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Archives

Appendix Solutions to Concept Application
Exercises

A.1
Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. A combined national-level engineering entrance examination has finally been proposed by the Ministry of Human Resource Development, Government of India. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulate. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject.

This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter:

Step 1: Go through the entire opening discussion about the fundamentals and concepts.
Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

Ghanshyam Tewani
CHAPTER 1

Number System, Inequalities and Theory of Equations

- Constant and Variables
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CONSTANT AND VARIABLES

In mathematics, a variable is a value that may change within the scope of a given problem or set of operations. In contrast, a constant is a value that remains unchanged, though often unknown or undetermined.

Dependent and Independent Variables

Variables are further distinguished as being either a dependent variable or an independent variable. Independent variables are regarded as inputs to a system and may take on different values freely.

Dependent variables are those values that change as a consequence to changes in other values in the system.

When one value is completely determined by another, or of several others, then it is called a function of the other value or values. In this case, the value of the function is a dependent variable and the other values are independent variables. The notation \( f(x) \) is used for the value of the function \( f \) with \( x \) representing the independent variable.

For example, \( y = f(x) = 3x^2 \), here we can take \( x \) as any real value, hence \( x \) is independent variable. But value of \( y \) depends on value of \( x \), hence \( y \) is dependent variable.

WHAT IS FUNCTION

To provide the classical understanding of functions, think of a function as a kind of machine. You feed the machine raw materials, and the machine changes the raw materials into a finished product based on a specific set of instructions. The kinds of functions we consider here, for the most part, take in a real number, change it in a formulaic way, and give out a real number (possibly the same as the one it took in). Think of this as an input-output machine; you give the function an input, and it gives you an output.

![Fig. 1.1](image)

For example, the squaring function takes the input 4 and gives the output value 16. The same squaring function takes the input 1 and gives the output value 1.

A function is always defined as "of a variable" which tells us what to replace in the formula for the function.

For example, \( f(x) = 3x + 2 \) tells us:

- The function \( f \) is a function of \( x \).
- To evaluate the function at a certain number, replace the \( x \) with that number.
- Replacing \( x \) with that number in the right side of the function will produce the function's output for that certain input.
- In English, the above definition of \( f \) is interpreted, "Given a number, \( f \) will return two more than the triple of that number."

Thus, if we want to know the value (or output) of the function at 3:

\[
f(x) = 3x + 2
f(3) = 3(3) + 2 = 11
\]

Thus, the value of \( f \) at 3 is 11.

Note that \( f(3) \) means the value of the dependent variable when "\( x \)" takes on the value of 3. So we see that the number 11 is the output of the function when we give the number 3 as the input. We refer to the input as the argument of the function (or the independent variable), and to the output as the value of the function at the given argument (or the dependent variable). A good way to think of it is the dependent variable \( f(x) \) depends on the value of the independent variable \( x \).

The formal definition of a function states that a function is actually a rule that associates elements of one set called the domain of the function with the elements of another set called the range of the function. For each value, we select from the domain of the function, there exists exactly one corresponding element in the range of the function. The definition of the function tells us which element in the range corresponds to the element we picked from the domain. Classically, the element picked from the domain is pictured as something that is fed into the function and the corresponding element in the range is pictured as the output. Since we "pick" the element in the domain whose corresponding element in the range we want to find, we have control over what element we pick and hence this element is also known as the "independent variable". The element mapped in the range is beyond our control and is "mapped to" by the function. This element is hence also known as the "dependent variable", for it depends on which independent variable we pick. Since the elementary idea of functions is better understood from the classical viewpoint, we shall use it hereafter. However, it is still important to remember the correct definition of functions at all times.

To make it simple, for the function \( f(x) \), all of the possible \( x \) values constitute the domain, and all of the values \( f(x) \) (\( y \) on the \( x-y \) plane) constitute the range.

**Example 1.1** A function is defined as \( f(x) = x^2 - 3x \).

(i) Find the value of \( f(2) \).

(ii) Find the value of \( x \) for which \( f(x) = 4 \).

**Sol.**

(i) \( f(2) = (2)^2 - 3(2) = -2 \)
(ii) \( f(x) = 4 \)

\[
\Rightarrow x^2 - 3x = 4 \quad \Rightarrow \quad x^2 - 3x - 4 = 0
\]

\[
\Rightarrow (x - 4)(x + 1) = 0 \quad \Rightarrow \quad x = 4 \text{ or } -1
\]

This means \( f(4) = 4 \) and \( f(-1) = 4 \).
**Example 1.2** If \( f \) is linear function and \( f(2) = 4, f(-1) = 3 \), then find \( f(x) \).

**Solution.** Let \( f(x) = ax + b \)

Given \( f(2) = 4 \) \( \Rightarrow 2a + b = 4 \) \( \quad (1) \)

Also \( f(-1) = 3 \) \( \Rightarrow -a + b = 3 \) \( \quad (2) \)

Solving (1) and (2) we get \( a = \frac{1}{3} \) and \( b = \frac{10}{3} \)

Hence, \( f(x) = \frac{x+10}{3} \)

**Example 1.3** A function is defined as \( f(x) = \frac{x^2+1}{3x-2} \). Can \( f(x) \) take a value 1 for any real \( x \)?

Also find the value/values of \( x \) for which \( f(x) \) takes the value 2.

**Solution.** Here \( f(x) = \frac{x^2 + 1}{3x-2} = 1 \)

\( \Rightarrow x^2 + 1 = 3x - 2 \)

\( \Rightarrow x^2 - 3x + 3 = 0 \)

Now this equation has no real roots as \( D < 0 \).

Hence, value of \( f(x) \) cannot be 1 for any real \( x \).

For \( f(x) = 2 \) we have \( \frac{x^2 + 1}{3x-2} = 2 \)

or \( x^2 + 1 = 6x - 4 \) or \( x^2 - 6x + 5 = 0 \)

or \( (x-1)(x-5) = 0 \)

or \( x = 1, 5 \)

**Example 1.4** Find the values of \( x \) for which the following functions are defined. Also find all possible values which functions take.

(i) \( f(x) = \frac{1}{x+1} \)

(ii) \( f(x) = \frac{x-2}{x-3} \)

(iii) \( f(x) = \frac{2x}{x-1} \)

**Solution.**

(i) \( f(x) = \frac{1}{x+1} \) is defined for all real values of \( x \) except when \( x + 1 = 0 \)

Hence, \( f(x) \) is defined for \( x \in R - [-1] \)

Let \( y = \frac{1}{x+1} \)

Here we cannot find any real \( x \) for which \( y = \frac{1}{x+1} = 0 \)

For \( y \) other than ‘0’, there exists a real number \( x \).

Hence, \( \frac{1}{x+1} \in R - [0] \).

(ii) \( f(x) = \frac{x-2}{x-3} \) is defined for all real values of \( x \) except when \( x - 3 = 0 \)

Hence, \( f(x) \) is defined for \( x \in R - [3] \)

Let \( y = \frac{x-2}{x-3} \)

(iii) \( f(x) = \frac{2x}{x-1} \)

**Example 1.5** If \( f(x) = \begin{cases} x^2, & x < 0 \\ 3x-2, & 0 \leq x \leq 2 \end{cases} \), then find the value of \( f(-1) + f(1) + f(3) \).

Also find the value/values of \( x \) for which \( f(x) = 2 \).

**Solution.** Here function is differently defined for three different intervals mentioned.

For \( x = -1 \), consider \( f(x) = x^2 \)

\( \Rightarrow f(-1) = (-1)^2 = 1 \)

For \( x = 1 \), consider \( f(x) = 3x - 2 \)

\( \Rightarrow f(1) = 3(1) - 2 = 1 \)

For \( x = 3 \), consider \( f(x) = x^2 + 1 \)

\( \Rightarrow f(3) = 3^2 + 1 = 10 \)

Hence, \( f(-1) + f(1) + f(3) = 1 + 1 + 10 = 12 \)

Also when \( f(x) = 2 \),

For \( x^2 = 2 \), \( x = \pm \sqrt{2} \), which is not possible as \( x < 0 \).

For \( 3x-2 = 2 \), \( x = 4/3 \), which is possible as \( 0 \leq x \leq 2 \).

For \( x^2 + 1 = 2 \), \( x = \pm 1 \), which is not possible as \( x > 2 \).

Hence, for \( f(x) = 2 \), we have \( x = 4/3 \).

**INTERVALS**

The set of numbers between any two real numbers is called interval. The following are the types of interval.
1.4 Algebra

**Closed Interval**

\[ x \in [a, b] = \{ x : a \leq x \leq b \} \]

\[ \begin{array}{c}
\bullet \\
\text{Fig. 1.2}
\end{array} \]

**Open Interval**

\[ x \in (a, b) \text{ or } [a, b) = \{ x : a < x < b \} \]

\[ \begin{array}{c}
\bullet \\
\text{Fig. 1.3}
\end{array} \]

**Semi–Open or Semi Closed Interval**

\[ x \in [a, b) \text{ or } (a, b] = \{ x : a \leq x < b \} \]

\[ \begin{array}{c}
\bullet \\
\text{Fig. 1.4}
\end{array} \]

\[ x \in ]a, b[ \text{ or } (a, b] = \{ x : a < x \leq b \} \]

\[ \begin{array}{c}
\bullet \\
\text{Fig. 1.5}
\end{array} \]

**Note:**
- A set of all real numbers can be expressed as \((-\infty, \infty)\).
- \( x \in (-\infty, a) \cup (b, \infty) \Rightarrow x \in R - [a, b] \)
- \( x \in (-\infty, a] \cup [b, \infty) \Rightarrow x \in R - (a, b) \)

**INEQUALITIES**

**Some Important Facts about Inequalities**

The following are some very useful points to remember:

(i) \( a \leq b \) either \( a < b \) or \( a = b \)

(ii) \( a < b \) and \( b < c \Rightarrow a < c \) (transitive property)

(iii) \( a < b \Rightarrow -a > -b \), i.e., inequality sign reverses if both sides are multiplied by a negative number.

(iv) \( a < b \) and \( c < d \Rightarrow a + c < b + d \) and \( a - d < b - c \).

(v) If both sides of inequality are multiplied (or divided) by a positive number, inequality does not change. When both of its sides are multiplied (or divided) by a negative number, inequality gets reversed.

\[ \begin{align*}
& a < b \Rightarrow ka < kb \text{ if } k > 0 \text{ and } ka > kb \text{ if } k < 0 \\
& 0 < a < b \Rightarrow a' < b' \text{ if } r > 0 \text{ and } a' > b' \text{ if } r < 0
\end{align*} \]

(vii) \( a + \frac{1}{a} \geq 2 \) for \( a > 0 \) and equality holds for \( a = 1 \)

(viii) \( a + \frac{1}{a} \leq -2 \) for \( a < 0 \) and equality holds for \( a = -1 \)

(ix) Squaring an inequality:

If \( a < b \), then \( a^2 < b^2 \) does not follow always:

Consider the following illustrations:

- \( 2 < 3 \Rightarrow 4 < 9 \), but \( -4 < 3 \Rightarrow 16 > 9 \)
- Also if \( x > 2 \Rightarrow x^2 > 4 \), but if \( x < 2 \Rightarrow x^2 \geq 0 \)
- If \( 2 < x < 4 \Rightarrow 4 < x^2 < 16 \)
- If \( -2 < x < 4 \Rightarrow 0 \leq x^2 < 16 \)

If \( -5 < x < 4 \Rightarrow 0 \leq x^2 < 25 \)

In fact \( a < b \Rightarrow a^2 < b^2 \) follows only when absolute value of \( a \) is less than the absolute value of \( b \) or distance of \( a \) from zero is less than the distance of \( b \) from zero on real number line.

(x) Law of reciprocal:

If both sides of inequality have same sign, while taking its reciprocal the sign of inequality gets reversed. i.e., \( a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b} \) and \( a < b < 0 \Rightarrow \frac{1}{a} > \frac{1}{b} \)

But if both sides of inequality have opposite sign, then while taking reciprocal sign of inequality does not change, i.e.

\[ a < 0 < b \Rightarrow \frac{1}{b} < \frac{1}{a} \text{ if } a \text{ and } b \text{ have same sign} \]

\[ \frac{1}{a} > \frac{1}{b} \text{ if } a \text{ and } b \text{ have opposite signs} \]

**Example 1.6**

Find the values of \( x^2 \) for the given values of \( x \).

(i) \( x < 2 \)  
(ii) \( x > -1 \)  
(iii) \( x \geq 2 \)  
(iv) \( x < -1 \)

**Sol.**

(i) When \( x < 2 \) we have \( x \in (-\infty, 0) \cup [0, 2) \)

- \( x \in [0, 2) \Rightarrow x^2 \in [0, 4) \)
- \( x \in (-\infty, 0) \Rightarrow x^2 \in (0, 4) \)

(ii) When \( x > -1 \) we have \( x \in (-1, 0) \cup [0, \infty) \)

- \( x \in [0, \infty) \Rightarrow x^2 \in [0, \infty) \)
- \( x \in (-1, 0) \Rightarrow x^2 \in (0, 1) \)

(iii) Here \( x \in [2, \infty) \)

\[ x^2 \in [4, \infty) \]

(iv) Here \( x \in (-\infty, -1) \)

\[ x^2 \in (1, \infty) \]

**Example 1.7**

Find the values of \( \frac{1}{x} \) for the given values of \( x \).

(i) \( x > 3 \)  
(ii) \( x < -2 \)  
(iii) \( x \in (-1, 3) \) \( - \{ 0 \} \)

**Sol.**

(i) We have \( 3 < x < \infty \)

\[ \Rightarrow \frac{1}{x} > \frac{1}{3} \text{ (} \to \infty \text{ means tends to infinity)} \]

\[ \Rightarrow \frac{1}{x} < -\frac{1}{3} \]

(ii) We have \( -\infty < x < -2 \)
\[ \Rightarrow \frac{1}{x} > \frac{1}{-2} \]
\[ \Rightarrow \frac{1}{x} > \frac{1}{-2} \]
\[ \Rightarrow 0 > \frac{1}{2} \]

(iii) \( x \in (-1, 3) \)
\[ \Rightarrow x \in (-1, 0) \cup (0, 3) \]

For \( x \in (-1, 0) \)
\[ \frac{1}{x} > \frac{1}{-1} \]
\[ \Rightarrow -1 > \frac{1}{x} \]

(Here \( \Rightarrow 0^+ \) means value of \( x \) approaches 0 from its left hand side or negative side)
\[ \Rightarrow \frac{1}{x} > -\infty \]
\[ \Rightarrow -\infty < \frac{1}{x} < 0 \]

For \( x \in (0, 3) \)
\[ \frac{1}{x} > \frac{1}{3} \]
\[ \Rightarrow 0 > \frac{1}{x} \]

(Here \( \Rightarrow 0^- \) means value of \( x \) approaches 0 from its right hand side or positive side)
\[ \Rightarrow \infty > \frac{1}{x} > \frac{1}{3} \]

\[ \Rightarrow \frac{1}{3} < \frac{1}{x} < \infty \]

From (1) and (2), \( \frac{1}{x} \in (-\infty, -1) \cup (\frac{1}{3}, \infty) \)

**Example 1.8** Find all possible values of the following expressions:

(i) \( \frac{1}{x^2 + 2} \)  
(ii) \( \frac{1}{x^2 - 2x + 3} \)  
(iii) \( \frac{1}{x^2 - x - 1} \)

**Sol.**

(i) We know that \( x^2 \geq 0 \forall x \in \mathbb{R} \). 
\[ \Rightarrow x^2 + 2 \geq 2, \forall x \in \mathbb{R} \]
\[ \text{or} \quad 2 \leq (x^2 + 2) < \infty \]
\[ \Rightarrow \frac{1}{2} \leq \frac{1}{x^2 + 2} > 0 \]
\[ \Rightarrow 0 < \frac{1}{x^2 + 2} \leq \frac{1}{2} \]

(ii) \( \frac{1}{x^2 - 2x + 3} = \frac{1}{(x-1)^2} + 2 \)

Now we know that \( (x-1)^2 \geq 0 \forall x \in \mathbb{R} \).

**Example 1.9** Find all possible values of the following expressions:

(i) \( \sqrt{x^2 - 4} \)  
(ii) \( \sqrt{9-x^2} \)  
(iii) \( \sqrt{x^2 - 2x + 10} \)

**Sol.**

(i) \( \sqrt{x^2 - 4} \)

Least value of square root is 0 when \( x^2 = 4 \) or \( x = \pm 2 \). Also \( x^2 - 4 \geq 0 \)

Hence, \( \sqrt{x^2 - 4} \in [0, \infty) \).

(ii) \( \sqrt{9-x^2} \)

Least value of square root is 0 when \( 9 - x^2 = 0 \) or \( x = \pm 3 \).

Also, the greatest value of \( 9 - x^2 = 9 \) when \( x = 0 \).

Hence, we have \( 0 \leq 9 - x^2 \leq 9 \Rightarrow \sqrt{9-x^2} \in [0, 3] \).

