Index

ALLEN CAREER INSTITUTE KOTA (RAJASTHAN)
Path to Success

HANDBOOK OF MATHEMATICS

LOGARITHM

LOGARITHM OF A NUMBER :

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$.

(a) $\log_a N = x$, read as log of N to the base a $\Leftrightarrow a^x = N$
If a = 10 then we write log N or $\log_{10} N$ and if a = e we write ln N or $\log_e N$ (Natural log)

(b) Necessary conditions : $N > 0$; $a > 0$; $a \neq 1$

(c) $\log_a 1 = 0$

(d) $\log_a a = 1$

(e) $\log_{1/a} a = -1$

(f) $\log_a(x.y) = \log_a x + \log_a y$; $x, y > 0$

(g) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$; $x, y > 0$

(h) $\log_a x^p = p\log_a x$; $x > 0$

(i) $\log_{a^q} x = \frac{1}{q}\log_a x$; $x > 0$

(j) $\log_a x = \frac{1}{\log_x a}$; $x > 0, x \neq 1$

(k) $\log_a x = \log_b x/\log_b a$; $x > 0$, $a, b > 0$, $b \neq 1$, $a \neq 1$

(l) $\log_a b.\log_b c.\log_c d = \log_a d$; $a, b, c, d > 0, \neq 1$

(m) $a^{\log_a x} = x$; $a > 0$, $a \neq 1$

(n) $a^{\log_b c} = c^{\log_b a}$; $a,b,c > 0$; $b \neq 1$

(o) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{cases}$

(p) $\log_a x = \log_a y \Rightarrow x = y$; $x, y > 0$; $a > 0$, $a \neq 1$

(q) $e^{\ln a^x} = a^x$

(r) $\log_{10} 2 = 0.3010$; $\log_{10} 3 = 0.4771$; $\ell n2 = 0.693$, $\ell n10 = 2.303$

(s) If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$

(t) If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$

(u) If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$

(v) If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$

TRIGONOMETRIC RATIOS & IDENTITIES

1. RELATION BETWEEN SYSTEM OF MEASUREMENT OF ANGLES :

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

1 Radian = $\frac{180}{\pi}$ degree $\approx$ 57°17'15" (approximately)

1 degree = $\frac{\pi}{180}$ radian $\approx$ 0.0175 radian

2. BASIC TRIGONOMETRIC IDENTITIES :

(a) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

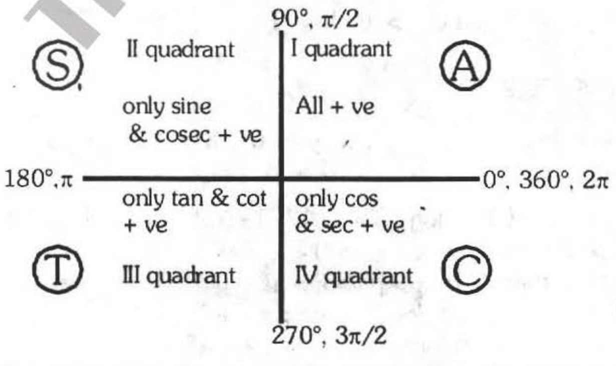
(b) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(c) If $\sec\theta + \tan\theta = k \Rightarrow \sec\theta - \tan\theta = \frac{1}{k} \Rightarrow 2\sec\theta = k + \frac{1}{k}$

(d) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(e) If $\text{cosec}\theta + \cot\theta = k \Rightarrow \text{cosec}\theta - \cot\theta = \frac{1}{k} \Rightarrow 2\,\text{cosec}\,\theta = k + \frac{1}{k}$

3. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



4. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b) $\sin(-\theta) = -\sin\theta$ $\quad$ $\cos(-\theta) = \cos\theta$

$\sin(90° - \theta) = \cos\theta$ $\quad$ $\cos(90° - \theta) = \sin\theta$

$\sin(90° + \theta) = \cos\theta$ $\quad$ $\cos(90° + \theta) = -\sin\theta$

$\sin(180° - \theta) = \sin\theta$ $\quad$ $\cos(180° - \theta) = -\cos\theta$

$\sin(180° + \theta) = -\sin\theta$ $\quad$ $\cos(180° + \theta) = -\cos\theta$

$\sin(270° - \theta) = -\cos\theta$ $\quad$ $\cos(270° - \theta) = -\sin\theta$

$\sin(270° + \theta) = -\cos\theta$ $\quad$ $\cos(270° + \theta) = \sin\theta$

Note :

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(ii) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$; $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in I$

5. IMPORTANT TRIGONOMETRIC FORMULAE :

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B \cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \dfrac{\cot B \cot A + 1}{\cot B - \cot A}$

(ix) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$.

(x) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$.

