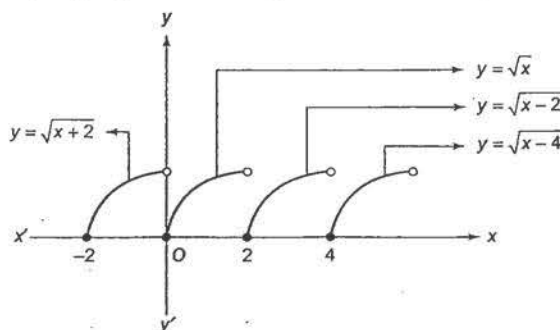


Fig. 1.92

3. Given $f(x) = x^2 - 2x = (x-1)^2 - 1$

$$\begin{aligned} \Rightarrow g(x) &= f(f(x) - 1) + f(5 - f(x)) \\ &= f[(x-1)^2 - 2] + f[6 - (x-1)^2] \\ &= [(x-1)^2 - 2 - 1]^2 - 1 + [6 - (x-1)^2 - 1]^2 - 1 \\ &= (x-1)^4 - 6(x-1)^2 + 9 - 1 + (x-1)^4 \\ &= 2(x-1)^4 - 6(x-1)^2 + 8 \end{aligned}$$

$$\begin{aligned}
 &= 2(x-1)^4 - 16(x-1)^2 + 32 \\
 &= 2[(x-1)^4 - 8(x-1)^2 + 16] \\
 &= 2[(x-1)^2 - 4]^2 \geq 0 \quad \forall x \in R
 \end{aligned}$$

4. Let two linear functions be $f(x) = ax + b$ and $g(x) = cx + d$

They map $[-1, 1] \rightarrow [0, 2]$ and mapping is onto

$$\Rightarrow f(-1) = 0 \text{ and } f(1) = 2 \text{ and } g(-1) = 2 \text{ and } g(1) = 0$$

$$\Rightarrow -a + b = 0 \text{ and } a + b = 2 \quad (1)$$

$$\text{and } -c + d = 2 \text{ and } c + d = 0 \quad (2)$$

$$\Rightarrow a = b = 1 \text{ and } c = -1, d = 1$$

$$\Rightarrow f(x) = x + 1 \text{ and } g(x) = -x + 1$$

$$\Rightarrow h(x) = \frac{x+1}{1-x} \Rightarrow h(h(x)) = \frac{\frac{x+1}{1-x} + 1}{\frac{x+1}{1-x} - 1} = \frac{1}{x}$$

$$\Rightarrow h(h(1/x)) = x$$

$$\Rightarrow |h(h(x)) + h(h(1/x))| = |x + 1/x| > 2.$$

$$5. f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & 3 \leq x < 4 \\ x - 4, & x \geq 4 \end{cases} \quad (1)$$

$$g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$$

$$\Rightarrow g(x) = \begin{cases} x - 3, & x < 3 \\ x - 3, & 3 \leq x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases} \quad (2)$$

From (1) and (2), we have

$$\frac{f(x)}{g(x)} = \begin{cases} \frac{x^2 - 4x + 3}{x - 3}, & x < 3 \\ \frac{x - 4}{x - 3}, & 3 < x < 4 \\ \frac{x - 4}{x^2 + 2x + 2}, & x \geq 4 \end{cases}$$

Clearly, $f(x)/g(x)$ is not defined at $x = 3$, hence the domain is $R - \{3\}$.