(iii) \( \sqrt{x^2 - 2x + 10} = \sqrt{(x-1)^2 + 9} \)

Here, the least value of \( \sqrt{(x-1)^2 + 9} \) is 3 when \( x - 1 = 0 \).

Also \( (x-1)^2 + 9 \geq 9 \Rightarrow \sqrt{(x-1)^2 + 9} \geq 3 \)

Hence, \( \sqrt{x^2 - 2x + 10} \in [3, \infty) \).
GENERALIZED METHOD OF INTERVALS FOR SOLVING INEQUALITIES

Let \( F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2} \cdots (x-a_n)^{k_n} \) where \( k_1, k_2, \ldots, k_n \in \mathbb{Z} \) and \( a_1, a_2, \ldots, a_n \) are fixed real numbers satisfying the condition

\[ a_1 < a_2 < a_3 < \cdots < a_{n-1} < a_n. \]

For solving \( F(x) > 0 \) or \( F(x) < 0 \), consider the following algorithm:

- We mark the numbers \( a_1, a_2, \ldots, a_n \) on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e., on the right of \( a_n \).
- Then we put the plus sign in the interval on the left of \( a_i \) if \( k_i \) is an even number and the minus sign if \( k_i \) is an odd number. In the next interval, we put a sign according to the following rule:
  * When passing through the point \( a_{n-1} \), the polynomial \( F(x) \) changes sign if \( k_{n-1} \) is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality \( F(x) > 0 \) is the union of all intervals in which we have put the plus sign and the solution of the inequality \( F(x) < 0 \) is the union of all intervals in which we have put the minus sign.

Frequently used inequalities

(i) \((x-a)(x-b) < 0 \Rightarrow x \in (a, b)\) where \( a < b \)

(ii) \((x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)\) where \( a < b \)

(iii) \(x^2 < a^2 \Rightarrow x \in [-a, a]\)

(iv) \(x^2 < a^2 \Rightarrow x \in (-\infty, -a) \cup [a, \infty)\)

(v) If \( ax^2 + bx + c < 0, (a > 0) \Rightarrow x \in (\alpha, \beta), \) where \( \alpha, \beta (\alpha < \beta) \) are the roots of the equation \( ax^2 + bx + c = 0 \).

(vi) If \( ax^2 + bx + c > 0, (a > 0) \Rightarrow x \in (-\infty, -\alpha) \cup (\beta, \infty), \) where \( \alpha, \beta (\alpha < \beta) \) are the roots of the equation \( ax^2 + bx + c = 0 \).

**Example 1.10** Solve \( x^2 - x - 2 > 0 \).

**Sol.** \( x^2 - x - 2 > 0 \)

\[ \Rightarrow (x - 2)(x + 1) > 0 \]

Now \( x^2 - x - 2 = 0 \Rightarrow x = 1, -2 \).

Now on number line (x-axis) mark \( x = -1 \) and \( x = 2 \).

Now when \( x > 2, x + 1 > 0 \) and \( x - 2 > 0 \)

\[ \Rightarrow (x + 1)(x - 2) > 0 \]

when \(-1 < x < 2, x + 1 > 0 \) but \( x - 2 < 0 \)

\[ \Rightarrow (x + 1)(x - 2) < 0 \]

when \( x < -1, x + 1 < 0 \) and \( x - 2 < 0 \)

\[ \Rightarrow (x + 1)(x - 2) > 0 \]

Hence, sign scheme of \( x^2 - x - 2 \) is

\[ \frac{1}{2} \quad -1 \quad -2 \quad + \quad + \]

**Fig. 1.6**

From the figure, \( x^2 - x - 2 > 0, x \in (-\infty, -1) \cup (2, \infty). \)

**Example 1.11** Solve \( x^2 - x - 1 < 0. \)

**Sol.** Let's first factorize \( x^2 - x - 1. \)

For that let \( x^2 - x - 1 = 0 \)

\[ \Rightarrow x = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2} \]

Now on number line (x-axis) mark \( x = \frac{1 \pm \sqrt{5}}{2} \)

**Fig. 1.7**

From the sign scheme of \( x^2 - x - 1 \) which shown in the given figure.

\( x^2 - x - 1 < 0 \Rightarrow x \in \left( \frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right) \)

**Example 1.12** Solve \( (x - 1)(x - 2)(1 - 2x) > 0. \)

**Sol.** We have \((x - 1)(x - 2)(1 - 2x) > 0 \)

or \(-x(x - 1)(2x - 1) > 0 \)

or \((x - 1)(x - 2)(2x - 1) < 0 \)

on number line mark \( x = 1/2, 1, 2 \)

**Fig. 1.8**

When \( x > 2 \), all factors \((x - 1), (2x - 1) \) are positive.

Hence, \((x - 1)(2x - 1) > 0 \) for \( x > 2 \).

Now put positive and negative sign alternatively as shown in figure.

Hence, solution set of \((x - 1)(x - 2)(1 - 2x) > 0 \) or \((x - 1)(x - 2)(2x - 1) < 0 \) is \((-\infty, 1/2) \cup (1, 2) \).

**Example 1.13** Solve \( (2x + 1)(x - 3)(x + 7) < 0. \)

**Sol.** \((2x + 1)(x - 3)(x + 7) < 0 \)

Sign scheme of \((2x + 1)(x - 3)(x + 7) \) is as follows:

**Fig. 1.9**

Hence, solution is \((-\infty, -7) \cup (-1/2, 3) \).

**Example 1.14** Solve \( \frac{2}{x} < 3. \)

**Sol.** \( \frac{2}{x} < 3 \)

\[ \Rightarrow \frac{2}{3} - \frac{1}{x} < 0 \] (We cannot cross multiply with \( x \) as \( x \) can be negative or positive)
\[ \Rightarrow \frac{2-3x}{x} < 0 \]
\[ \Rightarrow \frac{3x-2}{x} > 0 \]
\[ \Rightarrow \frac{(x-2/3)}{x} > 0 \]

Sign scheme of \( \frac{(x-2/3)}{x} \) is as follows:

Fig. 1.10

\[ \Rightarrow x \in (-\infty, 0) \cup (2/3, \infty) \]

**Example 1.15** Solve \( \frac{2x-3}{3x-5} \geq 3 \).

*Sol.*

\[ \frac{2x-3}{3x-5} \geq 3 \]
\[ \Rightarrow \frac{2x-3}{3x-5} - 3 \geq 0 \]
\[ \Rightarrow \frac{2x-3-9x+15}{3x-5} \geq 0 \]
\[ \Rightarrow \frac{-7x+12}{3x-5} \geq 0 \]
\[ \Rightarrow \frac{7x-12}{3x-5} \leq 0 \]

Sign scheme of \( \frac{7x-12}{3x-5} \) is as follows:

Fig. 1.11

\[ \Rightarrow x \in (5/3, 12/7) \]

\( x = 5/3 \) is not included in the solutions as at \( x = 5/3 \) denominator becomes zero.

**Example 1.16** Solve \( x > \sqrt{1-x} \).

*Sol.* Given inequality can be solved by squaring both sides. But sometimes squaring gives extraneous solutions which do not satisfy the original inequality. Before squaring we must restrict \( x \) for which terms in the given inequality are well defined.

\[ x > \sqrt{1-x} \] Here \( x \) must be positive.

\[ \sqrt{1-x} \text{ is defined only when } 1-x \geq 0 \text{ or } x \leq 1 \] (1)

Squaring given inequality but sides \( x^2 > 1-x \)

\[ \Rightarrow x^2 + x - 1 > 0 \]
\[ \Rightarrow \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right) > 0 \]
\[ \Rightarrow x < \frac{-1-\sqrt{5}}{2} \text{ or } x > \frac{-1+\sqrt{5}}{2} \] (2)

From (1) and (2), \( x \in \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) \) (as \( x \) is > +ve)

**Example 1.17** Solve \( \frac{2}{x^2-x+1} - \frac{1}{x+1} - \frac{2x-1}{x^2} \leq 0 \).

*Sol.*

\[ \frac{2}{x^2-x+1} - \frac{1}{x+1} - \frac{2x-1}{x^2} \leq 0 \]
\[ \Rightarrow \frac{2(x+1)-(x^2-x+1)-(2x-1)}{(x+1)(x^2-x+1)} \geq 0 \]
\[ \Rightarrow \frac{(x^2-2x)}{(x+1)(x^2-x+1)} \geq 0 \]
\[ \Rightarrow \frac{(x-2)(x+1)}{(x+1)(x^2-x+1)} \geq 0 \]
\[ \Rightarrow \frac{x-2}{x^2-x+1} \geq 0, \text{ where } x \neq 1 \]
\[ \Rightarrow x-2 \geq 0, \text{ where } x \neq 1 \]
\[ \Rightarrow x \leq 2, x \neq -1 \]

**Example 1.18** Solve \( x(x+2)^4(x-1)^4(2x-3)(x-3)^4 \geq 0 \).

*Sol.* Let \( E = x(x+2)^4(x-1)^4(2x-3)(x-3)^4 \).

Here for \( x, (x-1), (2x-3), (x-3)^4 \) exponents are odd, hence sign of \( E \) changes while crossing \( x = 0, 1, 3/2, 3 \).

Also for \( (x+2), (x-3) \) exponents are even, hence sign of \( E \) does not change while crossing \( x = -2 \) and \( x = 3 \).

Further for \( x > 3 \), all factors are positive, hence sign of the expression starts with positive sign from the right hand side.

The sign scheme of the expression is as shown in the following figure.

Fig. 1.12

Hence, for \( E \geq 0 \), we have \( x \in [0, 1] \cup [3/2, \infty) \)

**Example 1.19** Solve \( x(2x-1)(3x-9)(x-3) < 0 \).

*Sol.* Let \( E = x(2x-1)(3x-9)(x-3) \)

Here \( 2x-1 = 0 \Rightarrow x = 0 \) and when \( 3x-9 = 0 \Rightarrow x = 3 \)

Now mark \( x = 0, 2 \) and 3 on real number line.

Sign of \( E \) starts with positive sign from right hand side.

Also at \( x = 0 \), two factors are 0, and \( 2x-1 \), hence sign of \( E \) does not change while crossing \( x = 0 \).

Sign scheme of \( E \) is as shown in the following figure.

Fig. 1.13

From the figure, we have \( E < 0 \) for \( x \in (2, 3) \).
Example 1.20 Find all possible values of \( \frac{x^2 + 1}{x^2 - 2} \).

Sol. Let \( y = \frac{x^2 + 1}{x^2 - 2} \)

\[ y = \frac{x^2 - 2y = x^2 + 1}{x^2 - 2} \]

\[ x^2 = \frac{2y + 1}{y - 1} \]

Now \( x^2 \geq 0 \) \( \Rightarrow \) \( \frac{2y + 1}{y - 1} \geq 0 \)

Now \( x^2 \geq 0 \) \( \Rightarrow \) \( \frac{2y + 1}{y - 1} \geq 0 \)

\[ y \leq -1/2 \text{ or } y > 1 \]

Solving Irrational Inequalities

Example 1.21 Solve \( \sqrt{x - 2} \geq -1 \).

Sol. We must have \( x - 2 \geq 0 \) for \( \sqrt{x - 2} \) to get defined, thus \( x \geq 2 \).

Now \( \sqrt{x - 2} \geq -1 \), as square roots are always non-negative.

Hence, \( x \geq 2 \).

Note: Some students solve it by squaring both sides for which \( x - 2 \geq 1 \) or \( x \geq 3 \) which cause loss of interval \([2, 3)\).

Example 1.22 Solve \( \sqrt{x - 1} > \sqrt{3 - x} \).

Sol. \( \sqrt{x - 1} > \sqrt{3 - x} \) is meaningful if \( x - 1 \geq 0 \) and \( 3 - x \geq 0 \)

or \( 1 \leq x \leq 3 \)

Also \( \sqrt{x - 1} > \sqrt{3 - x} \)

Squaring, we have \( x - 1 > 3 - x \)

\[ x > 2 \]

From (1) and (2), we have \( 2 < x \leq 3 \).

Example 1.23 \( x + \sqrt{x} \geq \sqrt{x - 3} \).

Sol. \( x + \sqrt{x} \geq \sqrt{x - 3} \) is meaningful only when \( x \geq 0 \)

Now \( x + \sqrt{x} \geq \sqrt{x - 3} \)

\[ x \geq -3 \]

From (1) and (2), we have \( x \geq 0 \).

Example 1.24 \( (x^2 - 4) \sqrt{x^2 - 1} < 0 \).

Sol. \( (x^2 - 4) \sqrt{x^2 - 1} < 0 \)

We must have \( x^2 - 1 \geq 0 \)

or \( (x - 1)(x + 1) \geq 0 \)

or \( x \geq 1 \) or \( x \leq -1 \)

Also \( (x^2 - 4) \sqrt{x^2 - 1} < 0 \)

\[ x^2 - 4 < 0 \]

\[ \Rightarrow -2 < x < 2 \]

From (1) and (2), we have \( x \in (-2, -1) \cup (1, 2) \)

ABSOLUTE VALUE OF \( x \)

Absolute value of any real number \( x \) is denoted by \( |x| \) (read as modulus of \( x \)).

The absolute value is closely related to the notions of magnitude, distance, and norm in various mathematical and physical contexts.

From an analytic geometry point of view, the absolute value of a real number is that number's distance from zero along the real number line, and more generally the absolute value of the difference of two real numbers is the distance between them.

Let's look at the number line:

![Number Line](image)

Fig. 1.14

The absolute value of \( x \), denoted \( |x| \) (and which is read as "the absolute value of \( x \)"), is the distance of \( x \) from zero. This is why absolute value is never negative; absolute value only asks "how far?", not "in which direction?". This means that not only that \( |3| = 3 \), because 3 is three units to the right of zero, but also that \( |-3| = 3 \), because -3 is three units to the left of zero.

When the number inside the absolute value (the "argument" of the absolute value) was positive anyway, we did not change the sign when we took the absolute value. But when the argument was negative, we did change the sign.

If \( x > 0 \) (that is, if \( x \) is positive), then the value would not change when you take the absolute value. For instance, if \( x = 2 \), then you have \( |x| = |2| = 2 = x \). In fact, for any positive value of \( x \) (or if \( x \) equals zero), the sign would be unchanged, so:

For \( x \geq 0 \), \( |x| = x \)

On the other hand, if \( x < 0 \) (that is, if \( x \) is negative), then it will change its sign when you take the absolute value. For instance, if \( x = -4 \), then \( |x| = |-4| = +4 = -4 = x \). In fact, for any negative value of \( x \), the sign would have to be changed, so:

For \( x < 0 \), \( |x| = -x \)

Thus \( |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \)

Also \( \sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \)

i.e., \( 2 = \sqrt{2^2} = \sqrt{(-2)^2} = \sqrt{(-2)^2} = \sqrt{4} = 2 \)

This square root exists only for non-negative numbers and its value is also non-negative.

Some students consider \( \sqrt{4} = \pm 2 \), which is wrong.

In fact \( \sqrt{-4} = 1 - 2 = 2 \)
\[ \sqrt{1 - \sqrt{2}} = 1 - \sqrt{2} = \sqrt{2} - 1 \text{ etc.} \]

Also, some students write \( \sqrt{x^2} = \pm x \) which is wrong, in fact,
\[ \sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \]

Also \( a^2 < b^2 \Rightarrow \sqrt{a^2} < \sqrt{b^2} \Rightarrow |a| < |b| \)

Graph of function \( f(x) = y = |x| \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>±1</th>
<th>±2</th>
<th>±3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

**Fig. 1.15**

We can see that graph of \( y = |x| \) is in 1st and 2nd quadrant only where \( y \geq 0 \), hence \( |x| \geq 0 \).

**Example 1.25** Solve the following:
(i) \( |x| = 5 \)
(ii) \( x^2 - |x| - 2 = 0 \)

Sol.
(i) \( |x| = 5 \), i.e., those points on real number line which are at distance 5 units from “0”, which are -5 and 5.
Thus, \( |x| = 5 \Rightarrow x = \pm 5 \)
(ii) \( x^2 - |x| - 2 = 0 \)
\[ \Rightarrow |x|^2 - |x| - 2 = 0 \]
\[ \Rightarrow (|x| - 2)(|x| + 1) = 0 \]
\[ \Rightarrow |x| = 2 \text{ (since } |x| + 1 \neq 0) \]
\[ \Rightarrow x = \pm 2 \]

**Example 1.26** Find the value of \( x \) for which following expressions are defined:
(i) \( \frac{1}{\sqrt{x-\mid x \mid}} \)
(ii) \( \frac{1}{\sqrt{x+\mid x \mid}} \)

Sol.
(i) \( x - |x| = \begin{cases} x - x = 0, & \text{if } x \geq 0 \\ x + x = 2x, & \text{if } x < 0 \end{cases} \)

\[ \Rightarrow x - |x| \leq 0 \text{ for all } x \]
\[ \Rightarrow \frac{1}{\sqrt{x-|x|}} \text{ does not take real values for any } x \in \mathbb{R} \]
\[ \Rightarrow \frac{1}{\sqrt{x+|x|}} \text{ is not defined for any } x \in \mathbb{R} \]

(ii) \( x + |x| = \begin{cases} x + x = 2x, & \text{if } x \geq 0 \\ x - x = 0, & \text{if } x < 0 \end{cases} \)
\[ \Rightarrow \frac{1}{\sqrt{x+|x|}} \text{ is defined only when } x > 0 \]

What is the geometric meaning of \( |x - y| \)?
\( |x - y| \) is the distance between \( x \) and \( y \) on the real number line.