(xi) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

(xii) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

(xiii) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xiv) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(xv) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xvi) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(xvii) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(xviii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(xix) $1 + \cos 2\theta = 2\cos^2\theta$ or $\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$

(xx) $1 - \cos 2\theta = 2\sin^2\theta$ or $\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

(xxi) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

(xxii) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(xxiii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(xxiv) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(xxv) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(xxvi) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(xxvii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

(xxviii) $\sin(A+B+C)$

$= \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$

$= \Sigma \sin A \cos B \cos C - \Pi \sin A$

$= \cos A \cos B \cos C\,[\tan A + \tan B + \tan C - \tan A \tan B \tan C]$

(xxix) $\cos(A+B+C)$

$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

$= \Pi \cos A - \Sigma \sin A \sin B \cos C$

$= \cos A \cos B \cos C\,[1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$

(xxx) $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$$

(xxxi) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

(xxxii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

6. VALUES OF SOME T-RATIOS FOR ANGLES 18°, 36°, 15°, 22.5°, 67.5° etc.

(a) $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72° = \sin\frac{\pi}{10}$ $\cos 18 = \sqrt{\frac{\sqrt{5}+5}{8}}$

(b) $\cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54° = \cos\frac{\pi}{5}$

(c) $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \sin\frac{\pi}{12}$

(d) $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \cos\frac{\pi}{12}$

(e) $\tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot\frac{5\pi}{12}$

(f) $\tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot\frac{\pi}{12}$

(g) $\tan(22.5°) = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8} = \tan\frac{\pi}{8}$

(h) $\tan(67.5°) = \sqrt{2}+1 = \cot(22.5°)$

7. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(a) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(b) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $a, b > 0$

(c) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(d) Minimum value of $a^2\cos^2\theta + b^2\sec^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\cos\theta = b\sec\theta$ is true or not true $\{a, b > 0\}$

(e) Minimum value of $a^2\sin^2\theta + b^2\mathrm{cosec}^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\sin\theta = b\,\mathrm{cosec}\,\theta$ is true or not true $\{a, b > 0\}$

8. IMPORTANT RESULTS :

(a) $\sin\theta\sin(60°-\theta)\sin(60°+\theta) = \frac{1}{4}\sin 3\theta$

(b) $\cos\theta.\cos(60°-\theta)\cos(60°+\theta) = \frac{1}{4}\cos 3\theta$

(c) $\tan\theta\tan(60°-\theta)\tan(60°+\theta)=\tan 3\theta$

(d) $\cot\theta\cot(60°-\theta)\cot(60°+\theta)=\cot 3\theta$

(e) **(i)** $\sin^2\theta+\sin^2(60°+\theta)+\sin^2(60°-\theta)=\frac{3}{2}$

(ii) $\cos^2\theta+\cos^2(60°+\theta)+\cos^2(60°-\theta)=\frac{3}{2}$

(f) **(i)** If $\tan A+\tan B+\tan C=\tan A\tan B\tan C$,

then $A+B+C=n\pi, n\in I$

(ii) If $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$,

then $A+B+C=(2n+1)\frac{\pi}{2}, n\in I$

(g) $\cos\theta\cos 2\theta\cos 4\theta\ \ldots.\ \cos(2^{n-1}\theta)=\frac{\sin(2^n\theta)}{2^n\sin\theta}$

(h) $\cot A-\tan A=2\cot 2A$

9. CONDITIONAL IDENTITIES :

If $A+B+C=180°$, then :

(a) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(b) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

(c) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

(d) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(e) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(f) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

(g) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

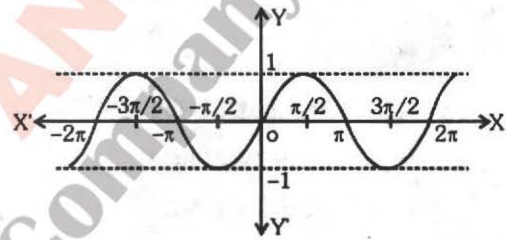
(h) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

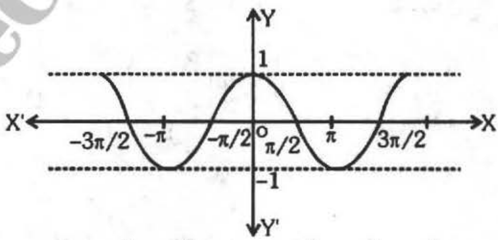
T-Ratio	Domain	Range	Period
sin x	R	[–1,1]	2π
cos x	R	[–1,1]	2π
tan x	$R-\{(2n+1)\pi/2\ ;\ n\in I\}$	R	π
cot x	$R-\{n\pi : n \in I\}$	R	π
sec x	$R-\{(2n+1)\pi/2 : n \in I\}$	$(-\infty,-1] \cup [1,\infty)$	2π
cosec x	$R-\{n\pi : n \in I\}$	$(-\infty,-1] \cup [1,\infty)$	2π