6. Given $f(x) = \log_2 \log_3 \log_4 \log_5 (\sin x + a^2)$

$f(x)$ is defined only if $\log_3 \log_4 \log_5 (\sin x + a^2) > 0, \forall x \in R$

$$\Rightarrow \log_4 \log_5 (\sin x + a^2) > 1, \forall x \in R$$

$$\Rightarrow \log_5 (\sin x + a^2) > 4, \forall x \in R$$

$$\Rightarrow (\sin x + a^2) > 5^4, \forall x \in R$$

$$\Rightarrow a^2 > 625 - \sin x, \forall x \in R$$

$$\Rightarrow a^2 \text{ must be greater than maximum value of } 625 - \sin x \text{ which is } 626 \text{ (when } \sin x = -1)$$

$$\Rightarrow a^2 > 626$$

$$\Rightarrow a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)$$

7. By remainder theorem, $P(a) = a, P(b) = b$ and $P(c) = c$.
Let the required remainder be $R(x)$, then $P(x) = (x-a)(x-b)(x-c)Q(x) + R(x)$, where $R(x)$ is a polynomial of degree at most 2.

$$\text{We get } R(a) = a, R(b) = b \text{ and } R(c) = c.$$

So, the equation $R(x) - x = 0$ has three roots a, b and c .

But its degree is at most 2, So, $R(x) - x$ must be zero polynomial (or identity). Hence, $R(x) = x$.

8.

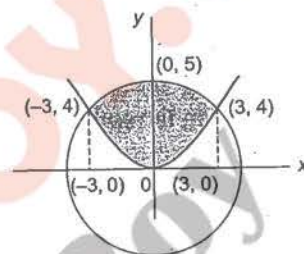


Fig. 1.93

The equation $x^2 + y^2 = 25$ represents a circle with centre (0, 0) and radius 5 and the equation $y = \frac{4}{9}x^2$ represents a

parabola with vertex (0, 0). Hence, $R \cap R'$ is the set of points indicated in the figure = $\{(x, y) : -3 \leq x \leq 3, 0 \leq y \leq 5\}$. Thus, the domain $R \cap R' = [-3, 3]$ and the range $R \cap R' = [0, 5]$.

$$9. \text{ Put } y = \frac{1}{x}$$

$$\Rightarrow 2 + f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1) \quad (1)$$

Now put $x = 1$

$$\Rightarrow 2 + (f(1))^2 = 3f(1)$$

$$\Rightarrow f(1) = 1 \text{ or } 2$$

But $f(1) \neq 1$, otherwise from the given relation $2 + f(x)f(1) = f(x) + f(1) + f(x)$ or $f(x) = 1$, which is not possible as given that $f(2) = 5$.

Hence, $f(1) = 2$.

$$\Rightarrow \text{From (1), we have } f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$$\Rightarrow f(x) = \pm x^n + 1$$

$$\Rightarrow f(2) = \pm 2^n + 1 = 5$$

$$\Rightarrow 2^n = 4 \Rightarrow n = 2$$

$$\Rightarrow f(x) = x^2 + 1$$

$$\Rightarrow f(f(2)) = f(5) = 26$$

$$\begin{aligned}
 10. f(x) &= f(b + (x - b)) \\
 &= f(b - (x - b)) \\
 &= f(2b - x) \\
 &= f(a + (2b - x - a)) \\
 &= f(a - (2b - x - a)) \\
 &= f(2a - 2b + x)
 \end{aligned}$$

Hence, $f(x)$ is periodic with period $2a - 2b$.

11. Given $f(xf(y)) = x^p y^q$

$$\Rightarrow x = \frac{\{f(xf(y))\}^{1/p}}{y^{q/p}} \quad (1)$$

Let $xf(y) = 1 \Rightarrow x = \frac{1}{f(y)}$, then from (1)

$$f(y) = \frac{y^{q/p}}{\{f(1)\}^{1/p}}$$

$$\Rightarrow f(1) = \frac{1}{\{f(1)\}^{1/p}}$$

$$\Rightarrow f(1) = 1$$

$$\Rightarrow f(y) = y^{q/p} \quad (2)$$

 Now, $f(xy^{q/p}) = x^p y^q$. Put $y^{q/p} = z$, we get

$$f(xz) = (xz)^p$$

$$\Rightarrow f(x) = x^p \quad (3)$$

 From (2) and (3) $x^p = x^{q/p} \Rightarrow p^2 = q$.