**Example 1.27** Solve the following:
(i) \( |x - 2| = 1 \)
(ii) \( 2|x + 1| - |x - 1| = 3 \)

Sol.
(i) \( |x - 2| = 1 \), i.e., those points on real number line which are at distance 1 units from 2.

As shown in the figure \( x = 1 \) and \( x = 3 \) are at distance 1 units from 2.

Hence, \( x = 1 \) or \( x = 3 \).
Thus \( |x - 2| = 1 \)
\[ \Rightarrow x - 2 = \pm 1 \]
\[ \Rightarrow x = 1 \text{ or } x = 3 \]

(ii) \( 2|x + 1| - |x - 1| = 3 \)
\[ \Rightarrow 2|x + 1| - |x + 1| - 3 = 0 \]
\[ \Rightarrow |x + 1|^2 - 3|x + 1| + 2|x + 1| - 3 = 0 \]
\[ \Rightarrow (2|x + 1| - 3)(|x + 1| + 1) = 0 \]
\[ \Rightarrow |x + 1| = 3/2 \]
\[ \Rightarrow x + 1 = \pm 3/2 \]
\[ \Rightarrow x = 1/2 \text{ or } x = -5/2 \]
\[ \Rightarrow |x - a| = \begin{cases} x - a, & x \geq a \\ a - x, & x < a \end{cases} \text{, where } a > 0 \]

In general, \( |f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases} \) where \( y = f(x) \) is any real-valued function.

**Example 1.28** Solve the following:
(i) \( |x - 2| = (x - 2) \)
(ii) \( |x + 3| = -x - 3 \)
(iii) \( |x^2| = x^2 - x \)
1.10 Algebra

(iv) \(|x^2 - x - 2| = 2 + x - x^2\)

Sol.
(i) \(|x - 2| = (x - 2)\), if \(x - 2 \geq 0\) or \(x \geq 2\)
(ii) \(|x + 3| = -x - 3\), if \(x + 3 \leq 0\) or \(x \leq -3\)
(iii) \(|x^2 - x| = x^2 - x\), if \(x^2 - x \geq 0\)
   \(\Rightarrow x(x - 1) \geq 0\)
   \(\Rightarrow x \in (-\infty, 0] \cup [1, \infty)\)
(iv) \(|x^2 - x - 2| = 2 + x - x^2\)
   \(\Rightarrow x^2 - x - 2 \leq 0\)
   \(\Rightarrow (x - 2)(x + 1) \leq 0\)
   \(\Rightarrow -1 \leq x \leq 2\)

Example 1.29 Solve \(1 - x = \sqrt{x^2 - 2x + 1}\).

Sol. \(1 - x = \sqrt{x^2 - 2x + 1}\)
   \(\Rightarrow 1 - x = \sqrt{(x - 1)^2}\)
   \(\Rightarrow 1 - x = x - 1\)
   \(\Rightarrow 1 - x \geq 0\)
   \(\Rightarrow x \leq 1\)

Example 1.30 Solve \(|3x - 2| = x\).

Sol. \(|3x - 2| = x\)
Case (i)
   When \(3x - 2 \geq 0\) or \(x \geq 2/3\)
   For which we have \(3x - 2 = x \) or \(x = 1\).
Case (ii)
   When \(3x - 2 < 0\) or \(x < 2/3\)
   For which we have \(2 - 3x = x \) or \(x = 1/2\).
   Hence, solution set is \(\{1/2, 1\}\).

Example 1.31 Solve \(|x| = x^2 - 1\).

Sol. \(x^2 - 1 = |x|\)
   \(\Rightarrow x^2 - 1 = x \) when \(x \geq 0\)
   or \(x^2 - 1 = -x \) when \(x < 0\)
   \(x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2} \) (as \(x \geq 0\))
   \(x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 + \sqrt{5}}{2} \) (as \(x < 0\))

Example 1.32 Solve
\[
\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1
\]

Sol. \(\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1\)
   \(\Rightarrow \sqrt{x - 1 - 4\sqrt{x - 1} + 4} + \sqrt{x - 1 - 6\sqrt{x - 1} + 9} = 1\)
   \(\Rightarrow \sqrt{(\sqrt{x - 1} - 2)^2} + \sqrt{(\sqrt{x - 1} - 3)^2} = 1\)
   \(\Rightarrow |\sqrt{x - 1} - 2| + |\sqrt{x - 1} - 3| = 1\)

Example 1.33 Prove that
\[
\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1} = \begin{cases} 
-2, & x < -1 \\
2x, & -1 \leq x \leq 1 \\
2, & x > 1 
\end{cases}
\]

Sol. \(\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1} = \)
   \(= \sqrt{(x + 1)^2} - \sqrt{(x - 1)^2}\)
   \(= |x + 1| - |x - 1|\)
   \(= \begin{cases} 
-2, & x < -1 \\
2x, & -1 \leq x \leq 1 \\
2, & x > 1 
\end{cases}
\]

Example 1.34

(i) For \(2 < x < 4\), find the values of \(|x|\).
(ii) For \(-3 \leq x \leq -1\), find the values of \(|x|\).
(iii) For \(-3 \leq x < 1\), find the values of \(|x|\).
(iv) For \(-5 < x < 7\), find the values of \(|x - 2|\).
(v) For \(1 \leq x \leq 5\), find the values of \(|2x - 7|\).

Sol.
(i) \(2 < x < 4\)
   Here values on real number line whose distance lies between 2 and 4.
   Here values of \(x\) are positive \(\Rightarrow |x| \in (2, 4)\)
(ii) \(-3 \leq x \leq -1\)
   Here values on real number line whose distance lies between 1 and 3 or at distance 1 or 3.
   \(\Rightarrow 1 \leq |x| \leq 3\)
(iii) \(-3 \leq x < 1\)
   For \(-3 \leq x < 0\), \(|x| \in (0, 3)\)
   For \(0 \leq x < 1\), \(|x| \in [0, 1]\)
   So for \(-3 \leq x < 1\), \(|x| \in [0, 1) \cup (0, 3] \) or \(x \in [0, 3]\)
(iv) \(-5 < x < 7\)
   \(\Rightarrow -7 < x - 2 < 5\)
   \(\Rightarrow 0 \leq |x - 2| < 7\)
Example 1.35 \[ \text{Find all possible values of} \]
\[
\begin{align*}
\text{(i) } & |x - 3| - 2 \quad \text{(ii) } 4 - |2x + 3| \\
\text{Sol.} \quad & (i) \quad \text{We know that } |x - 3| \geq 0 \quad \forall x \in R \\
& \Rightarrow |x - 3| - 2 \geq -2 \\
& \Rightarrow |x - 3| - 2 \in [-2, \infty) \\
\text{(ii) } & \text{We know that } |2x + 3| \geq 0 \quad \forall x \in R \\
& \Rightarrow -|2x + 3| \leq 0 \\
& \Rightarrow 4 - |2x + 3| \leq 4 \\
& \quad \text{or } 4 - |2x + 3| \in (-\infty, 4] \\
\end{align*}
\]

Example 1.36 \[ \text{Find all possible values of} \]
\[
\begin{align*}
\text{(i) } & \sqrt{|x - 1| - 2} \quad \text{(ii) } \sqrt{3 - |x - 1|} \quad \text{(iii) } \sqrt{4 - \sqrt{x^2}} \\
\text{Sol.} \quad & (i) \quad \text{We know that square roots are defined for non-negative values only.} \\
& \quad \text{It implies that we must have } |x - 1| - 2 \geq 0. \\
& \Rightarrow \sqrt{|x - 1| - 2} \geq 0 \\
\text{(ii) } & \sqrt{3 - |x - 1|} \text{ is defined when } 3 - |x - 1| \geq 0 \\
& \quad \text{But the maximum value of } 3 - |x - 1| \text{ is } 3 \text{ when } |x - 1| = 0. \\
& \quad \text{Hence, for } \sqrt{3 - |x - 1|} \text{ to get defined, } 0 \leq 3 - |x - 1| \leq 3. \\
& \Rightarrow \sqrt{3 - |x - 1|} \in \left[0, \sqrt{3}\right] \\
& \quad \text{Alternatively, } |x - 1| \geq 0. \\
& \Rightarrow -|x - 1| \leq 0 \\
& \Rightarrow 3 - |x - 1| \leq 3 \\
& \quad \text{But for } \sqrt{3 - |x - 1|} \text{ to get defined, we must have} \\
& 0 \leq 3 - |x - 1| \leq 3 \Rightarrow 0 \leq \sqrt{3 - |x - 1|} \leq \sqrt{3} \\
\text{(iii) } & \sqrt{4 - \sqrt{x^2}} = \sqrt{4 - 1x1} \\
& |x| \geq 0 \\
& \Rightarrow -|x| \leq 0 \\
& \Rightarrow 4 - |x| \leq 4 \\
& \quad \text{But for } \sqrt{4 - 1x1} \text{ to get defined } 0 \leq 4 - |x| \leq 4 \\
& \Rightarrow 0 \leq \sqrt{4 - 1x1} \leq 2 \\
\end{align*}
\]

Example 1.37 \[ \text{Solve } |x - 3| + |x - 2| = 1. \]
\[
\begin{align*}
\text{Sol.} \quad & |x - 3| + |x - 2| = 1 \\
& \Rightarrow |x - 3| + |x - 2| = (3 - x) + (x - 2) \\
& \Rightarrow x - 3 \leq 0 \text{ and } x - 2 \geq 0 \\
& \Rightarrow x \leq 3 \text{ and } x \geq 2 \\
& \Rightarrow 2 \leq x \leq 3 \\
\end{align*}
\]

Example 1.38 \[ \text{Solve } x^2 - 4|x| + 3 < 0. \]
\[
\begin{align*}
\text{Sol.} \quad & x^2 - 4|x| + 3 < 0 \\
& \Rightarrow (|x| - 1)(|x| - 3) < 0 \\
& \Rightarrow 1 < |x| < 3 \\
& \Rightarrow -3 < x < -1 \text{ or } 1 < x < 3 \\
& \Rightarrow x \in (-3, -1) \cup (1, 3) \\
\end{align*}
\]

Example 1.39 \[ \text{Solve } 0 < |x| < 2. \]
\[
\begin{align*}
\text{Sol.} \quad & \text{We know that } |x| \geq 0, \forall x \in R \\
& \quad \text{But given } |x| > 0 \Rightarrow x \neq 0 \\
& \quad \text{Now } 0 < |x| < 2 \\
& \Rightarrow x \in (-2, 2), x \neq 0 \\
\end{align*}
\]
Example 1.40 Solve $|3x - 2| < 4$.

Sol. $|3x - 2| < 4$

\[ -4 < 3x - 2 < 4 \]

\[ -2 < 3x < 6 \]

\[ -\frac{2}{3} < x < 2 \]

Example 1.41 Solve $1 \leq |x - 2| \leq 3$.

Sol. $1 \leq |x - 2| \leq 3$

\[ -3 \leq x - 2 \leq -1 \text{ or } 1 \leq x - 2 \leq 3 \]

\[ -1 \leq x \leq 1 \text{ or } 3 \leq x \leq 5 \]

\[ x \in [-1, 1] \cup [3, 5] \]

Example 1.42 Solve $0 < |x - 3| \leq 5$.

Sol. $0 < |x - 3| \leq 5$

\[ -5 < x - 3 < 0 \text{ or } 0 < x - 3 \leq 5 \]

\[ -2 < x < 3 \text{ or } 3 < x \leq 8 \]

\[ x \in (-2, 3) \cup (3, 8] \]

Example 1.43 Solve $|x - 1| - |2| < 5$.

Sol. $|x - 1| - 2 < 5$

\[ -5 < x - 1 - 2 < 5 \]

\[ -3 < x < 8 \]

\[ x \in (-3, 8) \]

Example 1.44 Solve $|x - 3| \geq 2$.

Sol. $|x - 3| \geq 2$

\[ x - 3 \leq -2 \text{ or } x - 3 \geq 2 \]

\[ x \leq 1 \text{ or } x \geq 5 \]

Example 1.45 Solve $|x| - 3 > 1$.

Sol. $|x| - 3 > 1$

\[ |x| - 3 > 1 \]

\[ x < 2 \text{ or } x > 4 \]

\[ x \in (-\infty, 2) \cup (4, \infty) \]

Example 1.46 Solve $|x - 1| + |x - 2| \geq 4$.

Sol. Let $f(x) = |x - 1| + |x - 2|$

<table>
<thead>
<tr>
<th>A</th>
<th>B, $f(x)$</th>
<th>C, $f(x) \geq 4$</th>
<th>D, $A \cap C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; 1$</td>
<td>$1 - x + 2 - x$</td>
<td>$3 - 2x \geq 4$</td>
<td>$x \leq -3/2$</td>
</tr>
<tr>
<td>$1 \leq x \leq 2$</td>
<td>$x - 1 + 2 - x$</td>
<td>$1 \geq 4$, not possible</td>
<td></td>
</tr>
<tr>
<td>$x &gt; 2$</td>
<td>$x - 1 + x - 2$</td>
<td>$2x - 3 \geq 4$</td>
<td>$x \geq 7/2$</td>
</tr>
</tbody>
</table>

Hence, solutions is $x \in (-\infty, -1/2) \cup (7/2, \infty)$.

Example 1.47 Solve $|x + 1| + |2x - 3| = 4$.

Sol. Let $f(x) = |x + 1| + |2x - 3|$.

<table>
<thead>
<tr>
<th>A</th>
<th>B, $f(x)$</th>
<th>C, $f(x) \geq 4$</th>
<th>D, $A \cap C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -1$</td>
<td>$-1 - x + 3 - 2x$</td>
<td>$2 - 3x = 4$</td>
<td>$x = -2/3$</td>
</tr>
<tr>
<td>$-1 \leq x \leq 3/2$</td>
<td>$x + 1 + 3 - 2x$</td>
<td>$4 - x = 0$</td>
<td></td>
</tr>
<tr>
<td>$x &gt; 3/2$</td>
<td>$x + 1 + 2x - 3$</td>
<td>$3x - 2 = 4$</td>
<td>$x = 2$</td>
</tr>
</tbody>
</table>

Hence, solutions set is $\{0, 2\}$.

Example 1.48 Solve $|x| + |x - 2| = 2$.

Sol. We have $|x| + |x - 2| = 2$

\[ |x| + |x - 2| = x - (x - 2) \]

\[ 0 \leq x \leq 2 \]

Example 1.49 Solve $|2x - 3| + |x - 1| = |x - 2|$.

Sol. $|2x - 3| + |x - 1| = |x - 2|$

\[ (2x - 3) + (x - 1) = -(x - 2) \]

\[ 1 \leq x \leq 3/2 \]

Example 1.50 Solve $|x^2 + x - 4| = |x^2 - 4| + |x|$.

Sol. $|x^2 + x - 4| = |x^2 - 4| + |x|$

\[ x(x^2 - 4) \geq 0 \]

\[ x(x^2 - 4) \geq 0 \]

\[ x \in [-2, 0] \cup (2, \infty) \]

Example 1.51 If $|\sin x + \cos x| = |\sin x| + |\cos x| (\sin x, \cos x \neq 0)$, then in which quadrant does $x$ lie?

Sol. Here we have $|\sin x + \cos x| = |\sin x| + |\cos x|$.

It implies that $\sin x$ and $\cos x$ must have the same sign.

Therefore, $x$ lies in the first or third quadrant.

Example 1.52 Is $|\tan x + \cot x| < |\tan x| + |\cot x|$ true for any $x$? If it is true, then find the values of $x$.

Sol. Since $\tan x$ and $\cot x$ have always the same sign,

$|\tan x + \cot x| < |\tan x| + |\cot x|$ does not hold true for any value of $x$.

Example 1.53 Solve $\left| \frac{x - 3}{x + 1} \right| \leq 1$.