11. GRAPH OF TRIGONOMETRIC FUNCTIONS :

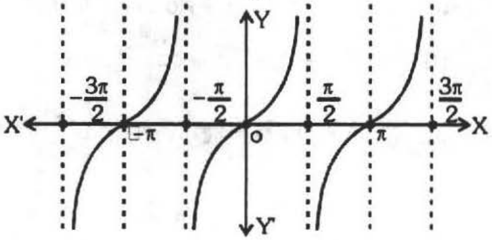
(a) y = sinx



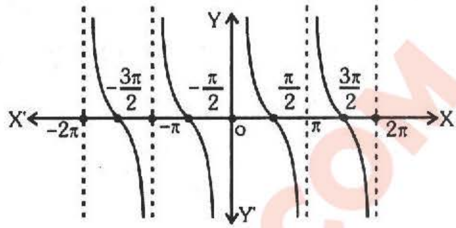
(b) y = cosx



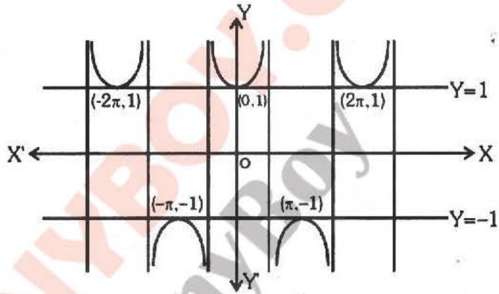
(c) y = tanx



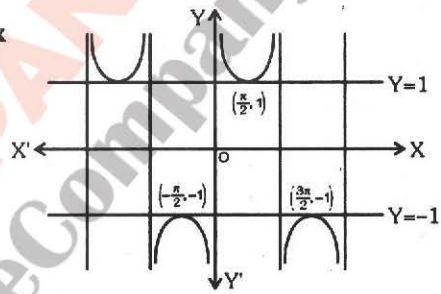
(d) y = cotx



(e) y = secx



(f) y = cosecx



12. IMPORTANT NOTE :

(a) The sum of interior angles of a polygon of n-sides

$= (n-2) \times 180° = (n-2)\pi$.

(b) Each interior angle of a regular polygon of n sides

$$= \frac{(n-2)}{n} \times 180° = \frac{(n-2)}{n}\pi.$$

(c) Sum of exterior angles of a polygon of any number of sides

$= 360° = 2\pi$.

TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equations is called a solution of the trigonometric equation.

(a) Principal solution :- The solution of the trigonometric equation lying in the interval $[0, 2\pi]$.

(b) General solution :- Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solutions of trigonometric equation.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\dfrac{\pi}{2}$ $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \dfrac{\pi}{2} = (4n+1)\dfrac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi, n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$,

then $\theta = n\pi \pm \alpha, n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n, n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$

$\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n, n \in I$

(l) If n is an odd integer then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \cos\frac{n\pi}{2} = 0$

(m) $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$, $\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$

4. GENERAL SOLUTION OF EQUATION $a\cos\theta + b\sin\theta = c$:

Consider, $a\sin\theta + b\cos\theta = c$ .... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has the solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxiliary argument ϕ, equation (i) reduces to

$$\sin(\theta + \phi) = \frac{c}{\sqrt{a^2+b^2}}$$

Now this equation can be solved easily.

5. GENERAL SOLUTION OF EQUATION OF FORM :

$a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2}x\cos^2 x + \dots\dots\dots + a_n\cos^n x = 0$

$a_0, a_1, \dots\dots\dots a_n$ are real numbers

Such an equation is solved by dividing equation by $\cos^n x$.

6. IMPORTANT TIPS :

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(e) Check that denominator is not zero at any stage while solving equations.

(f) **(i)** If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(ii) If $\cot\theta$ or $\text{cosec}\,\theta$ is involved in the equation, θ should not be integral multiple of π or 0.

(g) If two different trigonometric ratios such as $\tan\theta$ and $\sec\theta$ are involved then after solving we cannot apply the usual formulae for general solution because periodicity of the functions are not same.

(h) If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different value of θ, then solution does not exist.

QUADRATIC EQUATION

1. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by $\mathbf{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then ;

(i) $\alpha + \beta = -b/a$ **(ii)** $\alpha\beta = c/a$ **(iii)** $|\alpha - \beta| = \sqrt{D}/|a|$

(d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0.

2. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, c $\in$ R & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$).