12. $f(x-1) + f(x+1) = \sqrt{3}f(x) \quad (1)$

 Putting $x+2$ for x in relation (1) we get $f(x+1) + f(x+3) =$

$$\sqrt{3}f(x+2) \quad (2)$$

From (1) and (2), we get

$$\begin{aligned} f(x-1) + 2f(x+1) + f(x+3) &= \sqrt{3}(f(x) + f(x+2)) \\ &= \sqrt{3}(\sqrt{3}f(x+1)) \\ &= 3f(x+1) \end{aligned}$$

$$\Rightarrow f(x-1) + f(x+3) = f(x+1) \quad (3)$$

 Putting $x+2$ for x in relation (3), we get

$$f(x+1) + f(x+5) = f(x+3) \quad (4)$$

 Adding (3) and (4) in $f(x-1) = -f(x+5)$

$$\text{Now, put } x+1 \text{ for } x, f(x) = -f(x+6) \quad (5)$$

 Put $x+6$ in place of x in (5), we get $f(x+6) = -f(x+12)$

$$\therefore \text{from (5) again, } f(x) = -[-f(x+12)] = f(x+12)$$

 \therefore the period of $f(x)$ is 12.

13. $f(a+x) = b + [1 + b^3 - 3b^2 f(x) + 3b \{f(x)\}^2 - \{f(x)\}^3]^{1/3}$

$$= b + [1 + \{b - f(x)\}^3]^{1/3}$$

$$\Rightarrow f(a+x) - b = [1 - \{f(x) - b\}^3]^{1/3}$$

$$\Rightarrow \phi(a+x) = [1 - \{\phi(x)\}^3]^{1/3} \quad (1)$$

 where $\phi(x) = f(x) - b$

$$\Rightarrow \phi(2a+x) = [1 - \{\phi(x+a)\}^3]^{1/3} = \phi(x) \text{ from (1)}$$

$$\Rightarrow f(x+2a) - b = f(x) - b$$

$$\Rightarrow f(x+2a) = f(x)$$

 $\therefore f(x)$ is periodic with period $2a$.

14. $f(x, y) = f(2x + 2y, 2y - 2x)$
 (Replacing x by $2x + 2y$ and y by $2y - 2x$)
 $= f(2(2x + 2y) + 2(2y - 2x),$
 $2(2y - 2x) - 2(2x + 2y))$

$$f(x, y) = f(8x, -8y) = f(8(-8x), -8(8y))$$

$$= f(-64x, -64y)$$

$$= f(-64(-64x), -64y(-64y)) = f(2^{12}x, 2^{12}y)$$

$$f(x, 0) = f(2^{12}x, 0)$$

$$f(2^y, 0) = f(2^{12} \cdot 2^y, 0) = f(2^{12+y}, 0)$$

$$\Rightarrow g(y) = g(y + 12)$$

 Hence, $g(x)$ is periodic and its period is 12.

15. $y = \frac{x-a}{(x-b)(x-c)} \Rightarrow yx^2 - [(b+c)y+1]x + bcy + a = 0$

 Now x is real, $\Rightarrow D \geq 0$

$$\Rightarrow [(b+c)y-1]^2 - 4y(bcy-a) \geq 0, \forall y \in R,$$

 (as given that $f(x)$ is an onto function)

$$\Rightarrow (b-c)^2 y^2 - 2(b+c-2a)y + 1 \geq 0, \forall y \in R$$

$$D \leq 0$$

$$\Rightarrow 4(b+c-2a)^2 - 4(b-c)^2 \leq 0$$

$$\Rightarrow (b+c-2a-b+c)(b+c-2a+b-c) \leq 0$$

$$\Rightarrow (c-a)(b-a) \leq 0$$

$$\Rightarrow c \leq a \text{ and } b \geq a \text{ or } b > c \text{ and } c \geq a \text{ and } b \leq a$$

$$\Rightarrow c \leq a \leq b \text{ (as } b > c)$$

$$\Rightarrow a \in (b, c)$$

16. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, a_i \in I (i=0, 1, 2, \dots, n)$