Sol. $\left| \frac{x - 3}{x + 1} \right| \leq 1$

\[ -1 \leq \frac{x - 3}{x + 1} \leq 1 \]

\[ \frac{x - 3}{x + 1} \leq 0 \text{ and } 0 \leq \frac{x - 3}{x + 1} \]

\[ -4 \leq 0 \text{ and } 0 \leq \frac{2x - 2}{x + 1} \]
Example 1.54 Solve $|x^2 - 2x| + |x - 4| > |x^2 - 3x + 4|.$

Sol. We have $|x^2 - 2x| + |x - 4| = |x^2 - 2x + 4 - x|$  
$\Rightarrow (x^2 - 2x)(4 - x) < 0$  
$\Rightarrow x (x - 2)(x - 4) > 0$  
$\Rightarrow x \in (0, 2) \cup (4, \infty)$

Concept Application Exercise 1.1

1. If $f(x) = \frac{x + 3}{x - 1}$, then which of the following is greatest?  
   (a) $f(0)$  
   (b) $f(3)$  
   (c) $f(4)$  
   (d) $f(2)$

2. If $f(x)$ is a quadratic function such that $f(0) = -4$, $f(1) = -5$ and $f(-1) = -1$, then find the value of $f(3)$.

3. Find the value of $x^2$ for the following values of $x$:  
   (i) $[-5, -1]$  
   (ii) $(3, 6)$  
   (iii) $(-2, 3)$  
   (iv) $(-\infty, \infty)$  
   (v) $(-\infty, 4)$

4. Find the values of $1/x$ for the following values of $x$:  
   (i) $(2, 5)$  
   (ii) $[-5, -1]$  
   (iii) $(3, \infty)$  
   (iv) $(-\infty, -2]$  
   (v) $[-3, 4]$  

5. Which of the following is always true?  
   (a) If $a < b$, then $a^2 < b^2$  
   (b) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$  
   (c) If $a < b$, then $|a| < |b|$

6. Find the values of $x$ which satisfy the inequalities simultaneously:  
   (i) $-3 < 2x - 1 < 5$  
   (ii) $-1 < \frac{2x + 3}{5} \leq 3$

7. Find all the possible values which the following expressions take.  
   (i) $2 \cdot 5x - 3x - 4$  
   (ii) $\sqrt{x^2 - 7x + 6}$  
   (iii) $\frac{x^2 - 6}{x - 3}$

8. Solve $3x^2 + 3x + 1 < 0$.  
9. Solve $2x + 3)(x - 3)^2 + 3 < 0$.  
10. Solve $\sqrt{x - 3} + x + 7 \leq 0$.  
11. Find all positive values of $f(x) = \frac{1 - x^2}{x^2 + 3}$

SOME DEFINITIONS

Real Polynomial
Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and $x$ is a real variable. Then, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is called a real polynomial of real variable $x$ with real coefficients.

Complex Polynomial
If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and $x$ is a varying complex number, then $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is called a complex polynomial or a polynomial of complex coefficients.

Rational Expression or Rational Function
An expression of the form  
$\frac{P(x)}{Q(x)}$
where $P(x)$ and $Q(x)$ are polynomials in $x$ is called a rational expression.
In the particular case when \( Q(x) \) is a non-zero constant,

\[
P(x) \quad \frac{Q(x)}{Q(x)}
\]

reduces to a polynomial. Thus every polynomial is a rational expression but the converse is not true. Some of the examples are as follows:

\[
(1) \quad \frac{x^2 - 5x + 4}{x - 2} \quad (2) \quad x^2 - 5x + 4
\]

\[
(3) \quad \frac{1}{x} \quad (4) \quad \frac{x + 1}{x}, i.e., \quad \frac{x^2 + 1}{x}
\]

**Degree of a Polynomial**

A polynomial \( f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \), real or complex, is a polynomial of degree \( n \), if \( a_n \neq 0 \).

The polynomials \( 2x^2 - 7x^2 + x + 5 \) and \( (3 - 2i)x^2 - ix + 5 \) are polynomials of degree 3 and 2, respectively.

A polynomial of second degree is generally called a quadratic polynomial, and polynomials of degree 3 and 4 are known as cubic and bi-quadratic polynomials, respectively.

**Polynomial Equation**

If \( f(x) \) is a polynomial, then \( f(x) = 0 \) is called a polynomial equation.

If \( f(x) \) is a quadratic polynomial, then \( f(x) = 0 \) is called a quadratic equation. The general form of a quadratic equation is \( ax^2 + bx + c = 0 \), \( a \neq 0 \). Here, \( x \) is the variable and \( a, b, \) and \( c \) are called coefficients, real or imaginary.

**Roots of an Equation**

The values of the variable satisfying a given equation are called its roots.

Thus, \( x = a \) is a root of the equation \( f(x) = 0 \), if \( f(a) = 0 \).

For example, \( x = 1 \) is a root of the equation \( x^2 - 6x^2 + 11x - 6 = 0 \), because \( 1^2 - 6 \times 1^2 + 11 \times 1 - 6 = 0 \) (1).

Similarly, \( x = \omega \) and \( x = \omega^2 \) are roots of the equation \( x^2 + x + 1 = 0 \) as they satisfy it (where \( \omega \) is the complex cube root of unity).

**Solution Set**

The set of all roots of an equation, in a given domain, is called the solution set of the equation.

For example, the set \( \{1, 2, 3\} \) is the solution set of the equation \( x^2 - 6x^2 + 11x - 6 = 0 \).

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining all its roots.

**Example 1.55** If \( x = 1 \) and \( x = 2 \) are solutions of the equation \( x^2 + ax^2 + bx + c = 0 \) and \( a + b = 1 \), then find the value of \( b \).

**Sol.** Since \( x = 1 \) is a root of the given equation it satisfies the equation.

Hence, putting \( x = 1 \) in the given equation, we get

\[
a + b + c = -1
\]

but given that

\[
a + b = 1
\]

\[\Rightarrow\] \( c = -2\)

Now put \( x = 2 \) in the given equation, we have

\[
8 + 4a + 2b + 2 = 0
\]

\[\Rightarrow\] \( 6 + 2a + 2(a + b) = 0\)

\[\Rightarrow\] \( 6 + 2a + 2 = 0\)

\[\Rightarrow\] \( a = -4\)

\[\Rightarrow\] \( b = 5\)

**Example 1.56** Let \( f(x) = ax^2 + bx + c \), where \( a, b, c \in R \) and \( a \neq 0 \). It is known that \( f(5) = -3f(2) \) and that 3 is a root of \( f(x) = 0 \), then find the other root of \( f(x) = 0 \).

**Sol.** \( f(x) = ax^2 + bx + c \)

Given that \( f(5) = -3f(2) \)

\[
25a + 5b + c = -3(4a + 2b + c)
\]

or

\[
37a + 11b + 4c = 0
\]

Also \( x = 3 \) satisfies \( f(x) = 0 \)

\[
9a + 3b + c = 0
\]

or

\[
36a + 12b + 4c = 0
\]

[Subtracting (3) from (1), we have]

\[
a - b = 0
\]

\[\Rightarrow\] \( a = b \Rightarrow \) \( 12a + c = 0 \) or \( c = -12a \)

Hence, equation \( f(x) = 0 \) becomes

\[
ax^2 + bx - 12a = 0
\]

or

\[
x^2 + x - 12 = 0
\]

or

\[
(x - 3)(x + 4) = 0 \quad \text{or} \quad x = -4, 3
\]

**Example 1.57** A polynomial in \( x \) of degree three vanishes when \( x = 1 \) and \( x = -2 \), and has the values 4 and 28 when \( x = -1 \) and \( x = 2 \), respectively. Then find the value of polynomial when \( x = 0 \).

**Sol.** From the given data \( f(x) = (x - 1)(x + 2)(ax + b) \)

Now \( f(-1) = 4 \) and \( f(2) = 28 \)

\[
(-1 - 1)(-1 + 2)(-a + b) = 4
\]

and

\[
(2 - 1)(2 + 2)(2a + b) = 28
\]

\[
a - b = 2 \quad \text{and} \quad 2a + b = 7
\]

Solving, \( a = 3 \) and \( b = 1 \)

\[
f(x) = (x - 1)(x + 2)(3x + 1)
\]

\[
f(0) = -2
\]

**Example 1.58** If \( (1 - p) \) is a root of quadratic equation \( x^2 + px + (1 - p) = 0 \), then find its roots.

**Sol.** Since \( (1 - p) \) is a root of quadratic equation

\[
x^2 + px + (1 - p) = 0
\]

So \( (1 - p) \) satisfies the above equation

\[
(1 - p)^2 + p (1 - p) + (1 - p) = 0
\]

\[
(1 - p)(1 - p + p + 1) = 0
\]

\[
(1 - p)(2) = 0
\]

\[
\Rightarrow\] \( p = 1\)

On putting this value of \( p \) in Eq. (1), we get

\[
x^2 + x = 0
\]

\[\Rightarrow\] \( x(x + 1) = 0\)

\[\Rightarrow\] \( x = 0, -1\)
Example 1.59  The quadratic polynomial \( p(x) \) has the following properties:

- \( p(x) \) can be positive or zero for all real numbers.
- \( p(1) = 0 \) and \( p(2) = 2 \).

Then find the quadratic polynomial.

Sol. \( p(x) \) is positive or zero for all real numbers also \( p(1) = 0 \) then we have \( p(x) = k(x - 1)^2 \), where \( k > 0 \)

Now \( p(2) = 2 \)

\[ \Rightarrow k = 2 \]

\[ \therefore p(x) = 2(x - 1)^2 \]

GEOMETRICAL MEANING OF ROOTS (ZEROS) OF AN EQUATION

We know that a real number \( k \) is a zero of the polynomial \( f(x) \) if \( f(k) = 0 \). But why are the zeroes of a polynomial so important? To answer this, first we will see the geometrical representations of polynomials and the geometrical meaning of their zeroes.

We know that graph of the linear function \( y = f(x) = ax + b \) is a straight line.

Consider the function \( f(x) = x + 3 \).

![Fig. 1.20](image)

Now we can see that this graph cuts the \( x \)-axis at \( x = -3 \), where value of \( y = 0 \) or we can say \( x + 3 = 0 \) (or \( y = 0 \)) when value of \( x = -3 \). Thus, \( x = -3 \) which is a root (zero) of equation \( x + 3 = 0 \) is actually the value of \( x \) where graph of \( y = f(x) = x + 3 \) intersects the \( x \)-axis.

Consider the function \( f(x) = x^2 - x - 2 \), now for \( f(x) = 0 \) or \( x^2 - x - 2 = 0 \), we have \((x - 2)(x + 1) = 0 \) or \( x = -1 \) or \( x = 2 \). Then graph of \( f(x) = x^2 - x - 2 \) cuts the \( x \)-axis at two values of \( x \), \( x = -1 \) and \( x = 2 \).

Following is the graph of \( y = f(x) \).

![Fig. 1.21](image)

Consider the function \( f(x) = x^3 - 6x^2 + 11x - 6 \), now for \( f(x) = 0 \) we have \((x - 1)(x - 2)(x - 3) = 0 \) or \( x = 1, 2, 3 \). Then graph of \( y = f(x) \) cuts \( x \)-axis at three values of \( x \), \( x = 1, 2, 3 \).

Following is the graph of \( y = f(x) \).

![Fig. 1.22](image)

Consider the function \( f(x) = x^3 - 3x^2 + 2(x^2 - x + 1) \), now for \( f(x) = 0 \) we have \( x = 1 \) or \( x = 2 \), as \( x^2 - x + 1 = 0 \) is not possible for any real value of \( x \). Hence, \( f(x) = 0 \) has only two real roots and cuts \( x \)-axis for only two values of \( x \), \( x = 1 \) and \( x = 2 \).
Following is the graph of $y = f(x)$.

**Fig. 1.23**

Thus, roots of equation $f(x) = 0$ are actually those values of $x$ where graph $y = f(x)$ meets $x$-axis.

**Roots (Zeros) of the Equation $f(x) = g(x)$**

Now we know that zeros of the equation $f(x) = 0$ are the $x$-coordinates of the points where graph of $y = f(x)$ intersect the $x$-axis, where $y = 0$ or zeros are $x$-coordinate of the point of intersection of $y = f(x)$ and $y = 0$ ($x$-axis).

Consider the equation $x + 5 = 2$.

Let's draw the graph of $y = x + 5$ and $y = 2$, which are as shown in the following figure.

**Fig. 1.24**

Graph of $y = 2$ is a line parallel to $x$-axis at height 2 unit above $x$-axis. Now in the figure, we can see that graphs of $y = x + 5$ and $y = 2$ intersect at point $(-3, 2)$ where value of $x = -3$.

Also from $x + 5 = 2$, we have $x = 2 - 5$ or $x = -3$, which is a root of the equation $x + 5 = 2$. Thus root of the equation $x + 5 = 2$ occurs at point of intersection of graphs $y = x + 5$ and $y = 2$.

Consider the another example $x^2 - 2x = 2 - x$. Let's draw the graph of $y = x^2 - 2x$ and $y = 2 - x$ as shown in the following figure.

**Fig. 1.25**

Now in the figure, we can see that graphs of $y = x^2 - 2x$ and $y = 2 - x$ intersect at points $(1, 3)$ and $(2, 0)$ or where values of $x$ are $x = -1$ and $x = 2$, which are in fact zeros or roots of the equation $x^2 - 2x = 2 - x$ or $x^2 - x - 2 = 0$.

The given equation simplifies to $x^2 - x - 2 = 0$. So one can also locate the roots of the same equation by plotting the graph of $y = x^2 - x - 2$, then the roots of equation are $x$-coordinates of points where graph of $y = x^2 - x - 2$ intersects with the $x$-axis (where $y = 0$), as shown in the following figure.

**Fig. 1.26**

From the above discussion we understand that roots of the equation $f(x) = g(x)$ are the $x$-coordinate of the points of intersection of graphs $y = f(x)$ and $y = g(x)$.

**Example 1.60** In how many points graph of $y = x^3 - 3x^2 + 5x - 3$ intersect x-axis?

**Sol.** Number of point in which $y = x^3 - 3x^2 + 5x - 3$ intersect the x-axis is same as number of real roots of the equation $x^3 - 3x^2 + 5x - 3 = 0$. 

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Now we can see that \( x = 1 \) satisfies the equation, hence one root of the equation is \( x = 1 \).

Now dividing \( x^4 - 3x^2 + 5x - 3 \) by \( x - 1 \), we have quotient \( x^3 - 2x + 3 \).

Hence, equation reduces to \( (x - 1)(x^3 - 2x + 3) = 0 \).

Now \( x^3 - 2x + 3 = 0 \) or \( (x - 1)^3 + 2 = 0 \) is not true for any real value of \( x \).

Hence, the only root of the equation is \( x = 1 \).

Therefore, the graph of \( y = x^3 - 3x^2 + 5x - 3 \) cuts the x-axis in one point only.

**Example 1.61** In the following diagram, the graph of \( y = f(x) \) is given.

![Graph of y = f(x)](image)

**Answer the following questions:**

(a) what are the roots of the \( f(x) = 0 \)?
(b) what are the roots of the \( f(x) = 4 \)?
(c) what are the roots of the \( f(x) = 10 \)?

**Sol.**

(a) The root of the equation \( f(x) = 0 \) occurs for the values of \( x \) where the graphs of \( y = f(x) \) and \( y = 0 \) intersect.

From the diagram, for these point of intersection \( x = -1 \) and \( x = 2 \). Hence, roots of the equation \( f(x) = 0 \) are \( x = -1 \) and \( x = 2 \).

(b) The root of the equation \( f(x) = 4 \) occurs for the values of \( x \) where the graphs of \( y = f(x) \) and \( y = 4 \) intersect.

**Example 1.62** Which of the following pair of graphs intersect?

(i) \( y = x^2 - x \) and \( y = 1 \)
(ii) \( y = x^2 - 2x + 3 \) and \( y = \sin x \)
(iii) \( y = x^2 - x + 1 \) and \( y = x - 4 \)

**Sol.** \( y = x^2 - x \) and \( y = 1 \) intersect if \( x^2 - x = 1 \) \( \Rightarrow x^2 - x - 1 = 0 \), which has real roots.

\( y = x^2 - 2x + 3 \) and \( y = \sin x \) intersect if \( x^2 - 2x + 3 = \sin x \) or \( (x - 1)^2 + 2 = \sin x \), which is not possible as L.H.S. has the least value 2, while R.H.S. has the maximum value 1.

\( y = x^2 - x + 1 \) and \( y = x - 4 \) intersect if \( x^2 - x + 1 = x - 4 \) or \( x^2 - 2x + 5 = 0 \), which has non-real roots. Hence, graphs do not intersect.

**Example 1.63** Prove that graphs \( y = 2x - 3 \) and \( y = x^2 - x \) never intersect.

**Sol.** \( y = 2x - 3 \) and \( y = x^2 - x \) intersect only when \( x^2 - x = 2x - 3 \) or \( x^2 - 3x + 3 = 0 \)

Now discriminant \( D = (-3)^2 - 4(3) = -3 < 0 \)

Hence, roots of the equation are not real, or we can say that there is no real number for which \( 2x - 3 \) and \( x^2 - x \) are equal (or \( y = 2x - 3 \) and \( y = x^2 - x \) intersect).

Hence, proved.

**KEY POINTS IN SOLVING AN EQUATION**

**Domain of Equation**

It is a set of the values of independent variables \( x \) for which each function used in the equation is defined, i.e., it takes up finite real values. In other words, the final solution obtained while solving any equation must satisfy the domain of the expression of the parent equation.