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, c $\in$ Q & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

3. COMMON ROOTS OF TWO QUADRATIC EQUATIONS

(a) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's

Rule $\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$

Therefore, $\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(b) If both roots are same then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

4. ROOTS UNDER PARTICULAR CASES

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(b) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(c) If $a = c \Rightarrow$ roots are reciprocal to each other

(d) If $\left.\begin{matrix} a > 0 \ c < 0 \\ a < 0 \ c > 0 \end{matrix}\right\} \Rightarrow$ roots are of opposite signs

(e) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(f) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(g) If sign of a = sign of b $\neq$ sign of c $\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a $\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

5. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSION :

Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

$$y \in \left[\frac{-D}{4a}, \infty\right) \text{ if } a > 0 \quad \& \quad y \in \left(-\infty, \frac{-D}{4a}\right] \text{ if } a < 0.$$

6. LOCATION OF ROOTS :

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

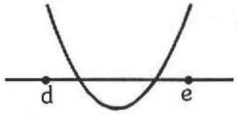
(a) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $\mathbf{D \geq 0;\ a.f(d) > 0\ \&\ (-b/2a) > d}$.

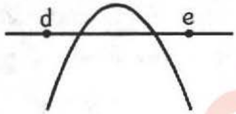
(b) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is $\mathbf{a.f(d) < 0}$.

(c) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d,e) i.e.. $d < x < e$ is $\mathbf{f(d).\ f(e) < 0}$

(d) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$D \geq 0$; $a.f(d) > 0$ & $af(e) > 0$; $d < (-b/2a) < e$





7. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

8. THEORY OF EQUATIONS :

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the roots of the equation ;

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+ a_{n-1}x + a_n = 0$ where a_0, $a_1, ...a_n$ are constants $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0},\ \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0},\ \sum \alpha_1\alpha_2\alpha_3$$

$$= -\frac{a_3}{a_0}, ..., \alpha_1\alpha_2\alpha_3......\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Note :

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+) ve}.

Ex. $x^3 - x^2 + x - 1 = 0$

(ii) Even degree polynomial whose last term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(iii) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

SEQUENCE & SERIES

1. ARITHMETIC PROGRESSION (AP) :

AP is sequence whose terms increase or decrease by a fixed number. This fixed number is called the **common difference**. If 'a' is the first term & 'd' is the common difference, then AP can be written as $a, a + d, a + 2d, \ldots\ldots\ldots\ldots a + (n - 1)d, \ldots\ldots\ldots$

(a) n^{th} term of this AP $\boxed{T_n = a + (n-1)d}$, where $d = T_n - T_{n-1}$

(b) The sum of the first n terms : $\boxed{S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + \ell]}$

where ℓ is the last term.

(c) Also n^{th} term $\boxed{T_n = S_n - S_{n-1}}$

Note :

(i) Sum of first n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. 2A

(ii) n^{th} term of an A.P. is of the form $An + B$ i.e. a linear expression in n, in such case the coefficient of n is the common difference of the A.P. i.e. A

(iii) Three numbers in AP can be taken as $a - d, a, a + d$; four numbers in AP can be taken as $a - 3d, a - d, a + d, a + 3d$ five numbers in AP are $a - 2d, a - d, a, a + d, a + 2d$ & six terms in AP are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) If for A.P. p^{th} term is q, q^{th} term is p, then r^{th} term is $= p + q - r$ & $(p + q)^{th}$ term is 0.

(v) If $a_1, a_2, a_3 \ldots\ldots$ and $b_1, b_2, b_3 \ldots\ldots$ are two A.P.s, then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \ldots\ldots$ are also in A.P.

(vi) (a) If each term of an A.P. is increased or decreased by the same number, then the resulting sequence is also an A.P. having the same common difference.

(b) If each term of an A.P. is multiplied or divided by the same non zero number (k), then the resulting sequence is also an A.P. whose common difference is kd & d/k respectively, where d is common difference of original A.P.

(vii) Any term of an AP (except the first & last) is equal to half the sum of terms which are equidistant from it.

$$T_r = \frac{T_{r-k} + T_{r+k}}{2},\ k < r$$

2. GEOMETRIC PROGRESSION (GP) :

GP is a sequence of numbers whose first term is non-zero & each of the succeeding terms is equal to the preceeding terms multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4 , ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term $\boxed{T_n = a\, r^{n-1}}$

(b) Sum of the first n terms $\boxed{S_n = \frac{a(r^n - 1)}{r-1},\ \text{if } r \neq 1}$

(c) Sum of infinite GP when $|r| < 1$ & $n \to \infty$, $r^n \to 0$

$$\boxed{S_\infty = \frac{a}{1-r};\ |r| < 1}$$

(d) Any 3 consecutive terms of a GP can be taken as a/r, a, ar; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(e) If a, b, c are in GP $\Rightarrow b^2 = ac \Rightarrow$ loga, logb, logc, are in A.P.