Now, $f(a) = a_0 + a_1 a + a_2 a^2 + \dots + a_n a^n = b$

$$f(b) = a_0 + a_1 b + a_2 b^2 + \dots + a_n b^n = c$$

$$f(x) = a_0 + a_1 c + a_2 c^2 + \dots + a_n c^n = a$$

$$\therefore f(a) - f(b) = (a-b)f_1(a, b) = b-c,$$

 where $f_1(a, b)$ is an integer

Similarly, $(b-c)f_1(b, c) = c-a$

and $(c-a)f_1(c, a) = a-b$

 Multiplying all these, we get $f_1(a, b)f_1(b, c)f_1(c, a) = 1$

$$\Rightarrow f_1(a, b) = f_1(b, c) = f_1(c, a) = 1$$

$$\Rightarrow a-b = b-c, c-a = a-b \text{ and } c-a = a-b$$

$$\Rightarrow a = b = c \text{ which is a contradiction.}$$

Hence, no such polynomial exists.

17. Clearly, from graph $g(x) = \begin{cases} x^2, & -2 \leq x \leq -1 \\ 1-x, & -1 < x \leq -1/4 \\ \frac{1}{2}x, & -1/4 < x < 0 \\ 1+x, & 0 \leq x < 1 \\ x^2, & 1 \leq x \leq 2 \end{cases}$

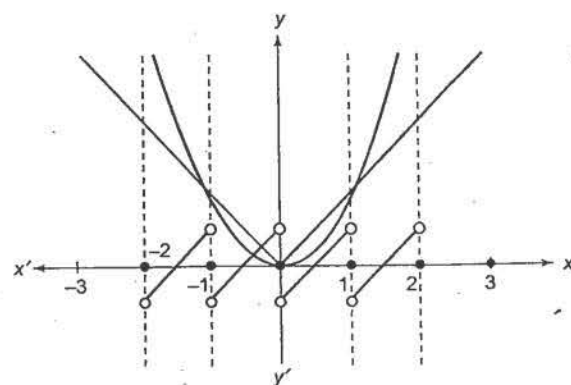


Fig. 1.94

18. Given $f(x-f(y)) = f(f(y)) + xf(y) + f(x) - 1$ (1)

Putting $x = f(y) = 0$, then $f(0) = f(0) + 0 + f(0) - 1$

$\therefore f(0) = 1$ (2)

Again putting $x = f(y) = \lambda$ in (1)

Then $f(0) = f(\lambda) + \lambda^2 + f(\lambda) - 1$

$\Rightarrow 1 = 2f(\lambda) + \lambda^2 - 1$ {from (2)}

$\therefore f(\lambda) = \frac{2 - \lambda^2}{2} = 1 - \frac{\lambda^2}{2}$

Hence, $f(x) = 1 - \frac{x^2}{2}$ is the unique function.

19. Since $f(x) = (2 \cos x - 1)(2 \cos 2x - 1)(2 \cos 2^2 x - 1) \dots (2 \cos 2^{n-1} x - 1)$

$$\begin{aligned} \therefore f(x) &= \frac{(2 \cos x + 1)(2 \cos x - 1)(2 \cos 2x - 1) \times \dots (2 \cos 2^{n-1} x - 1)}{(2 \cos x + 1)} \\ &= \frac{(4 \cos^2 x - 1)(2 \cos 2x - 1)(2 \cos 2^2 x - 1) \dots (2 \cos 2^{n-1} x - 1)}{(2 \cos x + 1)} \\ &= \frac{(2 \cos 2x + 1)(2 \cos 2x - 1)(2 \cos 2^2 x - 1) \dots (2 \cos 2^{n-1} x - 1)}{(2 \cos x + 1)} \\ &= \frac{(4 \cos^2 2x - 1)(2 \cos 2^2 x - 1) \dots (2 \cos 2^{n-1} x - 1)}{(2 \cos x + 1)} \\ &= \frac{(2 \cos 2^2 x + 1)(2 \cos 2^2 x - 1) \dots (2 \cos 2^{n-1} x - 1)}{(2 \cos x + 1)} \end{aligned}$$