**Example 1.64** Solve \( \frac{x^2 - 2x - 3}{x + 1} = 0 \).

**Sol.** Equation \( \frac{x^2 - 2x - 3}{x + 1} = 0 \) is solvable over \( R - \{ -1 \} \)

Now \( \frac{x^2 - 2x - 3}{x + 1} = 0 \)

\( \Rightarrow x^2 - 2x - 3 = 0 \) or \( (x - 3)(x + 1) = 0 \)

\( \Rightarrow x = 3 \) (as \( x \in R - \{ -1 \} \))

**Example 1.65** Solve \( \sqrt{x^2 - 1} = 0 \).

**Sol.** Given equation is solvable for \( x^2 - 1 \geq 0 \) or \( x \in (-\infty, -1] [1, \infty) \)

\( (x^2 - 4x)\sqrt{x^2 - 1} = 0 \)

\( \Rightarrow x(x - 2)(x + 2)\sqrt{x^2 - 1} = 0 \)

\( \Rightarrow x = 0, -2, 2, -1, 1 \)
Example 1.66: Solve \( \frac{2x-3}{x-1} + 1 = \frac{6x-x^2-6}{x-1} \).

**Solution:**

\[
\frac{2x-3}{x-1} + 1 = \frac{6x-x^2-6}{x-1}
\]

\[
\Rightarrow \frac{3x-4}{x-1} = \frac{6x-x^2-6}{x-1}, \quad x \neq 1
\]

\[
\Rightarrow 3x-4 = 6x-x^2-6, \quad x \neq 1
\]

\[
\Rightarrow x^2-3x+2 = 0, \quad x \neq 1
\]

\[
\Rightarrow x = 2
\]

**Extraneous Roots**

While simplifying the equation, the domain of the equation may expand and give the extraneous roots.

For example, consider the equation \( \sqrt{x} = x - 2 \). For solving, we first square it

\[
\sqrt{x} = x - 2
\]

\[
\Rightarrow x = (x-2)^2 \quad \text{[on squaring both sides]}
\]

\[
\Rightarrow x^2 - 5x + 4 = 0
\]

\[
\Rightarrow (x-1)(x-4) = 0
\]

\[
\Rightarrow x = 1, 4
\]

We observe that \( x = 4 \) satisfies the given equation but \( x = 1 \) does not satisfy it.

**Hence, \( x = 4 \) is the only solution of the given equation.**

The domain of actual equation is \([2, \infty)\).

While squaring the equation, domain expands to \( \mathbb{R} \), which gives extra root \( x = 1 \).

**Loss of Root**

Cancellation of common factors from both sides of equation leads to loss of root.

For example, consider an equation \( x^2 - 2x = x - 2 \)

\[
\Rightarrow x(x-2) = x - 2
\]

\[
\Rightarrow x = 1
\]

Here we have cancelled factor \( x - 2 \) which causes the loss of root, \( x = 2 \).

The correct way of solving is:

\[
x^2 - 2x = x - 2
\]

\[
x^2 - 3x + 2 = 0
\]

\[
(x-1)(x-2) = 0
\]

\[
\Rightarrow x = 1 \text{ and } x = 2
\]

**Graphs of Polynomial Functions**

When the polynomial function is written in standard form, \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) (\( a_n \neq 0 \)), the leading term is \( a_nx^n \). In other words, the leading term is the term that the variable has its highest exponent. The degree of a term of a polynomial function is the exponent on the variable. The degree of the polynomial is the largest degree of all of its terms.

For drawing the graph of the polynomial function, we consider the following tests.
If \( n \) is even and the leading coefficient \( a_n \) is negative, the graph falls to the left and to the right.

**Test 2: Roots (Zeros) of Polynomial**

In other words, when a polynomial function is set equal to zero and has been completely factored and each different factor is written with the highest appropriate exponent, depending on the number of times that factor occurs in the product, the exponent on the factor that the zero is a solution for it gives the multiplicity of that zero.

The exponent indicates how many times that factor would be written out in the product, this gives us a multiplicity.

**Multiplicity of Zeros and the \( x \)-intercept**

If \( r \) is a zero of even multiplicity:
This means the graph touches the \( x \)-axis at \( r \) and turns around.
This happens because the sign of \( f(x) \) does not change from one side to the other side of \( r \).
See the graph of \( f(x) = (x - 2)^2(x - 1)(x + 1) \).

If \( r \) is a zero of odd multiplicity:
This means the graph crosses (also touches if exponent is more than 1) the \( x \)-axis at \( r \). This happens because the sign of \( f(x) \) changes from one side to the other side or \( r \).
See the graph of \( f(x) = (x - 1)(3x - 2)(x - 3)^3 \).

Thus, in general, polynomial function graphs consist of a smooth line with a series of hills and valleys. The hills and valleys are called **turning points**. The maximum possible number of turning points is one less than the degree of the polynomial.
The point where graph has turning point, derivative of function \( f(x) \) becomes zero, which provides point of local minima or local maxima. Knowledge of derivative provides great help in drawing the graph of the function, hence finding its point of intersection with \( x \)-axis or roots of the equation \( f(x) = 0 \). Also we know that geometrically the derivative of function at any point of the graph of the function is equal to the slope of tangent at that point to the curve.

Consider the following graph of the function \( y = f(x) \) as shown in the following figure.
In the figure, we can see that tangent to the curve at point for which \( x < -1 \) and \( x > 1 \) makes acute angle with the positive direction of \( x \)-axis, hence derivative is positive for these points. For \( -1 < x < 1 \), tangent to the curve makes obtuse angle with the positive direction of \( x \)-axis, hence derivative is negative at these points. At \( x = -1 \) and \( x = 1 \), tangent is parallel to \( x \)-axis, where derivative is zero.

Here \( x = -1 \) is called point of maxima, where derivative changes sign from positive to negative (from left to right), and \( x = 1 \) is called point of minima, where derivative changes sign from negative to positive (from left to right).

At point of maxima and minima, derivative of the function is zero.

**Example 1.67** Using differentiation method check how many roots of the equation \( x^3 - x^2 + x - 2 = 0 \) are real?

**Sol.** Let \( y = f(x) = x^3 - x^2 + x - 2 \)

\[
\frac{dy}{dx} = 3x^2 - 2x + 1
\]

Let \( 3x^2 - 2x + 1 = 0 \), now this equation has non-real roots, i.e., derivative never becomes zero or graph of \( y = f(x) \) has no turning point.

Also when \( x \to \infty, f(x) \to \infty \) and when \( x \to -\infty, f(x) \to -\infty \)

Further \( 3x^2 - 2x + 1 > 0 \) \( \forall \) \( x \in \mathbb{R} \)

Thus graph of the function is as shown in the following figure.

Also \( f(0) = -2 \); hence graph cuts the \( x \)-axis for some positive value of \( x \).

Hence, the only root of the equation is positive.

Thus we can see that differentiation and then graph of the function is much important in analyzing the equation.

**Example 1.68** Analyze the roots of the following equations:

(i) \( 2x^3 - 9x^2 + 12x - (9/2) = 0 \)

(ii) \( 2x^3 - 9x^2 + 12x - 3 = 0 \)

**Sol.**

(i) Let \( f(x) = 2x^3 - 9x^2 + 12x - (9/2) \)

Then \( f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2) \)

Now \( f(x) = 0 \Rightarrow x = 1 \) and \( x = 2 \).

Hence, graph has turn at \( x = 1 \) and at \( x = 2 \).

Also \( f(1) = 2 - 9 + 12 - (9/2) > 0 \)

and \( f(2) = 16 - 36 + 24 - (9/2) < 0 \)

Hence, graph of the function \( y = f(x) \) is as shown in the following figure.

![Fig. 1.36(a)](image)

As shown in the figure, graph cuts \( x \)-axis at three distinct point.

Hence, equation \( f(x) = 0 \) has three distinct roots.

(ii) For \( 2x^3 - 9x^2 + 12x - 3 = 0 \), \( f(x) = 2x^3 - 9x^2 + 12x - 3 \)

\( f'(x) = 0 \Rightarrow x = 1 \) and \( x = 2 \)

Also \( f(1) = 2 - 9 + 12 - 3 = 2 \) and \( f(2) = 16 - 36 + 24 - 3 = 1 \)

Hence, graph of \( y = f(x) \) is as shown in the following figure.
Thus from the graph, we can see that \( f(x) = 0 \) has only one real root, though \( y = f(x) \) has two turning points.

**Example 1.69** Find how many roots of the equation \( x^4 + 2x^2 - 8x + 3 = 0 \) are real.

**Sol.** Let \( f(x) = x^4 + 2x^2 - 8x + 3 \)
\[ f'(x) = 4x^3 + 4x - 8 = 4(x - 1)(x^2 + x + 2) \]
Now \( f'(x) = 0 \Rightarrow x = 1 \)
Hence graph of \( y = f(x) \) has only one turn (maxima/minima).
Now \( f(1) = 1 + 2 - 8 + 3 < 0 \)
Also when \( x \to \pm \infty, f(x) \to \infty \)
Then graph of the function is as shown in the following figure.

**Fig. 1.37**

Hence, equation \( f(x) = 0 \) has only two real roots.

**EQUATIONS REDUCIBLE TO QUADRATIC**

**Example 1.70** Solve \( \sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1 \).

**Sol.** Let \( 5x^2 - 6x = y \). Then,
\[ \sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1 \]

\[ y = \sqrt{y^2 + y - 56} \]
\[ y^2 = y^2 + y - 56 \]
\[ y = y - 56 \]
\[ y = 56 \]
\[ 5x^2 - 6x = 56 \]
\[ 5x^2 - 6x - 56 = 0 \]
\[ (5x + 14)(x - 4) = 0 \]
\[ x = 4, -\frac{14}{5} \]
Clearly, both the values satisfy the given equation. Hence, the roots of the given equation are 4 and \(-14/5\).

**Example 1.71** Solve \((x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1\).

**Sol.** We have,
\[ (x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1 \]
\[ (x^2 - 5x + 7)^2 - (x^2 - 5x + 7) = 0 \]
\[ y^2 - y = 0, \text{ where } y = x^2 - 5x + 7 \]
\[ y(y - 1) = 0 \]
\[ y = 0, 1 \]
Now,
\[ y = 0 \]
\[ x^2 - 5x + 7 = 0 \]
\[ \Rightarrow x = \frac{5 \pm \sqrt{25 - 28}}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5 \pm i\sqrt{3}}{2} \]

where \( i = \sqrt{-1} \) and
\[ y = 1 \]
\[ x^2 - 5x + 6 = 0 \]
\[ (x - 3)(x - 2) = 0 \]
\[ 3, 2 \]
Hence, the roots of the equation are 2, 3, \((5 + i\sqrt{3})/2\) and \((5 - i\sqrt{3})/2\).

**Example 1.72** Solve the equation \( 4^t - 5 \times 2^t + 4 = 0 \).

**Sol.** We have,
\[ 4^t - 5 \times 2^t + 4 = 0 \]
\[ (2^t)^2 - 5(2^t) + 4 = 0 \]
\[ y^2 - 5y + 4 = 0, \text{ where } y = 2^t \]
\[ (y - 1)(y - 4) = 0 \]
\[ y = 1, 4 \]
\[ 2^t = 1, 2^t = 4 \]
\[ t = 0, 2 \]
Hence, the roots of the given equation are 0 and 2.

**Example 1.73** Solve the equation \( 12x^3 - 56x^2 + 89x^2 - 56x + 12 = 0 \).

**Sol.** The given equation is
\[ 12x^3 - 56x^2 + 89x^2 - 56x + 12 = 0 \]


Examples of solving and evaluating expressions.

Example 1.76: Solve the equation \((x + 2) (x + 3) (x + 8) = 4x^2\).

Sol. \((x + 2) (x + 3) (x + 8) = 4x^2\)

\[\Rightarrow (x + 2) (x + 3) = 4\]

\[\Rightarrow x^2 + 5x + 6 = 0\]

\[\Rightarrow (x + 2) (x + 3) = 0\]

\[\Rightarrow x = -2, -3\]

Example 1.74: Solve the equation \(3x^2 + 4x^2 = 25\).

Sol. \(3x^2 + 4x^2 = 25\)

\[\Rightarrow 7x^2 = 25\]

\[\Rightarrow x^2 = \frac{25}{7}\]

\[\Rightarrow x = \pm 5/\sqrt{7}\]

Example 1.75: Solve the equation \((x - 1)^4 + (x - 5)^4 = 82\).

Sol. Let \(y = (x - 1) + (x - 5) = x - 3\)

\[\Rightarrow y = x + 3\]

Putting \(x = y + 3\) in the given equation, we obtain \((y + 2)^4 + (y - 2)^4 = 82\)

\[\Rightarrow (y^2 + 4y + 4)^2 + (y^2 - 4y + 4)^2 = 82\]

\[\Rightarrow 2(y^2 + 4y + 4)^2 = 82\]

\[\Rightarrow y^4 + 8y^2 + 16 = 41\]

\[\Rightarrow y^4 + 4y^2 - 25 = 0\]

\[\Rightarrow (y^2 + 5)^2 - 25 = 0\]

\[\Rightarrow y^2 + 5 = \pm 5\]

\[\Rightarrow y = \pm 5\]

Hence, the roots of the given equation are \(3\pm 5\), \(2\), \(-2\), and \(4\).
Example 1.78: Solve \( \sqrt{x + 5} + \sqrt{x + 21} = \sqrt{6x + 40} \).

Solution:
\[
\sqrt{x + 5} + \sqrt{x + 21} = \sqrt{6x + 40} \\
\Rightarrow (\sqrt{x + 5} + \sqrt{x + 21})^2 = 6x + 40 \\
\Rightarrow (x + 5) + (x + 21) + 2 \sqrt{(x + 5)(x + 21)} = 6x + 40 \\
\Rightarrow (x + 5)(x + 21) = 2x + 7 \\
\Rightarrow 3x^2 + 2x - 56 = 0 \\
\Rightarrow (3x + 14)(x - 4) = 0 \\
\Rightarrow x = 4 \text{ or } x = -\frac{14}{3} \\
\]
Clearly, \( x = -\frac{14}{3} \) does not satisfy the given equation. Hence, \( x = 4 \) is the only root of the given equation.

Concept Application Exercise 1.2

1. Prove that the graph of \( y = x^2 + 2 \) and \( y = 3x - 4 \) never intersect.
2. In how many points the line \( y + 14 = 0 \) cuts the curve whose equation is \( x(x^2 + x + 1) + y = 0 \)?
3. Consider the following graphs:

![Graph](image)

Answer the following questions:
- (i) sum of roots of the equation \( f(x) = 0 \)
- (ii) product of roots of the equation \( f(x) = 4 \)
- (iii) the absolute value of the difference of the roots of equation \( f(x) = x + 2 \)

4. Solve \( x^2 + 3x + 2 = 0 \).
5. Solve \( \sqrt{x - 2} + \sqrt{4 - x} = 2 \).
6. Solve \( x - 2(x^2 - 4x - 5) = 0 \).
7. Solve the equation \( x(x + 2)(x^2 - 1) = -1 \).

8. Find the value of \( \frac{2 + \frac{1}{2}}{2 + \frac{1}{2} + \frac{1}{2} + \cdots} \).
9. Solve \( 4^4 + 6^4 = 9^4 \).
10. Solve \( 3^2 + 7^2 + 7^2 = 9 \).
11. Find the number of real roots of the equation \( (x - 1)^3 + (x - 2)^3 + (x - 3)^3 = 0 \).
12. Solve \( \sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5 \).
13. If \( x = \sqrt{7 + 4\sqrt{3}} \), prove that \( x + 1/x = 4 \).
14. Solve \( \sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1 \).
15. Solve \( \sqrt{x^2 + 4x - 21} + \sqrt{x^2 - x - 6} = \sqrt{5x^2 - 5x - 39} \).
16. How many roots of the equation \( 3x^6 + 6x^4 + x^2 + 6x + 3 = 0 \) are real?
17. Find the value of \( k \) if \( x^2 - 3x + a = 0 \) has three real distinct roots.
18. Analyze the roots of the equation \( (x - 1)^3 + (x - 2)^3 + (x - 3)^3 + (x - 4)^3 + (x - 5)^3 = 0 \) by differentiation method.
19. In how many points the graph of \( f(x) = x^2 + 2x^2 + 3x + 4 \) meets \( x \)-axis.

REMAINDER AND FACTOR THEOREMS

Remainder Theorem

The remainder theorem states that if a polynomial \( f(x) \) is divided by a linear function \( x - k \), then the remainder is \( f(k) \).

Proof:

In any division,

\[
\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}
\]

Let \( Q(x) \) be the quotient and \( R \) be the remainder. Then,

\[
f(x) = (x - k) Q(x) + R
\]

\[
f(k) = (k - k) Q(x) + R = 0 + R = R
\]

Note: If a \( n \)-degree polynomial is divided by a \( m \)-degree polynomial, then the maximum degree of the remainder polynomial is \( m - 1 \).

Example 1.79: Find the remainder when \( x^3 + 4x^2 - 7x + 6 \) is divided by \( x - 1 \).

Solution: Let \( f(x) = x^3 + 4x^2 - 7x + 6 \). The remainder when \( f(x) \) is divided by \( x - 1 \) is

\[
f(1) = 1^3 + 4 \times 1^2 - 7 \times 1 + 6 = 4
\]

Example 1.80: If the expression \( ax^4 + bx^3 - x^2 + 2x + 3 \) has remainder \( 4x + 3 \) when divided by \( x^2 + x - 2 \), find the value of \( a \) and \( b \).