Note :

(i) In an G.P. product of k^{th} term from beginning and k^{th} term from the last is always constant which equal to product of first term and last term.

(ii) Three numbers in **G.P.** : **a/r, a, ar**
Five numbers in **G.P.** : $a/r^2, a/r, a, ar, ar^2$
Four numbers in **G.P.** : $a/r^3, a/r, ar, ar^3$
Six numbers in **G.P.** : $a/r^5, a/r^3, a/r, ar, ar^3, ar^5$

(iii) If each term of a **G.P.** be raised to the same power, then resulting series is also a **G.P.**

(iv) If each term of a G.P. be multiplied or divided by the same non-zero quantity, then the resulting sequence is also a G.P.

(v) If a_1, a_2, a_3 ..... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2 ..... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ...... will also form a G.P. common ratio will be $r_1 r_2$ and $\frac{r_1}{r_2}$ respectively.

(vi) In a positive G.P. every term (except first) is equal to square root of product of its two terms which are equidistant from it.
i.e. $T_r = \sqrt{T_{r-k}T_{r+k}}$, $k < r$

(vii) If $a_1, a_2, a_3.....a_n$ is a **G.P.** of **non zero, non negative terms,** then $\log a_1, \log a_2,.....\log a_n$ is an **A.P.** and **vice-versa.**

3. HARMONIC PROGRESSION (HP) :

A sequence is said to HP if the reciprocals of its terms are in AP.

If the sequence a_1, a_2, a_3, ..., a_n is an HP then $1/a_1$, $1/a_2$,........., $1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots\ldots \frac{1}{a+(n-1)d}$

Note : No term of any H.P. can be zero. If a, b, c are in

$$HP \Rightarrow b = \frac{2ac}{a+c} \text{ or } \frac{a}{c} = \frac{a-b}{b-c}$$

4. MEANS

(a) Arithmetic mean (AM) :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

n-arithmetic means between two numbers :

If a,b are any two given numbers & a, A_1, A_2, ..., A_n, b are in AP then $A_1, A_2, \ldots A_n$ are the n AM's between a & b, then

$A_1 = a + d,\ A_2 = a + 2d, \ldots\ldots, A_n = a + nd,$ where $d = \dfrac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

(b) Geometric mean (GM) :

If a, b, c are in GP, b is the GM between a & c, $b^2 = ac$, therefore $b = \sqrt{ac}$

n-geometric means between two numbers :

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in GP then $G_1, G_2, G_3, \ldots G_n$ are n GMs between a & b.

$G_1 = ar, G_2 = ar^2, \ldots\ldots G_n = ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b

(c) Harmonic mean (HM) :

If a, b, c are in HP, then b is HM between a & c, then $b = \dfrac{2ac}{a+c}$.

Important note :

(i) If A, G, H, are respectively AM, GM, HM between two positive number a & b then

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(a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \geq G \geq H$

(c) $A = G = H \Rightarrow a = b$

(ii) Let $a_1, a_2, \ldots\ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \dfrac{a_1 + a_2 + \ldots\ldots + a_n}{n}$

$$G = (a_1 a_2 \ldots\ldots\ldots a_n)^{1/n} \text{ and } H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots\ldots = a_n$

5. ARITHMETICO - GEOMETRIC SERIES :

Sum of First n terms of an Arithmetico-Geometric Series :

Let $S_n = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots\ldots + [a + (n-1)d]r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a + (n-1)d]\ r^n}{1-r}, \ r \neq 1$

Sum to infinity :

If $|r| < 1$ & $n \to \infty$ then $\lim\limits_{n\to\infty} r^n = 0 \Rightarrow S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

6. SIGMA NOTATIONS

Theorems :

(a) $\sum\limits_{r=1}^{n}(a_r \pm b_r) = \sum\limits_{r=1}^{n} a_r \pm \sum\limits_{r=1}^{n} b_r$ **(b)** $\sum\limits_{r=1}^{n} k a_r = k\sum\limits_{r=1}^{n} a_r$

(c) $\sum\limits_{r=1}^{n} k = nk$; where k is a constant.

7. RESULTS

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

PERMUTATION & COMBINATION

1. FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actually counting):

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of

(a) Simultaneous occurrence of both events in a definite order is m x n. This can be extended to any number of events (known as multiplication principle).

(b) Happening of exactly one of the events is m + n (known as addition principle).

2. FACTORIAL :

A Useful Notation : $n! = n(n-1)(n-2)\ ......\ 3.\ 2.\ 1$;

$n! = n.\ (n-1)!$ where $n \in W$

$0! = 1! = 1$

$(2n)! = 2^n\ .\ n!\ [1.\ 3.\ 5.\ 7.......(2n-1)]$

Note that factorials of negative integers are not defined.