Proceeding in similarly way

$$\begin{aligned} f(x) &= \frac{(2 \cos 2^{n-1} x + 1)(2 \cos 2^{n-1} x - 1)}{(2 \cos x + 1)} \\ &= \frac{(4 \cos^2 2^{n-1} x - 1)}{(2 \cos x + 1)} = \frac{(2 \cos 2^n x + 1)}{(2 \cos x + 1)} \\ \Rightarrow f\left(\frac{2\pi k}{2^n \pm 1}\right) &= \frac{2 \cos\left(\frac{2^{n+1} \pi k}{2^n \pm 1}\right) + 1}{2 \cos\left(\frac{2\pi k}{2^n \pm 1}\right) + 1} \\ &= \frac{2 \cos\left(2\pi k \mp \frac{2\pi k}{2^n \pm 1}\right) + 1}{2 \cos\left(\frac{2\pi k}{2^n \pm 1}\right) + 1} \\ &= \frac{2 \cos\left(\frac{2\pi k}{2^n \pm 1}\right) + 1}{2 \cos\left(\frac{2\pi k}{2^n \pm 1}\right) + 1} = 1 \end{aligned}$$

20. $f(x) = \frac{a^x}{a^x + \sqrt{a}}$

$\Rightarrow f(1-x) = \frac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \frac{a^1}{a^1 + \sqrt{a}a^x} = \frac{\sqrt{a}}{\sqrt{a} + a^x}$

$\Rightarrow f(x) + f(1-x) = 1$

Also, $f\left(\frac{1}{2}\right) = \frac{1}{2}$

$\Rightarrow \sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$

$$\begin{aligned} &= 2 \left[f\left(\frac{1}{2n}\right) + f\left(\frac{2}{2n}\right) + \dots + f\left(\frac{n-1}{2n}\right) \right. \\ &\quad \left. + f\left(\frac{n}{2n}\right) + f\left(\frac{n+1}{2n}\right) + \dots \right. \\ &\quad \left. + f\left(\frac{2n-1}{2n}\right) \right] \\ &= 2 \left[\left[f\left(\frac{1}{2n}\right) + f\left(\frac{2n-1}{2n}\right) \right] + \left[f\left(\frac{2}{2n}\right) + f\left(\frac{2n-2}{2n}\right) \right] \right. \\ &\quad \left. + \dots + \left[f\left(\frac{n-1}{2n}\right) + f\left(\frac{n+1}{2n}\right) \right] + f\left(\frac{1}{2}\right) \right] \\ &= 2 \left[\left[f\left(\frac{1}{2n}\right) + f\left(1 - \frac{1}{2n}\right) \right] + \left[f\left(\frac{2}{2n}\right) + f\left(1 - \frac{2}{2n}\right) \right] \right. \\ &\quad \left. + \dots + \left[f\left(\frac{n-1}{2n}\right) + f\left(1 - \frac{n-1}{2n}\right) \right] + \frac{1}{2} \right] \\ &= 2[1 + 1 + 1 + \dots + (n-1) \text{ times}] + 1 \\ &= 2n-1 \end{aligned}$$

Objective Type

1. b. $f: N \rightarrow N, f(n) = 2n + 3$.

Here, the range of the function is $\{5, 6, 7, \dots\}$ or $N - \{1, 2, 3, 4\}$

which is a subset of N (co-domain).

Hence, function is into.

Also, it is clear that $f(n)$ is one-one or injective.

Hence, $f(n)$ is injective only.

2. b. $f(x) = \sin(\log(x + \sqrt{1+x^2}))$

$\Rightarrow f(-x) = \sin[\log(-x + \sqrt{1+x^2})]$

$\Rightarrow f(-x) = \sin \log \left((\sqrt{1+x^2} - x) \frac{(\sqrt{1+x^2} + x)}{(\sqrt{1+x^2} + x)} \right)$

$\Rightarrow f(-x) = \sin \log \left[\frac{1}{(x + \sqrt{1+x^2})} \right]$