Solution: Let \( f(x) = ax^4 + bx^3 - x^2 + 2x + 3 \). Now, \( x^2 + x - 2 = (x + 2)(x - 1) \).

Given, \( f(-2) = a(-2)^4 + b(-2)^3 - (-2)^2 - (-2)^2 + 2(-2) + 3 \)}
\[ \begin{align*}
&= 4(-2) + 3 \\
&= 16a - 8b - 4 - 4 + 3 = -5 \\
&\Rightarrow 2a - b = 0 \\
\text{Also,} \\
&f(1) = a + b - 1 + 2 + 3 = 4(1) + 3 \\
&\Rightarrow a + b = 3 \\
\text{From (1) and (2), } a = 1, b = 2.
\end{align*} \]

**Factor Theorem**

**Factor Theorem Is a Special Case of Remainder Theorem**

Let,
\[ f(x) = (x - k) Q(x) + R \]
\[ \Rightarrow f(x) = (x - k) Q(x) + f(k) \]

When \( f(k) = 0, f(x) = (x - k) Q(x). \) Therefore, \( f(x) \) is exactly divisible by \( x - k. \)

**Example 1.81** Given that \( x^2 + x - 6 \) is a factor of \( 2x^4 + x^3 - ax^2 + bx + a + b - 1, \) find the values of \( a \) and \( b. \)

**Sol.** We have,
\[ x^2 + x - 6 = (x + 3)(x - 2). \]
Let,
\[ f(x) = 2x^4 + x^3 - ax^2 + bx + a + b - 1 \]
Now,
\[ f(-3) = 2(-3)^4 + (-3)^3 - a(-3)^2 - 3b + a + b - 1 = 0 \]
\[ \Rightarrow 134 - 8a - 2b = 0 \]
\[ \Rightarrow 4a + b = 67 \]
\[ f(2) = 2(2)^4 + 2^3 - a(2)^2 + 2b + a + b - 1 = 0 \]
\[ \Rightarrow 39 - 3a + 3b = 0 \]
\[ \Rightarrow a - b = 13 \]
From (1) and (2), \( a = 16, b = 3. \)

**Example 1.82** Use the factor theorem to find the value of \( k \) for which \( (a + 2b), \) where \( a, b \neq 0 \) is a factor of \( a^4 + 32b^4 + a^2b(k + 3). \)

**Sol.** Let \( f(a) = a^4 + 32b^4 + a^2b(k + 3). \)
Now,
\[ f(-2b) = (-2b)^4 + 32b^4 + (-2b)^2b(k + 3) = 0 \]
\[ \Rightarrow 8b^4 - 8b^4(k + 3) = 0 \]
\[ \Rightarrow 8b^4(1 - k) = 0 \]
Since \( b \neq 0, \) so \( 3 - k = 0 \) or \( k = 3. \)

**Example 1.83** If \( c, d \) are the roots of the equation \( (x - a)(x - b) - k = 0, \) prove that \( a, b \) are the roots of the equation \( (x - c)(x - d) + k = 0. \)

**Sol.** Since \( c \) and \( d \) are the roots of the equation \( (x - a)(x - b) - k = 0, \) therefore,
\[ (x - a)(x - b) - k = (x - c)(x - d) \]
\[ \Rightarrow (x - a)(x - b) = (x - c)(x - d) + k \]

\[ \Rightarrow (x - c)(x - d) + k = (x - a)(x - b) \]
Clearly, \( a \) and \( b \) are roots of the equation \( (x - a)(x - b) = 0. \)
Hence, \( a, b \) are roots of \( (x - c)(x - d) + k = 0. \)

**Concept Application Exercise 1.3**

1. Given that \( 2x^3 + 3x^2 - 4x + p \) has a remainder of \( 5 \) when divided by \( x + 2, \) find the value of \( p. \)
2. Determine the value of \( k \) for which \( x + 2 \) is a factor of \( (x + 1)^2 + (2x + b)^2. \)
3. Find the value of \( p \) for which \( x + 1 \) is a factor of \( x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p - 9)x + 6. \)
4. Find the remaining factors for this value of \( p. \)
5. If \( x^2 + ax + 1 \) is a factor of \( ax^3 + bx + c, \) then find the conditions.
6. If \( x^2 - x^2 + x + b \) is divisible by \( x - 2, \) then find the value of \( f(2). \)

**Identity**

A relation which is true for every value of the variable is called an identity.

**Example 1.84** If \( (a^2 - 1)x^3 + (a - 1)x + a^2 - 4a + 3 = 0 \) be an identity in \( x, \) then find the value of \( a. \)

**Sol.** The given relation is satisfied for all real values of \( x, \) so all the coefficients must be zero. Then,
\[ a^2 - 1 = 0 \Rightarrow a = \pm 1 \]
\[ a - 1 = 0 \Rightarrow a = 1 \]
\[ a^2 - 4a + 3 = 0 \Rightarrow a = 1, 3 \]

**Example 1.85** Show that \( \frac{(x + b)(x + c)}{(a - c)(b - c)} + \frac{(x + c)(x + a)}{(b - a)(c - a)} + \frac{(x + a)(x + b)}{(a - c)(b - c)} = 1 \) is an identity.

**Sol.** Given relation is
\[ \frac{(x + b)(x + c)}{(a - c)(b - c)} + \frac{(x + c)(x + a)}{(b - a)(c - a)} + \frac{(x + a)(x + b)}{(a - c)(b - c)} = 1 \] (1)

When \( x = -a, \)
\[ \text{L.H.S.} = \frac{(b - a)(a - c)}{(b - a)(c - a)} = 1 = R.H.S. \]

Similarly, when \( x = -b, \)
\[ \text{L.H.S.} = \frac{(c - b)(a - b)}{(c - b)(a - b)} = 1 = R.H.S. \]

When \( x = -c, \)
\[ \text{L.H.S.} = \frac{(a - c)(b - c)}{(a - c)(b - c)} = 1 = R.H.S. \]
Thus, the highest power of $x$ occurring in relation (1) is 2 and this relation is satisfied by three distinct values $a$, $b$, and $c$ of $x$; therefore, it is an equation but an identity.

**Example 1.86** A certain polynomial $P(x)$, $x \in R$ when divided by $x-a$, $x-b$ and $x-c$ leaves remainders $a$, $b$, and $c$, respectively. Then find the remainder when $P(x)$ is divided by $(x-a) (x-b) (x-c)$ where $a$, $b$, $c$ are distinct.

**Sol.** By remainder theorem, $P(a) = a$, $P(b) = b$ and $P(c) = c$.

Let the required remainder be $R(x)$. Then,

$$P(x) = (x-a) (x-b) (x-c) Q(x) + R(x)$$

where $R(x)$ is a polynomial of degree at most 2. We get $R(a) = a$, $R(b) = b$ and $R(c) = c$. So, the equation $R(x) - x = 0$ has three roots $a$, $b$, and $c$. But its degree is at most 2. So, $R(x) - x$ must be zero polynomial (or identity). Hence $R(x) = x$.

**QUADRATIC EQUATION**

**Quadratic Equation with Real Coefficients**

Consider the quadratic equation

$$ax^2 + bx + c = 0$$

where $a$, $b$, $c \in R$ and $a \neq 0$.

Roots of the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, we observe that the nature of the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by $D$ and is known as the discriminant of the quadratic equation [Eq. (1)]

We also observe the following results:

$$D = 0 \quad \Rightarrow \quad \text{Roots are equal: } a = \beta = -b/2a$$

$$D \neq 0 \quad \Rightarrow \quad \text{Roots are unequal}$$

For $a$, $b$, $c \in R$ and $D > 0$ Roots are real and distinct

For $a$, $b$, $c \in R$ and $D < 0$ Roots are imaginary: $a = p + iq$, $\beta = p - iq$

where $i = \sqrt{-1}$

**Example 1.87** If $a, b, c \in R$ and $2b = a + c$, then check the nature of roots of equation $ax^2 + 2bx + c = 0$.

**Sol.** Given equation is $ax^2 + 2bx + c = 0$. Hence,

$$D = 4b^2 - 4ac$$

$$= (a + c)^2 - 4ac$$

$$= (a - c)^2 > 0$$

Thus, the roots are real and distinct.

**Example 1.88** If the roots of the equation $a(b - c)x^2 + bx(c-a)x + c(a-b) = 0$ are equal, show that $2b = 1/a + 1/c$.

**Sol.** Since the roots of the given equation are equal, therefore its discriminant is zero, i.e.,

$$b^2 (c-a)^2 - 4a (b-c) c(a-b) = 0$$

$$\Rightarrow b^2 (c^2 + a^2 - 2ac) - 4ac (ba - ca - b^2 + bc) = 0$$

$$\Rightarrow a^2b^2 + b^2c^2 - 4a^2c^2 + 2b^2 ac - 4a^2 bc - 4abc^2 = 0$$

$$\Rightarrow (ab + bc - 2ac)^2 = 0$$

$$\Rightarrow ab + bc - 2ac = 0$$

$$\Rightarrow \frac{1}{c} = \frac{2}{b} = \frac{1}{a} \quad \text{[Dividing both sides by abc]}$$

**Example 1.89** Prove that the roots of the equation $(a^4 + b^4)x^2 + 4abcdx + (c^4 + d^4) = 0$ cannot be different, if real.

**Sol.** The discriminant of the given equation is

$$D = 16a^2b^2c^2d^2 - 4(a^4 + b^4)(c^4 + d^4)$$

$$= -4[(a^4 + b^4)(c^4 + d^4) - 4a^2c^2b^2d^2]$$

$$= -4[(a^4c^4 + b^4d^4 + b^4c^4 + a^4d^4) - 4a^2c^2b^2d^2]$$

$$= -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2]$$

Since roots of the given equation are real, therefore

$$D \geq 0$$

$$\Rightarrow -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \geq 0$$
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\[ (a^2c^2 - b^2d^2) + (a^2d^2 - b^2c^2) \leq 0 \]

\[ (a^2c^2 - b^2d^2) + (a^2d^2 - b^2c^2) = 0 \quad (2) \]

(since sum of two positive quantities cannot be negative)

From (1) and (2), we get \( D = 0 \). Hence, the roots of the given quadratic equation are not different, if real.

**Example 1.90**  If the roots of the equation \( x^2 - 8x + a^2 - 6a = 0 \) are real distinct, then find all possible values of \( a \).

**Sol.** Since the roots of the given equation are real and distinct, we must have

\[ D > 0 \]

\[ 64 - 4(a^2 - 6a) > 0 \]

\[ 4[16 - a^2 + 6a] > 0 \]

\[ -4(a^2 - 6a - 16) > 0 \]

\[ a^2 - 6a - 16 < 0 \]

\[ (a - 8)(a + 2) < 0 \]

\[ -2 < a < 8 \]

Hence, the roots of the given equation are real if \( a \) lies between -2 and 8.

**Example 1.91**  Find the quadratic equation with rational coefficients whose one root is \( 1/(2 + \sqrt{5}) \).

**Sol.** If the coefficients are rational, then irrational roots occur in conjugate pair. Given that if one root is \( a = 1/(2 + \sqrt{5}) = \sqrt{5} - 2 \), then the other root is \( b = 1/(2 - \sqrt{5}) = -(2 + \sqrt{5}) \).

Sum of roots \( a + b = -4 \) and product of roots \( ab = -1 \). Thus, required equation is \( x^2 + 4x - 1 = 0 \).

**Example 1.92**  If \( f(x) = ax^2 + bx + c \), \( g(x) = -ax^2 + bx + c \), where \( ac \neq 0 \), then prove that \( f(x) g(x) = 0 \) has at least two real roots.

**Sol.** Let \( D_1 \) and \( D_2 \) be discriminants of \( ax^2 + bx + c = 0 \) and \( -ax^2 + bx + c = 0 \), respectively. Then,

\[ D_1 = b^2 - 4ac, \quad D_2 = b^2 + 4ac \]

Now, \[ ac \neq 0 \] which implies either \( ac > 0 \) or \( ac < 0 \).

If \( ac > 0 \), then \( D_1 > 0 \). Therefore, roots of \( -ax^2 + bx + c = 0 \) are real.

If \( ac < 0 \), then \( D_1 < 0 \). Therefore, roots of \( ax^2 + bx + c = 0 \) are real.

Thus, \( f(x) g(x) \) has at least two real roots.

**Example 1.93**  If \( a, b, c \in R \) such that \( a + b + c = 0 \) and \( a \neq c \), then prove that the roots of \( (b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0 \) are real and distinct.

**Sol.** Given equation is

\[ (b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0 \]

or

\[ (-2a)x^2 + (-2b)x + (-2c) = 0 \]

or

\[ ax^2 + bx + c = 0 \]

\[ D = b^2 - 4ac \]

\[ = (-c - a)^2 - 4ac \]

\[ = (c - a)^2 > 0 \]

Hence, roots are real and distinct.

**Example 1.94**  If \( \cos \theta, \sin \phi, \sin \theta \) are in G.P., then check the nature of roots of \( x^2 + 2 \cot \phi x + 1 = 0 \).

**Sol.** We have,

\[ \sin^2 \phi = \cos \theta \sin \theta \]

The discriminant of the given equation is

\[ D = 4 \cot^2 \phi - 4 \]

\[ = 4 \left[ \frac{\cos^2 \phi \sin^2 \phi}{\sin^2 \phi} \right] \]

\[ = 4 \left[ 1 - 2 \sin^2 \phi \right] \]

\[ = \frac{4(1 - 2 \sin \phi \cos \theta)}{\sin^2 \phi} \]

\[ = \frac{2(\sin \theta \cos \phi - \cos \theta)^2}{\sin^2 \phi} \]

\[ \geq 0 \]

**Example 1.95**  If \( a, b \) and \( c \) are odd integers, then prove that roots of \( ax^2 + bx + c = 0 \) cannot be rational.

**Sol.** Discriminant \( D = b^2 - 4ac \). Suppose the roots are rational. Then, \( D \) will be a perfect square.

Let \( b^2 - 4ac = d^2 \). Since \( a, b \) and \( c \) are odd integers, \( d \) will be odd. Now,

\[ d^2 = 4ac \]

Let \( b = 2k + 1 \) and \( d = 2m + 1 \). Then

\[ b^2 - d^2 = (b - d)(b + d) \]

\[ = 2(k - m)(2k + m + 1) \]

Now, either \( (k - m) \) or \( (k + m + 1) \) is always even. Hence \( b^2 - d^2 \) is always a multiple of 8. But, \( 4ac \) is only a multiple of 4 (not of 8), which is a contradiction. Hence, the roots of \( ax^2 + bx + c = 0 \) cannot be rational.

**Quadratic Equations with Complex Coefficients**

Consider the quadratic equation \( ax^2 + bx + c = 0 \), where \( a, b, c \) are complex numbers and \( a \neq 0 \). Roots of equation are given by

\[ \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

\[ \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

Here, nature of roots should not be analyzed by sign of \( b^2 - 4ac \).

**Note:** In case of quadratic equations with real coefficients, imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation \( 4x^2 - 4x + 1 = 0 \) has both roots equal to \( 1/2 \).
Concept Application Exercise 1.4

1. Find the values of \( a \) for which the roots of the equation \( x^2 + a^2 = 8x + 6a \) are real.

2. Find the condition if the roots of \( ax^2 + bx + c = 0 \) and \( bx^2 - 2cax + ab - b = 0 \) are simultaneously real.

3. If \( a < c < b \), then check the nature of roots of the equation \( (a - b)^2 x^2 + 2(a - b - 2c)x + 1 = 0 \).

4. If \( a + b + c = 0 \) then check the nature of roots of the equation \( 4ax^3 + 3bx + 2c = 0 \) where \( a, b, c \in \mathbb{R} \).

5. Find the greatest value of a non-negative real number \( \lambda \) for which both the equations \( 2x^2 + (\lambda - 1)x + 8 = 0 \) and \( x^2 - 8x + \lambda + 4 = 0 \) have real roots.

Relations Between Roots and Coefficients

Let \( \alpha \) and \( \beta \) be the roots of quadratic equation \( ax^2 + bx + c = 0 \). Then by factor theorem,

\[
\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\]

Comparing coefficients, we have \( \alpha + \beta = \frac{-b}{a} \) and \( \alpha \beta = \frac{c}{a} \).

Thus, we find that

\[
\alpha + \beta = -\frac{b}{a} = \text{coef of} \ x \quad \text{and} \quad \alpha \beta = \frac{c}{a} = \text{constant term of} \ x^2
\]

Also, if sum of roots is \( S \) and product of roots is \( P \), then quadratic equation is given by \( x^2 - Sx + P = 0 \).

Example 1.96

Form a quadratic equation whose roots are \(-4\) and \(6\).

Sol. We have sum of the roots, \( S = -4 + 6 = 2 \) and product of the roots, \( P = -4 \times 6 = -24 \). Hence, the required equation is

\[
x^2 - Sx + P = 0 \Rightarrow x^2 - 2x - 24 = 0
\]

Example 1.97

Form a quadratic equation with real coefficients whose one root is \(3 - 2i\).