3. PERMUTATION :

(a) ${}^nP_r$ denotes the number of permutations of n different things, taken r at a time ($n \in N, r \in W, n \geq r$)

$${}^nP_r = n(n-1)(n-2)\ ....\ (n-r+1) = \frac{n!}{(n-r)!}$$

(b) The number of permutations of n things taken all at a time when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n-(p+q+r)$ are all different is : $\frac{n!}{p!\ q!\ r!}$.

(c) The number of permutation of n different objects taken r at a time, when a particular object is always to be included is $r!\ .\ {}^{n-1}C_{r-1}$

(d) The number of permutation of n different objects taken r at a time, when repetition be allowed any number of times is $n \times n \times n$ .... r times $= n^r$.

(e) (i) The number of circular permutations of n different things taken all at a time is ; $(n-1)! = \frac{^nP_n}{n}$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

(ii) The number of circular permutation of n different things taking r at a time distinguishing clockwise & anticlockwise arrangement is $\frac{^nP_r}{r}$

4. COMBINATION :

(a) ^nC_r denotes the number of combinations of n different things taken r at a time, and $^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}$ where $r \le n$; $n \in N$ and $r \in W$. ^nC_r is also denoted by $\binom{n}{r}$ or A_r^n or $C(n, r)$.

(b) The number of combination of n different things taking r at a time.

(i) when p particular things are always to be included $= {}^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded $= {}^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$

(c) Given n different objects , the number of ways of selecting atleast one of them is, $^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots.. + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(d) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +......things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1)\ldots\ldots -1$.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is $(p + 1)(q + 1)(r + 1)2^n - 1$

5. DIVISORS :

Let $N = p^a. q^b. r^c$ ....where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is $= (a + 1)(b + 1)(c + 1)\ldots\ldots$

(b) The sum of these divisors is $= (p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots$

(c) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}(a+1)(b+1)(c+1)\ldots\ldots$ if N is not a perfect square

$\frac{1}{2}\left[(a+1)(b+1)(c+1)\ldots\ldots+1\right]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

6. DIVISION AND DISTRIBUTION :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups containing m & n things respectively is : $\frac{(m+n)!}{m!\ n!}$ $(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of subdivision is $\frac{(2n)!}{n!\ n!\ 2!}$; for in any one way it is possible to inter change the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways = $\frac{(2n)!}{n!\ n!\ (2!)} \times 2!$.

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is $\frac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups = $\frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each ,m groups containing q objects each is equal to $\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ \ell!m!}$

Here $\ell p + mq = n$

(d) Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them = p^n

(e) Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ; ${}^{n+p-1}C_n$.

7. DEARRANGEMENT :

Number of ways in which n letters can be placed in n directed envelopes so that no letter goes into its own envelope is

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\ldots+(-1)^n\frac{1}{n!}\right]$$

8. IMPORTANT RESULT :

(a) Number of rectangle of any size in a square of size $n \times n$ is

$\sum_{r=1}^{n} r^3$ & number of square of any size is $\sum_{r=1}^{n} r^2$.

(b) Number of rectangle of any size in a rectangle of size $n \times p$

$(n < p)$ is $\frac{np}{4}(n+1)(p+1)$ & number of squares of any size is

$\sum_{r=1}^{n}(n+1-r)(p+1-r)$

(c) If there are n points in a plane of which m(<n) are collinear :

(i) Total number of lines obtained by joining these points is ${}^nC_2 - {}^mC_2 + 1$

(ii) Total number of different triangle ${}^nC_3 - {}^mC_3$

(d) Maximum number of point of intersection of n circles is ${}^nP_2$ & n lines is ${}^nC_2$.

BINOMIAL THEOREM

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_ny^n$

$= \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$, where $n \in N$.

1. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE :

(a) General term: The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r} . y^r$

(b) Middle term :

The middle term (s) is the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2} . x^{n/2} . y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(c) Term independent of x :

Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

2. If $(\sqrt{A} + B)^n = I + f$, where I & n are positive integers & $0 \le f < 1$, then

(a) $(I + f).\, f = K^n$ if n is odd & $A - B^2 = K > 0$

(b) $(I + f)(1 - f) = k^n$ if n is even & $\sqrt{A} - B < 1$

3. SOME RESULTS ON BINOMIAL COEFFICIENTS :

(a) ${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $x + y = n$

(b) ${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} ...... + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + ... = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(h) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + ... + C_{n-r}C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

Remember : $(2n)! = 2^n . n! [1.3.5 ...... (2n-1)]$

4. Greatest coefficient & greatest term in expansion of $(x + a)^n$:

(a) If n is even greatest coefficient is ${}^{n}C_{n/2}$

If n is odd greatest coefficient is ${}^{n}C_{\frac{n-1}{2}}$ or ${}^{n}C_{\frac{n+1}{2}}$

(b) For greatest term :

$$\text{Greatest term} = \begin{cases} T_p \ \& \ T_{P+1} & \text{if } \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{ is an integer} \\ T_{q+1} & \text{if } \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{ is non integer and} \in (q, q+1), q \in I \end{cases}$$

5. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\infty$ provided $|x| < 1$.

Note :

(i) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots \infty$

(ii) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots \infty$

(iii) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots \infty$

(iv) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots \infty$

6. EXPONENTIAL SERIES :

(a) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\infty$; where x may be any real or complex number & $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$

(b) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + \ldots\ldots\infty$, where $a > 0$.

7. LOGARITHMIC SERIES :

(a) $\ell n(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty$, where $-1 < x \le 1$

(b) $\ell n(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\infty$, where $-1 \le x < 1$

(c) $\ell n\frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\infty\right) |x| < 1$

COMPLEX NUMBER

1. DEFINITION :

Complex numbers are defined as expressions of the form a + ib where a , b $\in$ R & $i = \sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

Purely real	Purely imaginary	Imaginary
if b = 0	if a = 0	if b $\neq$ 0

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(iv) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

2. CONJUGATE COMPLEX :

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\mathrm{Re}(z)$

(ii) $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$ which is real

(iv) If z is purely real then $z - \bar{z} = 0$

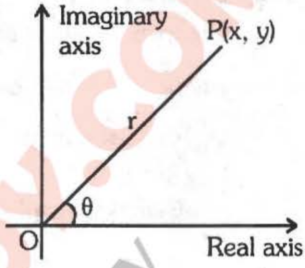
(v) If z is purely imaginary then $z + \bar{z} = 0$

3. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number $z = x + iy$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x , y).

Length OP is called **modulus** of the complex number denoted by $|z|$ & θ is called the **argument or amplitude**.



e.g. $|z| = \sqrt{x^2 + y^2}$ & $\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x–axis)

Geometrically $|z|$ represents the distance of point P from origin. $(|z| \geq 0)$

(b) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(c) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

4. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $\overline{(\bar{z})} = z$ **(b)** $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(c) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ **(d)** $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(e) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$ **(f)** If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

5. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z| \geq 0$ **(b)** $|z| \geq \text{Re}(z)$ **(c)** $|z| \geq \text{Im}(z)$

(d) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ **(e)** $z\bar{z} = |z|^2$ **(f)** $|z_1 z_2| = |z_1|.|z_2|$

(g) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|},\ z_2 \neq 0$ **(h)** $|z^n| = |z|^n$

(i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$
or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(j) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(k) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(l) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(m) If $\left|z + \dfrac{1}{z}\right| = 0$ $(a > 0)$, then $\max |z| = \dfrac{a + \sqrt{a^2 + 4}}{2}$

& $\min |z| = \dfrac{1}{2}\left(\sqrt{a^2 + 4} - a\right)$

6. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi\ ;\ k \in I$

(b) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi\ ;\ k \in I$

(c) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(d) $\log(z) = \log(re^{i\theta}) = \log r + i\theta = \log|z| + i\,\text{amp}(z)$

7. DE'MOIVER'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of roots of a complex quantity should be determined using theory of equation.

8. CUBE ROOT OF UNITY :

(a) The cube roots of unity are 1 , $\omega = \dfrac{-1+i\sqrt{3}}{2} = e^{i2\pi/3}$

& $\omega^2 = \dfrac{-1-i\sqrt{3}}{2} = e^{i4\pi/3}$

(b) $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$, in general

$$1+\omega^r+\omega^{2r} = \begin{cases} 0 & r \text{ is not integral multiple of } 3 \\ 3 & r \text{ is multiple of } 3 \end{cases}$$

(c) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

9. SQUARE ROOT OF COMPLEX NUMBER :

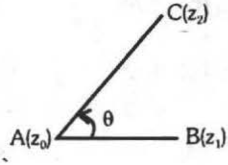
$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right\} \text{ for } b > 0$$

$$\& \quad \pm\left\{\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right\} \text{ for } b < 0 \text{ where } |z| = \sqrt{a^2+b^2}.$$

10. ROTATION :

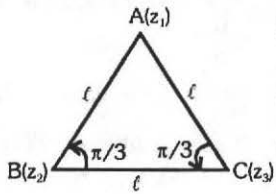
$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|}e^{i\theta}$$

Take θ in anticlockwise direction



11. RESULT RELATED WITH TRIANGLE :

(a) Equilateral triangle :