Sol. Since the complex roots always occur in pairs, so the other root is \(3 + 2i\). The sum of the roots is \((3 + 2i) + (3 - 2i) = 6\). The product of the roots is \((3 + 2i)(3 - 2i) = 9 - 4(2^2) = 9 - 4 = 5\).

Hence, the equation is

\[
x^2 - Sx + P = 0 \Rightarrow x^2 - 6x + 13 = 0
\]

Example 1.98

If roots of the equation \( ax^2 + bx + c = 0 \) are \( \alpha \) and \( \beta \), find the equation whose roots are

(i) \( \frac{1}{\alpha}, \frac{1}{\beta} \)

(ii) \(-\alpha, -\beta\)

(iii) \(\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}\)

Sol. Here in all cases functions of \( \alpha \) and \( \beta \) are symmetric.

(i) Let \( \frac{1}{\alpha} = y \Rightarrow \alpha = \frac{1}{y} \)

(ii) \(-\alpha, -\beta\)

(iii) \(\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}\)

Example 1.100

If the roots of the quadratic equation \( x^2 + px + q = 0 \) are \( \tan 30^\circ \) and \( \tan 15^\circ \), respectively, then find the value of \( 2 + q - p \).

Sol. The equation \( x^2 + px + q = 0 \) has roots \( \tan 30^\circ \) and \( \tan 15^\circ \).

Therefore,

\[
\tan 30^\circ + \tan 15^\circ = p \quad (1)
\]

\[
\tan 30^\circ \tan 15^\circ = q \quad (2)
\]

Now,

\[
\tan 45^\circ = \tan(30^\circ + 15^\circ)
\]

\[
1 = \frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ}
\]

\[
1 = \frac{-p}{1-q} \quad [\text{Using (1) and (2)}]
\]

\[
1 - q = -p \Rightarrow q - p = 1
\]
Example 1.101 If the sum of the roots of the equation $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ is zero, then prove that the product of the roots is $(-\frac{1}{2})(a^2 + b^2)$.

Sol. We have,
\[
\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}
\]
\[
x^2 + (a + b - 2c)x + (ab - bc - ca) = 0
\]
Let $\alpha, \beta$ be the roots of this equation. Then,
\[
a + \beta = -(a + b - 2c)
\]
and $ab = bc - ca$

It is given that
\[
a + \beta = 0
\]
\[
\Rightarrow -(a + b - 2c) = 0
\]
\[
c = \frac{a + b}{2}
\]
\[
\Rightarrow ab = \frac{a^2 + b^2}{2}
\]

Example 1.102 Solve the equation $x^2 + px + 45 = 0$. It is given that the squared difference of its roots is equal to 144.

Sol. Let $\alpha, \beta$ be the roots of the equation $x^2 + px + 45 = 0$.

Then,
\[
\alpha + \beta = -p
\]
\[
\alpha \beta = 45
\]

It is given that
\[
(\alpha - \beta)^2 = 144
\]
\[
\Rightarrow (\alpha + \beta)^2 - 4\alpha \beta = 144
\]
\[
\Rightarrow p^2 - 4 \times 45 = 144
\]
\[
\Rightarrow p^2 = 324
\]
\[
\Rightarrow p = \pm 18
\]

Substituting $p = 18$ in the given equation, we obtain
\[
x^2 + 18x + 45 = 0
\]
\[
\Rightarrow (x + 3)(x + 15) = 0
\]
\[
x = -3, -15
\]

Substituting $p = -18$ in the given equation, we obtain
\[
x^2 + 18x + 45 = 0
\]
\[
\Rightarrow (x - 3)(x - 15) = 0
\]
\[
x = 3, 15
\]

Hence, the roots of the given equation are $-3, -15$ or $3, 15$.

Example 1.103 If the ratio of the roots of the equation $x^2 + px + q = 0$ are equal to the ratio of the roots of the equation $x^2 + bx + c = 0$, then prove that $p^2 = b^2q$.

Sol. Let $\alpha, \beta$ be the roots of $x^2 + px + q = 0$ and $\gamma, \delta$ be the roots of the equation $x^2 + bx + c = 0$. Then,
\[
\alpha + \beta = -p, \quad \beta = q
\]
\[
\gamma + \delta = -b, \quad \gamma = c
\]
We have,
\[
\frac{\alpha - \gamma}{\beta - \delta} \Rightarrow \frac{\alpha + \beta}{\gamma + \delta} \Rightarrow \frac{\gamma + \delta}{\alpha - \beta} = \gamma - \delta
\]

[Using componendo and dividendo]
\[
\Rightarrow \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2} = \frac{q^2 - 4\alpha \beta}{q^2 + 4\alpha \beta}
\]
\[
\Rightarrow \frac{(\alpha + \beta)^2 - 4\alpha \beta}{(\alpha + \beta)^2} = \frac{q^2 - 4\alpha \beta}{q^2 + 4\alpha \beta}
\]
\[
\Rightarrow 1 - 4\alpha \beta \Rightarrow \frac{4\alpha \beta}{(\alpha + \beta)^2} = \frac{q^2 - 4\alpha \beta}{q^2 + 4\alpha \beta}
\]
\[
\Rightarrow \frac{\alpha \beta}{(\alpha + \beta)^2} = \frac{q^2}{(\alpha + \beta)^2} - \frac{4\alpha \beta}{(\alpha + \beta)^2}
\]
\[
\Rightarrow \frac{\alpha \beta}{b^2} \Rightarrow p^2 = b^2 \Rightarrow p^2 = b^2 q
\]

Example 1.104 If $\sin \theta, \cos \theta$ be the roots of $ax^2 + bx + c = 0$, then prove that $b^2 = a^2 + 2ac$.

Sol. We have,
\[
\sin \theta + \cos \theta = \frac{b}{a}, \quad \sin \theta \cos \theta = \frac{c}{a}
\]

Now, we know that
\[
\sin^2 \theta + \cos^2 \theta = 1
\]
\[
\Rightarrow (\sin \theta + \cos \theta)^2 - 2\sin \theta \cos \theta = 1
\]
\[
\Rightarrow \frac{b^2}{a^2} = 1 + 2 \cdot \frac{c}{a} \Rightarrow b^2 = a^2 + 2ac
\]

Example 1.105 If $a$ and $b$ ($\neq 0$) are the roots of the equation $x^2 + ax + b = 0$, then find the least value of $x^2 + ax + b$ ($x \in R$).

Sol. Since $a$ and $b$ are the roots of the equation $x^2 + ax + b = 0$, so
\[
a + b = -a, \quad ab = b
\]

Now,
\[
ab = b \Rightarrow (a - 1)b = 0 \Rightarrow a = 1 \quad (\because b \neq 0)
\]
Putting $a = 1$ in $a + b = -a$, we get $b = -2$. Hence,
\[
x^2 + ax + b = x^2 + x - 2 = (x + 1/2)^2 - 1/4 - 2 = (x + 1/2)^2 - 9/4
\]
which has a minimum value $-9/4$.

Example 1.106 If the sum of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is $-1$, then find the product of the roots.

Sol. Let $\alpha, \beta$ be roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$. Then,
\[
\alpha + \beta = -\frac{2a + 3}{a + 1} = -1 \Rightarrow a = -2
\]

Now, product of the roots is $(3a + 4)(a + 1) = (6 + 4)(-2 + 1) = 2$. 
Example 1.107: Find the value of ‘a’ for which one root of the quadratic equation \((a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0\) is twice as large as the other.

Sol. Let the roots be \(\alpha\) and \(2\alpha\). Then,

\[
\frac{\alpha + 2\alpha}{a^2 - 5a + 3} = \frac{2}{a^2 - 5a + 3}
\]

\[
\Rightarrow 2 \left(1 - \frac{3a^2}{a^2 - 5a + 3}\right) = \frac{2}{a^2 - 5a + 3}
\]

\[
\Rightarrow \frac{(1-3a)^2}{a^2 - 5a + 3} = 9
\]

\[
\Rightarrow 9a^2 - 6a + 1 = 9a^2 - 45a + 27
\]

\[
\Rightarrow 39a = 26 \Rightarrow a = \frac{2}{3}
\]

Example 1.108: If the difference between the roots of the equation \(x^2 + ax + 1 = 0\) is less than \(\sqrt{5}\), then find the set of possible values of \(a\).

Sol. If \(\alpha, \beta\) are roots of \(x^2 + ax + 1 = 0\), then

\[
1 - a = \sqrt{(a + \beta)^2 - 4a\beta} < \sqrt{5}
\]

\[
\Rightarrow a^2 - 4 < \sqrt{5}
\]

\[
\Rightarrow a^2 - 4 < 5
\]

\[
\Rightarrow a^2 < 9
\]

\[
\Rightarrow a < 3
\]

\[
\therefore a \in (-3, 3)
\]

Example 1.109: Find the values of the parameter ‘a’ such that the roots \(\alpha, \beta\) of the equation \(2x^2 + 6x + a = 0\) satisfy the inequality \(a\beta + \beta\alpha < 2\).

Sol. We have \(\alpha + \beta = -3\) and \(\alpha\beta = a/2\). Now,

\[
\frac{\alpha + \beta}{\alpha\beta} < 2
\]

\[
\Rightarrow \frac{\alpha^2 - \beta^2}{\alpha\beta} < 2
\]

\[
\Rightarrow (\alpha + \beta)(\alpha - \beta) = 2 \alpha\beta < 2
\]

\[
\Rightarrow \frac{9 - a}{a/2} < 2
\]

\[
\Rightarrow \frac{9 - a}{a} < 1
\]

\[
\Rightarrow \frac{9 - a}{a} - 1 < 0
\]

\[
\Rightarrow \frac{2a - 9}{a} < 0
\]

\[
\Rightarrow a < 0 \text{ or } a > \frac{9}{2}
\]

Example 1.110: If the harmonic mean between roots of \((5 + \sqrt{2})x^2 - bx + 8 + 2\sqrt{5} = 0\) is 4, then find the value of \(b\).

Sol. Let \(\alpha, \beta\) be the roots of the given equation whose H.M. is 4. Then,

\[
4 = \frac{2\alpha\beta}{\alpha + \beta}
\]

\[
\Rightarrow 4 = \frac{2x}{\frac{5 + \sqrt{5}}{5 + \sqrt{5}}}
\]

\[
\Rightarrow 2 = \frac{8 + 2\sqrt{5}}{b}
\]

\[
\Rightarrow b = 4 + \sqrt{5}
\]

Example 1.111: If \(\alpha, \beta\) are the roots of the equation \(2x^2 - 3x - 6 = 0\), find the equation whose roots are \(\alpha^2 + 2\) and \(\beta^2 + 2\).

Sol. Since \(\alpha, \beta\) are roots of the equation \(2x^2 - 3x - 6 = 0\), so

\[
\alpha + \beta = 3/2 \text{ and } \alpha\beta = -3
\]

\[
\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{9}{4} + 6 = \frac{33}{4}
\]

Now,

\[
(\alpha^2 + 2) + (\beta^2 + 2) = (\alpha + \beta)^2 + 4 = \frac{33}{4} + 4 = \frac{49}{4}
\]

and

\[
(\alpha^2 + 2)(\beta^2 + 2) = (\alpha^2 + \beta^2) + 4 = \left(3\right)^2 + 2\left(\frac{33}{4}\right) + 4 = \frac{59}{2}
\]

So, the equation whose roots are \(\alpha^2 + 2\) and \(\beta^2 + 2\) is

\[
x^2 - x[(\alpha^2 + 2) + (\beta^2 + 2)] + (\alpha^2 + 2)(\beta^2 + 2) = 0
\]

\[
\Rightarrow x^2 - \frac{49}{4}x + \frac{59}{2} = 0
\]

\[
\Rightarrow 4x^2 - 49x + 118 = 0
\]

Example 1.112: If \(\alpha \neq \beta\) and \(\alpha^2 = 5\alpha - 3\) and \(\beta^2 = 5\beta - 3\), find the equation whose roots are \(\alpha/\beta\) and \(\beta/\alpha\).

Sol. We have \(\alpha^2 = 5\alpha - 3\) and \(\beta^2 = 5\beta - 3\). Hence, \(\alpha, \beta\) are roots of \(x^2 = 5x - 3\), i.e., \(x^2 - 5x + 3 = 0\). Therefore,

\[
\alpha + \beta = 5 \text{ and } \alpha\beta = 3
\]

Now,

\[
S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}
\]

\[
= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}
\]

\[
= \frac{25 - 6}{3} = \frac{19}{3}
\]
and 

\[ P = \frac{\alpha}{\beta} \frac{\beta}{\alpha} = 1 \]

So, the required equation is

\[ x^2 - Sx + P = 0 \]

\[ \Rightarrow \quad x^2 - \frac{19}{3} x + 1 = 0 \]

\[ \Rightarrow \quad 3x^2 - 19x + 3 = 0 \]

**Example 1.113** If \( \alpha, \beta \) are the roots of the equation \( ax^2 + bx + c = 0 \), then find the roots of the equation \( ax^2 - bx(x - 1) + c(x - 1)^2 = 0 \) in terms of \( \alpha \) and \( \beta \).

**Solution.** \( ax^2 - bx(x - 1) + c(x - 1)^2 = 0 \)

\[ \Rightarrow \quad \frac{ax^2}{x(a - x)} + \frac{bx}{1 - x} + \frac{c}{1 - x} = 0 \]

Now, \( \alpha \) is a root of \( ax^2 + bx + c = 0 \). Then let

\[ \alpha = \frac{x}{1 - x} \]

\[ \Rightarrow \quad x = \alpha(1 + \alpha) \]

Hence, the roots of (1) are \( \alpha/(1 + \alpha), \beta/(1 + \beta) \).

---

**Concept Application Exercise 1.5**

1. If the product of the roots of the equation \( (a + 1)x^2 + (2a + 3)x + (3a + 4) = 0 \) is 2, then find the sum of roots.
2. Find the value of \( \alpha \) for which the sum of the squares of the roots of the equation \( x^2 - (a - 2)x - 1 = 0 \) assumes the least value.
3. If \( x_1 \) and \( x_2 \) are the roots of \( x^2 + (\sin \theta - 1)x - 1/2 \cos^2 \theta = 0 \), then find the maximum value of \( x_1^2 + x_2^2 \).
4. If \( \tan \theta \) and \( \sec \theta \) are the roots of \( ax^2 + bx + c = 0 \), then prove that \( a^2 = b(4ac - b^2) \).
5. If the roots of the equation \( 12x^2 - mx + 5 = 0 \) are in the ratio \( 2:3 \), then find the value of \( m \).
6. If \( \alpha, \beta \) are the roots of \( x^2 + px + q = 0 \) and \( \gamma, \delta \) are the roots of \( x^2 + qx + r = 0 \), then prove that \( \alpha^2 + \beta^2 = (\alpha - \gamma)(\beta - \delta)(\alpha + \beta)(\beta + \delta) \).
7. If the equation formed by decreasing each root of \( ax^2 + bx + c = 0 \) by 1 is \( 2x^2 + 3x + 2 = 0 \), find the condition.
8. If \( \alpha, \beta \) be the roots of \( x^2 - (a - 1)x - b = 0 \), then find the value of \( \frac{1}{\alpha} + \frac{1}{\beta} \).
9. If \( \alpha, \beta \) be the roots of \( x^2 + ax + b = 0 \), then find the value of \( \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \).
10. If \( \alpha, \beta \) are roots of \( 375x^2 - 25x - 2 = 0 \) and \( \gamma, \delta \) are roots of \( \gamma + \delta = \alpha + \beta \), then find the value of \( \sum_{i=1}^{\infty} s_i \).
11. If \( \alpha \) and \( \beta \) are the roots of the equation \( 2x^2 + 2(a + b)x + a^2 + b^2 = 0 \), then find the equation whose roots are \( (\alpha + \beta)^2 \) and \( -\beta - \beta^2 \).
12. If the sum of the roots of an equation is 2 and sum of their cubes is 98, then find the equation.
13. Let \( \alpha, \beta \) be the roots of \( x^2 + bx + 1 = 0 \). Then find the equation whose roots are \( -\alpha(1 + \beta) \) and \( -\beta(1 + \alpha) \).

---

**COMMON ROOT(S)**

**Condition for One Common Root**

Let us find the condition that the quadratic equations \( a_1x^2 + b_1x + c_1 = 0 \) and \( a_2x^2 + b_2x + c_2 = 0 \) may have a common root. Let \( \alpha \) be the common root of the given equations. Then,

\[ a_1\alpha^2 + b_1\alpha + c_1 = 0 \]

and

\[ a_2\alpha^2 + b_2\alpha + c_2 = 0 \]

Solving these two equations by cross-multiplication, we have

\[ \alpha = \frac{b_2c_1 - b_1c_2}{a_2b_1 - a_1b_2} \quad \text{from first and third} \]

and

\[ \alpha = \frac{c_2a_1 - c_1a_2}{a_1b_2 - a_2b_1} \quad \text{from second and third} \]

\[ \Rightarrow \quad \alpha = \frac{(c_2a_1 - c_1a_2)^2}{b_2c_1 - b_1c_2} = (b_2c_1 - b_1c_2)(a_1b_2 - a_2b_1) \]

This condition can easily be remembered by cross-multiplication method as shown in the following figure.