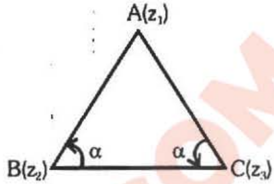


$$\frac{z_1 - z_2}{\ell} = \frac{z_3 - z_2}{\ell}e^{i\pi/3} \quad \text{......(i)}$$

Also $\quad \dfrac{z_2 - z_3}{\ell} = \dfrac{z_1 - z_3}{\ell}.e^{i\pi/3} \quad$ ......(ii) from (i) & (ii)

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

or $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$



(b) Isosceles triangle :

$$4\cos^2\alpha\,(z_1 - z_2)(z_3 - z_1) = (z_3 - z_2)^2$$

(c) Area of triangle ΔABC given by modulus of $\frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$

12. EQUATION OF LINE THROUGH POINTS z_1 & z_2 :

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_2 - z_1) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

$$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2)i + \bar{z}(z_2 - z_1)i + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

Let $(z_2 - z_1)i = a$, then equation of line is $\boxed{\bar{a}z + a\bar{z} + b = 0}$ where $a \in$ C & $b \in$ R.

Note :

(i) Complex slope of line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}}$

(ii) Two lines with slope μ_1 & μ_2 are parallel or perpendicular if $\mu_1 = \mu_2$ or $\mu_1 + \mu_2 = 0$

(iii) Length of perpendicular from point $A(\alpha)$ to line $\bar{a}z + a\bar{z} + b = 0$ is $\frac{|\bar{a}\alpha + a\bar{\alpha} + b|}{2|a|}$.

13. EQUATION OF CIRCLE :

(a) Circle whose centre is z_0 & radii = r

$$|z - z_0| = r$$

(b) General equation of circle

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$$

centre '$-a$' & radii $= \sqrt{|a|^2 - b}$

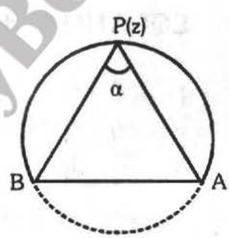
(c) Diameter form $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

or $\quad \arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\dfrac{\pi}{2}$

(d) Equation $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ represent a circle if $k \neq 1$ and a straight line if $k = 1$.

(e) Equation $|z - z_1|^2 + |z - z_2|^2 = k$

represent circle if $k \geq \dfrac{1}{2}|z_1 - z_2|^2$



(f) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha \qquad 0 < \alpha < \pi,\ \alpha \neq \dfrac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ & $B(z_2)$

14. STANDARD LOCI :

(a) $|z - z_1| + |z - z_2| = 2k$ (a constant) represent

(i) if $2k > |z_1 - z_2| \Rightarrow$ An ellipse

(ii) If $2k = |z_1 - z_2| \Rightarrow$ A line segment

(iii) If $2k < |z_1 - z_2| \Rightarrow$ No solution

(b) Equation $||z - z_1| - |z - z_2|| = 2k$ (a constant) represent

(i) If $2k < |z_1 - z_2| \Rightarrow$ A hyperbola

(ii) If $2k = |z_1 - z_2| \Rightarrow$ A line ray

(iii) $2k > |z_1 - z_2| \Rightarrow$ No solution

DETERMINANT

1. MINORS :

The minor of a given element of determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have " 9 minors".

2. COFACTORS :

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element : $C_{ij} = (-1)^{i+j} . M_{ij}$;

Important Note :

Consider $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let A_1 be cofactor of a_1, B_2 be cofactor of b_2 and so on, then,

(i) $a_1A_1 + b_1B_1 + c_1C_1 = a_1A_1 + a_2A_2 + a_3A_3 = \ldots\ldots\ldots\ldots = \Delta$

(ii) $a_2A_1 + b_2B_1 + c_2C_1 = b_1A_1 + b_2A_2 + b_3A_3 = \ldots\ldots\ldots\ldots = 0$

3. PROPERTIES OF DETERMINANTS:

(a) The value of a determinants remains unaltered, if the rows & columns are interchanged.

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

(c) If a determinant has any two rows (or columns) identical or in same proportion, then its value is zero.

(d) If all the elements of any row (or columns) be multiplied by the same number, then the determinant is multiplied by that number.

(e) $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(f) The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column) e.g.

$$\text{Let} \quad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}. \text{ Then } D' = D.$$

Note : While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

(g) If the elements of a determinant Δ are rational function of x and two rows (or columns) become identical when x = a, then x – a is a factor of Δ.

Again, if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of Δ.

(h) If $D(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$, where f_r, g_r, h_r; r = 1, 2, 3 are three differential function.

$$\text{then } \frac{d}{dx}D(x) = \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g'_1 & g'_2 & g'_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h'_1 & h'_2 & h'_3 \end{vmatrix}$$