![Fig. 1.40](image)

(Bigger cross product)²

= Product of the two smaller crosses

This is the condition required for a root to be common to two quadratic equations. The common root is given by

\[ \alpha = \frac{c_2a_1}{a_1b_2 - a_2b_1} \]

or

\[ \alpha = \frac{b_2c_1}{c_1a_2 - c_2a_1} \]

**Note:** The common root can also be obtained by making the coefficient of \( x^2 \) common to the two given equations and then subtracting the two equations. The other roots of the given equations can be determined by using the relations between their roots and coefficients.

**Condition for Both the Common Roots**

Let \( \alpha, \beta \) be the common roots of the quadratic equations \( a_1x^2 + b_1x + c_1 = 0 \) and \( a_2x^2 + b_2x + c_2 = 0 \). Then, both the equations are identical, hence,
Example 1.114 Determine the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

Sol. Let $\alpha$ be the common root of the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$. Then, $\alpha$ must satisfy both the equations. Therefore,

$3\alpha^2 + 4m\alpha + 2 = 0$
$2\alpha^2 + 3\alpha - 2 = 0$

Using cross-multiplication method, we have

$(-6 - 4)^2 = (9 - 8m)(-8m - 6)$
$50 = (8m - 9)(4m + 3)$
$32m^2 - 12m - 77 = 0$
$32m^2 - 56m + 44m - 77 = 0$
$8m(4m - 7) + 11(4m - 7) = 0$
$(8m + 11)(4m - 7) = 0$
$m = \frac{11}{8}, \frac{7}{4}$

Example 1.115 If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have common roots and $a, b, c \in \mathbb{N}$, then find the minimum value of $a + b + c$.

Sol. The roots of $x^2 + 3x + 5 = 0$ are non-real. Thus given equations will have two common roots. We have,

$a_1 = b = c = \lambda$
$a_2 = \frac{1}{3}, \frac{5}{3}$
$a_1 + b + c = 9\lambda$
Thus minimum value of $a + b + c$ is 9.

Example 1.116 If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and $a, b$ and $c$ are non-zero real numbers then find the value of $(a^2 + b^2 + c^2)abc$.

Sol. Given that $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Hence,

$(bc - a^2)^2 = (ab - c^2)(ac - b^2)$
$b^2c^2 + a^4 - 2a^2bc = a^2b^2 - ab^2 - ac^2 + b^2c^2$
$a^4 + ab^2 + ac^2 = 3a^2bc$

Example 1.117 $a, b, c$ are positive real numbers forming a G.P. If $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then prove that $a, b, c$ are in A.P.

Sol. For first equation $D = 4b^2 - 4ac = 0$ (as given $a, b, c$ are in G.P.). The equation has equal roots which are equal to $-2b/a$ each. Thus, it should also be the root of the second equation. Hence,

$d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$
$d\frac{b^2}{a^2} - 2\frac{be}{a} + f = 0$ (i.e. $b^2 = ac$)

d\frac{ac}{a^2} - 2\frac{be}{a} + f = 0$

d\frac{c}{a} - 2\frac{eb}{ac} = \frac{2c}{b}$

Example 1.118 If the equations $x^2 + ax + 12 = 0, x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root, then find the values of $a$ and $b$.

Sol. We have,

$x^2 + ax + 12 = 0$ (1)
$x^2 + bx + 15 = 0$ (2)

Adding (1) and (2), we get

$2x^2 + (a + b)x + 27 = 0$

Now subtracting it from the third given equation, we get

$x^2 - 9 = 0 \Rightarrow x = 3, -3$

Thus, common positive root is 3. Hence,

$9 + 12 + 3a = 0$
$a = -7$ and $9 + 3b + 15 = 0$
$b = -8$

Example 1.119 The equations $ax^2 + bx + a = 0$ and $x^2 - 2x^2 + 2x - 1 = 0$ have two roots common. Then find the value of $a + b$.

Sol. By observation, $x = 1$ is a root of equation $x^2 - 2x^2 + 2x - 1 = 0$. Thus we have

$(x - 1)(x^2 + x + 1) = 0$

Now roots of $x^2 - x + 1 = 0$ are non-real.

Then equation $ax^2 + bx + a = 0$ has both roots common with

$x^2 - x + 1 = 0$. Hence, we have

$\frac{a}{1} = \frac{b}{-1} = \frac{a}{1}$
or $a + b = 0$

Concept Application Exercise 1.6

1. If $x^2 + ax + bx + ca = 0$ and $x^2 + bx + ca = 0$ have a common root, then prove that their other roots satisfy the equation $x^2 + ca + ab = 0$. 
RELATION BETWEEN COEFFICIENT AND ROOTS OF N-DEGREE EQUATIONS

- Let \( \alpha \) and \( \beta \) be roots of quadratic equation \( ax^2 + bx + c = 0 \).
  Then by factor theorem
  \[
  ax^2 + bx + c = a(x - \alpha)(x - \beta)
  \]
  Comparing coefficients, we have
  \[
  a + \beta = -\frac{b}{a} \quad \text{and} \quad a\beta = \frac{c}{a}
  \]

- Let \( \alpha, \beta, \gamma \) are roots of cubic equation \( ax^3 + bx^2 + cx + d = 0 \).
  Then,
  \[
  ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)
  \]
  Comparing coefficients, we have
  \[
  a + \beta + \gamma = -\frac{b}{a} \quad \text{and} \quad a\beta + a\gamma + a\beta = \frac{c}{a} \quad \text{and} \quad a\beta\gamma = \frac{d}{a}
  \]

- If \( \alpha, \beta, \gamma, \delta \) are roots of \( ax^4 + bx^3 + cx^2 + dx + e = 0 \), then
  \[
  a + \beta + \gamma + \delta = -\frac{b}{a} \quad \text{and} \quad a\beta + a\gamma + a\delta + a\beta = \frac{c}{a} \quad \text{and} \quad a\beta\gamma + a\beta\delta + a\gamma\delta = \frac{d}{a} \quad \text{and} \quad a\beta\gamma\delta = \frac{e}{a}
  \]

In general, if \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the roots of equation \( ax^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0 \), then sum of the roots is
  \[
  \alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}
  \]
  Sum of the product taken two at a time is
  \[
  \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n = \frac{a_{n-2}}{a_n}
  \]
  Sum of the product taken three at a time is \( -\alpha_1a_0 \) and so on.
  Product of all the roots is
  \[
  \alpha_1\alpha_2\cdots\alpha_n = (-1)^n \frac{a_0}{a_n}
  \]

Note:
- A polynomial equation of degree \( n \) has \( n \) roots (real or imaginary).
- If all the coefficients are real then the imaginary roots occur in conjugate pairs, i.e., number of imaginary roots is always even.
- If the degree of a polynomial equation is odd, then the number of real roots will also be odd. It follows that at least one of the roots will be real.

SOLVING CUBIC EQUATION

By using factor theorem together with some intelligent guessing, we can factorise polynomials of higher degree.
In summary, to solve a cubic equation of the form \( ax^3 + bx^2 + cx + d = 0 \),
1. obtain one factor \( (x - \alpha) \) by trial and error
2. factorize \( ax^3 + bx^2 + cx + d = 0 \) as \( (x - \alpha)(bx^2 + kx + s) = 0 \)
3. solve the quadratic expression for other roots

Example 1.120: If \( \alpha, \beta, \gamma \) are the roots of the equation \( x^3 + 4x + 1 = 0 \), then find the value of \( (\alpha + \beta + \gamma)^3 + (\beta + \gamma + \alpha)^3 + (\gamma + \alpha + \beta)^3 \).

Sol. For the given equation \( \alpha + \beta + \gamma = 0 \),
\[
\alpha\beta + \beta\gamma + \gamma\alpha = 4, \quad \alpha\beta\gamma = -1
\]
Now,
\[
(\alpha + \beta + \gamma)^3 + (\beta + \gamma + \alpha)^3 + (\gamma + \alpha + \beta)^3 = (\alpha + \beta + \gamma)^3 + 3(\alpha + \beta + \gamma)^2(\beta + \gamma + \alpha) + 3(\alpha + \beta + \gamma)(\beta + \gamma + \alpha)^2 + (\beta + \gamma + \alpha)^3
\]
\[
= a\beta + \beta\gamma + \gamma\alpha = 4 + 3(-1)^2 + 3(-1)^2 + (-1)^3
\]
\[
= 4
\]

Example 1.121: Let \( \alpha + i\beta \) (\( \alpha, \beta \in R \)) be a root of the equation \( x^3 + qx + r = 0 \), \( q, r \in R \). Find a real cubic equation, independent of \( \alpha \) and \( \beta \), whose one root is \( 2\alpha \).

Sol. If \( \alpha + i\beta \) is a root then, \( \alpha - i\beta \) will also be a root. If the third root is \( \gamma \), then
\[
(\alpha + i\beta) + (\alpha - i\beta) + \gamma = 0
\]
\[
\Rightarrow \gamma = -2\alpha
\]
But \( \gamma \) is a root of the given equation \( x^3 + qx + r = 0 \). Hence,
\[
(\alpha + i\beta)^3 + (\alpha - i\beta)^3 + \gamma^3 = 0
\]
\[
\Rightarrow (2\alpha)^3 + (2\alpha)^3 - 2\alpha^3 = 0
\]
Therefore, \( 2\alpha \) is a root of \( t^3 + qt - r = 0 \), which is independent of \( \alpha \) and \( \beta \).

Example 1.122: In equation \( x^4 - 2x^3 + 4x^2 + 6x - 21 = 0 \) if two of its roots are equal in magnitude but opposite in sign, find the roots.

Sol. Given that \( \alpha + \beta = 0 \) but \( \alpha + \beta + \gamma + \delta = 2 \). Hence,
\[
\gamma + \delta = 2
\]
Let \( c\alpha = p \) and \( c\delta = q \). Therefore, given equation is equivalent to \( (x^2 + p)(x^2 - 2x + q) = 0 \). Comparing the coefficients, we get
Example 1.123: \[ p + q = 4, \quad -2p = 6, \quad pq = -21. \] Therefore, \( p = -3, \quad q = 7 \) and they satisfy \( pq = -21. \) Hence,
\[ (x^2 - 3)(x^2 - 2x + 7) = 0 \]
Therefore, the roots are \( \sqrt{3} \) and \( \sqrt{3}i, \) (where \( i = \sqrt{-1} \)).

Example 1.124: Solve the equation \( x^3 - 13x^2 + 15x + 189 = 0 \) if one root exceeds the other by 2.

Solution: Let the roots be \( a, \alpha + 2, \beta. \) Sum of the roots is \( 2\alpha + \beta + 2 = 13. \)
\[ \beta = 11 - 2\alpha \] (1)

Sum of the product of roots taken two at a time is
\[ a(\alpha + 2) + (\alpha + 2)\beta + \beta a = 15 \] or
\[ a^2 + 2a + 2(\alpha + 2) + \beta a = 15 \] (2)

Product of the roots is
\[ \alpha(\alpha + 2) = -189 \] (3)

Eliminating \( \beta \) from (1) and (2), we get
\[ a^2 + 2a + 2(\alpha + 2) + (11 - 2a) = 15 \] or
\[ 3a^2 - 20a - 7 = 0 \]
\[ (a - 7)(3a + 1) = 0 \]
\[ a = 7 \text{ or } -\frac{1}{3} \]
\[ \beta = -3, \frac{35}{3} \]

Out of these values, \( a = 7, \beta = -3 \) satisfy the third relation \( \alpha(\alpha + 2) = -189, \) i.e., \(-21)(9) = -189. Hence, the roots are \( 7, 2, -3 \) or \( 7, 9, -3. \)

REPEATED ROOTS

In equation \( f(x) = 0, \) if \( f(x) \) is a polynomial function, and if it has roots \( a, \alpha, \beta, \ldots \) or \( \alpha \) is a repeated root, then \( f(x) = 0 \) is equivalent to \( (x - \alpha)^n(x - \beta) \cdots = 0, \) from which we can conclude that \( f'(x) = 0 \) or \( 2(x - \alpha)[(x - \beta) \cdots] + (x - \alpha)[(x - \beta) \cdots] = 0 \) or \( (x - \alpha)[(x - \beta) \cdots] + (x - \alpha)[(x - \beta) \cdots] = 0 \) has root \( \alpha. \)

Thus if a root occurs twice in equation then it is common in equations \( f(x) = 0 \) and \( f'(x) = 0. \)

Similarly, if a root occurs thrice in equation, then it is common in the equations \( f(x) = 0, \) \( f'(x) = 0 \) and \( f''(x) = 0. \)

Example 1.125: If \( x - c \) is a factor of order \( m \) of the polynomial \( f(x) \) of degree \( n \) \((1 < m < n), \) then find the polynomials for which \( x - c \) is a root.

Solution: From the given information we have \( f(x) = (x - c)^n g(x), \) where \( g(x) \) is polynomial of degree \( n - m. \) Then \( x = c \) is common root for the equations \( f(x) = 0, f'(x) = 0, f''(x) = 0, \ldots, f^{(m-1)}(x) = 0, \) where \( f^{(i)}(x) \) represents \( i^{th} \) derivative of \( f(x) \) w.r.t. \( x. \)

Example 1.126: If \( ax^3 + bx^2 + c_x + d_1 = 0 \) and \( ax^3 + bx^2 + c_2 + d_2 = 0 \) have a pair of repeated roots common, then prove that
\[
\begin{vmatrix}
3a_1 & 2b_1 & c_1 \\
3a_2 & 2b_2 & c_2 \\
a_1b_1 - a_2b_2 & c_1 - c_2 & d_1 - d_2
\end{vmatrix} = 0
\]

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Solution: If \( f(x) = ax^3 + bx^2 + c_x + d_1 = 0 \) has roots \( a, \alpha, \beta, \) then \( g(x) = bx^2 + c_x + d_2 = 0 \) must have roots \( a, \alpha, \beta. \) Hence,
\[ a, \alpha^2 + b, \alpha^2 + c, \alpha + d_1 = 0 \] (1)
\[ a, \beta^2 + b, \beta^2 + c, \beta + d_2 = 0 \] (2)

Now, \( \alpha \) is also a root of equations
\[ 3a_1x^2 + 2bx + c_1 = 0 \] (3)
\[ 3a_2x^2 + 2bx + c_2 = 0 \] (4)

Also, from \( a_1 \times (1) - a_2 \times (2), \) we have
\[ (a_1b_1 - a_2b_2)a + (c_1 - c_2)a + d_1 - d_2 = 0 \] (5)

Eliminating \( a \) from (3), (4), and (5), we have
\[
\begin{vmatrix}
3a_1 & 2b_1 & c_1 \\
3a_2 & 2b_2 & c_2 \\
a_1b_1 - a_2b_2 & c_1 - c_2 & d_1 - d_2
\end{vmatrix} = 0
\]

Concept Application Exercise 1.7

1. If \( b^2 < 2ac, \) then prove that \( ax^2 + bx + c + d = 0 \) has exactly one real root.

2. If two roots of \( x^2 - ax^2 + bx + c = 0 \) are equal in magnitude but opposite in signs, then prove that \( ab = c. \)

3. If \( a, \beta, \gamma \) are the roots of \( x^2 + 8 = 0, \) then find the equation whose roots are \( a^2 + \beta^2 + \gamma^2. \)

4. If \( a, \beta, \gamma \) are the roots of the equation \( x^2 - px + q = 0, \) then find the cubic equation whose roots are \( a\beta, \beta\gamma, \) and \( \alpha\gamma. \)

5. If the roots of equation \( ax^2 + bx + c = 0 \) are \( a_1, a_2, \) and \( a_3, (a, b \neq 0), \) then find the equation whose roots are
\[ a_1a_2 + a_2a_3 + a_3a_1, \alpha_1a_2 + a_2a_3 + a_3\alpha_1, \alpha_3a_1 + a_1a_2 + a_2\alpha_3, \alpha_3a_1 + a_1a_2 + a_2\alpha_3, \alpha_3a_1 + a_1a_2 + a_2\alpha_3, \alpha_3a_1 + a_1a_2 + a_2\alpha_3. \]

QUADRATIC EXPRESSION IN TWO VARIABLES

The general quadratic expression \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c \) can be factorized into two linear factors. Given quadratic expression is
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c \] (1)

Corresponding equation is
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \] (2)

or
\[ ax^2 + 2(hy + g)x + hy^2 + 2fy + c = 0 \]
\[ a \neq 0 \quad \text{or} \quad \frac{x = -2(hy + g) \pm \sqrt{(hy + g)^2 - 4(hy + g)y^2 + 2fy + c}}{2a} \]
\[ \Rightarrow x = \frac{-(hy + g) \pm \sqrt{hy^2 + g^2 + 2ghy - aby^2 - 2afy - ac}}{a} \]
\[ \Rightarrow x = hy + g = \pm \sqrt{hy^2 + g^2 + 2ghy - aby^2 - 2afy - ac} \] (3)

Now, expression (1) can be resolved into two linear factors if \((h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac\) is a perfect square and \(h^2 - ab > 0. \) But \((h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac\) will be a perfect square if